The moduli stack of vector bundles on a curve

Norbert Hoffmann
norbert.hoffmann@fu-berlin.de

Abstract

This expository text tries to explain briefly and not too technically the notions of stack and algebraic stack, emphasizing as an example the moduli stack of vector bundles on an algebraic curve.

Introduction

The aim of this text is to explain what an algebraic stack is, in particular what the moduli stack of vector bundles on an algebraic curve is. (At the Allahabad conference, a shorter form of this introduction to stacks was followed by a brief report on the paper [6] about the birational type of moduli stacks for vector bundles with some extra structure.)

Algebraic stacks were introduced by Deligne-Mumford [3] and by M. Artin [1]. In order to motivate them, we recall the notion of fine moduli scheme and the usual problem with its existence in the presence of automorphisms in section 1. Then we define stacks as some sort of sheaves of groupoids in section 2, and finally we discuss the algebraicity notions of Deligne-Mumford and of M. Artin in section 3.

Warning: In the present text, several technicalities are suppressed or oversimplified (in order not to obscure the basic ideas). For example, open coverings and gluing are used freely without discussing the topology, where the correct thing would be to specify an appropriate Grothendieck topology and replace gluing by descent. Moreover, we use a simplified notion of prestack in order to avoid the technically better, but less suggestive notion of fibered category; cf. [5, Exposé VI]. Also finiteness conditions are not systematically taken into account; for example, we omit the condition that an algebraic stack has to be quasi-separated.

Full technical details about algebraic stacks can be found in the textbook [7] of Laumon and Moret-Bailly, or in a book in preparation by Behrend, Conrad, Edidin, Fulton, Fantechi, Göttsche and Kresch [2]. Expository texts have also been written by Vistoli [10], Sorger [9] and Gomez [4].

1 Fine moduli schemes and why they don't exist

Let $k = \bar{k}$ be an algebraically closed field, and let C be a connected smooth projective algebraic curve of genus $g \geq 2$ over k. We consider (algebraic) vector bundles E of fixed rank r on C. Their basic discrete invariant is the degree $\deg(E) \in \mathbb{Z}$, defined as the degree of the line bundle $\det(E) := \Lambda^r E$.

(Recall that the degree of a line bundle L on C is by definition the number of zeros minus the number of poles of any nonzero rational section s of L, both counted with multiplicities; this difference does not depend on the choice of s.)

"Classifying" vector bundles E on C of given rank r and degree d means understanding the set of isomorphism classes

$$\mathrm{Bun}_{r,d} := \{E \text{ vector bdl. on } C \text{ of rank } r, \text{ deg. } d\} / \cong .$$

This set is roughly the set of points of some algebraic variety, the *moduli space* of such vector bundles E. More precisely, Seshadri [8] constructed in 1967 a connected smooth quasiprojective variety

$$\mathfrak{Bun}_{r,d}^{\mathrm{stab}}$$

of dimension $r^2(g-1)+1$ over k whose set of k-valued points is ... not quite $\mathrm{Bun}_{r,d}$, but at least the subset $\mathrm{Bun}_{r,d}^{\mathrm{stab}} \subseteq \mathrm{Bun}_{r,d}$ consisting of the isomorphism classes of stable vector bundles E.

(Recall that a nonzero vector bundle E on C is called *stable* if

$$\deg(E')/\mathrm{rank}(E') < \deg(E)/\mathrm{rank}(E)$$

holds for every proper subbundle $0 \neq E' \subsetneq E$.)

How to give the set $\mathrm{Bun}_{r,d}$ (or $\mathrm{Bun}_{r,d}^{\mathrm{stab}}$) the structure of an algebraic variety over k? One way to do this is to decide, for every variety S over k, which maps of the underlying point sets $S(k) \to \mathrm{Bun}_{r,d}$ are actually morphisms of k-varieties. The natural answer is this: such a map should be a morphism of k-varieties if and only if it comes from a vector bundle $\mathcal{E}$ on $C \times_k S$, in the sense that it sends every point $s \in S(k)$ to the isomorphism class of the restriction $\mathcal{E}_s := \mathcal{E}|_{C \times \{s\}}$. This line of thought motivates the notion of fine moduli scheme, which we recall now.

Definition 1.1. *Let S be a k-scheme. We denote by*

$$\mathrm{Bun}_{r,d}(S) := \{\mathcal{E} \text{ vector bdl. on } C \times_k S \text{ of rank } r, \text{ deg. } d\}/\cong$$

the set of isomorphism classes of rank r vector bundles $\mathcal{E}$ on $C \times_k S$ with constant degree d; here the degree $\deg(\mathcal{E})$ is by definition the locally constant function on S which assigns to each point $s \in S$ the integer $\deg(\mathcal{E}_s)$.

Every morphism of k-schemes $f : T \to S$ induces a map

$$f^* : \mathrm{Bun}_{r,d}(S) \longrightarrow \mathrm{Bun}_{r,d}(T), \quad [\mathcal{E}] \mapsto [f^*\mathcal{E}]$$

by pullback; thus we get a contravariant functor

$$\mathrm{Bun}_{r,d}(\_) : \mathsf{Schemes}/k \longrightarrow \mathsf{Sets}$$

from the category of schemes over k to the category of sets.

Definition 1.2. *A scheme M over k is a* fine moduli scheme *for vector bundles E of rank r and degree d on C if M represents the functor $\mathrm{Bun}_{r,d}(\_)$.*

More explicitly, M is such a fine moduli scheme of vector bundles if

$$\{\varphi : S \to M \ k\text{-morphism}\} = \{\mathcal{E} \text{ vector bdl. on } C \times_k S \text{ of rank } r, \text{ deg. } d\}/\cong \quad (1)$$

where the equality sign means the existence of a bijection which is functorial in S.

By Yoneda's lemma, the fine moduli scheme M is unique up to unique isomorphisms if it exists. Unfortunately, it doesn't exist, i. e. the functor $\mathrm{Bun}_{r,d}(\_)$ is not representable. To see why, we compare how both sides of (1) behave under gluing. So assume given an open covering $S = \bigcup_i U_i$ of the k-scheme S.

- For any k-scheme M, a k-morphism $\varphi : S \to M$ is given by
  - a k-morphism $\varphi_i : U_i \to M$ for each i

  satisfying the condition
  - $\varphi_i = \varphi_j$ on the intersection $U_i \cap U_j$ for all i, j.

- A vector bundle $\mathcal{E}$ on $C \times_k S$ is given by
  - a vector bundle $\mathcal{E}_i$ on $C \times_k U_i$ for each i, and
  - an isomorphism $\alpha_{ij} : \mathcal{E}_i \xrightarrow{\sim} \mathcal{E}_j$ over $C \times_k (U_i \cap U_j)$ for each i, j,

  satisfying the condition
  - $\alpha_{il} = \alpha_{jl} \circ \alpha_{ij}$ over $C \times_k (U_i \cap U_j \cap U_l)$ for all i, j, l.

We see that both sides of (1) behave fundamentally different under gluing. To make this more precise, we recall some terminology.

Definition 1.3. *i) A* presheaf *is a contravariant functor $F : \mathsf{Schemes}/k \to \mathsf{Sets}$.*
ii) A presheaf F is a sheaf *if the following condition holds for every open covering of k-schemes $S = \bigcup_i U_i$:*
Given a tuple $(\varphi_i)_i$ of elements $\varphi_i \in F(U_i)$ such that φ_i and φ_j have the same image in $F(U_i \cap U_j)$ for all i, j, there is a unique element $\varphi \in F(S)$ whose image in $F(U_i)$ is φ_i for all i.

Now the above observations about gluing can be summarized as follows. The presheaf represented by any scheme M over k is a sheaf, but it is easy to check that the presheaf $\mathrm{Bun}_{r,d}(\_)$ is not a sheaf. Hence the latter is not representable, i. e. there is no fine moduli scheme.

This is a typical problem. Functors of isomorphism classes like $\mathrm{Bun}_{r,d}(\_)$ are usually not representable, the standard exception being isomorphism classes of objects without automorphisms. (In that case, the above isomorphisms α_{ij} are unique if they exist, and the cocycle relation $\alpha_{il} = \alpha_{jl} \circ \alpha_{ij}$ is automatically satisfied, so the presheaf of isomorphism classes is actually a sheaf, and in fact in many cases representable.)

The classical solution for this problem is the following. Instead of representing the functor $\mathrm{Bun}_{r,d}(\_)$, we approximate it by a representable functor, as closely as possible in a specific sense. The representing scheme is then called a *coarse moduli scheme.* For example, Seshadri's variety $\mathfrak{Bun}_{r,d}^{\mathrm{stab}}$ is a coarse moduli scheme of stable vector bundles E on C with rank r and degree d.

A more radical solution for this problem is given by the notion of stack. It is motivated by the above observation that objects (e. g. vector bundles) don't glue like maps to a set. Even if we are just interested in isomorphism classes, we need to keep track of actual isomorphisms (in particular of automorphisms) in order to understand gluing. This leads to the idea that a moduli space should not be an underlying *set* (of isomorphism classes) endowed with some geometric structure, it should be an underlying *groupoid* (of objects and their isomorphisms) endowed with some geometric structure. That's roughly what a stack is.

2 Stacks

Definition 2.1. *A* groupoid *is a category in which every morphism is invertible.*

Example 2.2. The vector bundles E on C of rank r and degree d, together with the isomorphisms of vector bundles as morphisms, form a groupoid.

Example 2.3. If a group G acts on a set X, then one has a quotient groupoid X/G: its objects are the elements $x \in X$, its morphisms from x to x' are the elements $g \in G$ satisfying $g \cdot x = x'$, and its composition law is the multiplication in G.

Note that the isomorphism class of the object x in X/G is precisely the orbit of the element x in X, whereas the automorphism group of the object x in X/G is precisely the stabilizer of the element x in X.

When replacing sets by groupoids, presheaves (in the sense of definition 1.3) get replaced by prestacks, and sheaves get replaced by stacks; more precisely:

Definition 2.4. *A* prestack $\mathcal{M}$ *over* k *consists of*

- *a groupoid* $\mathcal{M}(S)$ *for each* k*-scheme* S*,*
- *a functor* $f^* : \mathcal{M}(S) \to \mathcal{M}(T)$ *for each* k*-morphism* $f : T \to S$*, and*
- *an isomorphism of functors* $c_{f,g} : g^* f^* \xrightarrow{\sim} (f \circ g)^*$ *for each composable pair of* k*-morphism* $g : U \to T$ *and* $f : T \to S$*,*

such that the isomorphisms $c_{f,g}$ *satisfy the following two compatibility conditions:*

- $c_{f,\mathrm{id}_T} = \mathrm{id}_{f^*}$ *and* $c_{\mathrm{id}_S,f} = \mathrm{id}_{f^*}$ *for all* k*-morphisms* $f : T \to S$*.*
- *The diagram of isomorphisms of functors*

$$\begin{tikzcd} h^* g^* f^* \arrow[r, "c_{g,h} f^*"] \arrow[d, "h^* c_{f,g}"'] & (g \circ h)^* f^* \arrow[d, "c_{f,g \circ h}"] \\ h^*(f \circ g)^* \arrow[r, "c_{f \circ g,h}"'] & (f \circ g \circ h)^* \end{tikzcd}$$

commutes for all triples of k*-morphisms* $h : V \to U$*,* $g : U \to T$ *and* $f : T \to S$*.*

Example 2.5. The prestack $\mathcal{B}un_{r,d}$ of vector bundles E on C with rank r and degree d consists of the following data.

- Objects of the groupoid $\mathcal{B}un_{r,d}(S)$ are vector bundles $\mathcal{E}$ on $C \times_k S$ with rank r and (constant) degree d.
- Morphisms of the groupoid $\mathcal{B}un_{r,d}(S)$ are isomorphisms $\alpha : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ of vector bundles on $C \times_k S$.
- The functor $f^* : \mathcal{B}un_{r,d}(T) \to \mathcal{B}un_{r,d}(S)$ is the usual pullback of vector bundles along the k-morphism $f : T \to S$.
- $c_{f,g}(\mathcal{E})$ is the usual canonical isomorphism between $g^* f^* \mathcal{E}$ and $(f \circ g)^* \mathcal{E}$ whenever $g : U \to T$ and $f : T \to S$ is a composable pair of k-morphisms.

If $\mathcal{M}$ is a prestack and $f : T \hookrightarrow S$ is an embedding of k-schemes, then we will write $\mathcal{E}|_T$ and $\alpha|_T$ for the images in $\mathcal{M}(T)$ of objects $\mathcal{E}$ and morphisms α in $\mathcal{M}(S)$. If $g : U \hookrightarrow T$ is another embedding, then the isomorphism $c_{f,g}$ allows us to identify $(\mathcal{E}|_T)|_U$ with $\mathcal{E}|_U$ and $(\alpha|_T)|_U$ with $\alpha|_U$.

Just like a sheaf is a presheaf whose sections can be glued, a stack is a prestack whose objects and morphisms can be glued; more precisely:

Definition 2.6. *A prestack* $\mathcal{M}$ *over* k *is a* stack *if the following two conditions hold for every open covering of* k*-schemes* $S = \bigcup_i U_i$*:*

- *Given two objects* $\mathcal{E}, \mathcal{E}'$ *of* $\mathcal{M}(S)$ *and for each* i *a morphism* $\alpha_i : \mathcal{E}|_{U_i} \to \mathcal{E}'|_{U_i}$ *in* $\mathcal{M}(U_i)$ *such that*

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j} \text{ in } \mathcal{M}(U_i \cap U_j) \text{ for all } i, j,$$

there is a unique morphism $\alpha : \mathcal{E} \to \mathcal{E}'$ *in* $\mathcal{M}(S)$ *such that* $\alpha_i = \alpha|_{U_i}$ *for all* i*.*

- *Given for each i an object $\mathcal{E}_i$ in $\mathcal{M}(U_i)$ and for each i, j an isomorphism $\alpha_{ij} : \mathcal{E}_i|_{U_i \cap U_j} \to \mathcal{E}_j|_{U_i \cap U_j}$ in $\mathcal{M}(U_i \cap U_j)$ such that*

  $$\alpha_{il} = \alpha_{jl} \circ \alpha_{ij} \text{ in } \mathcal{M}(U_i \cap U_j \cap U_l) \text{ for all } i, j, l,$$

  there is an object $\mathcal{E}$ in $\mathcal{M}(S)$ and for each i an isomorphism $\beta_i : \mathcal{E}|_{U_i} \to \mathcal{E}_i$ in $\mathcal{M}(U_i)$ such that $\beta_j = \alpha_{ij} \circ \beta_i$ in $\mathcal{M}(U_i \cap U_j)$ for all i, j.

Example 2.7. The gluing behavior of vector bundles described in section 1 means that the prestack $\mathcal{B}un_{r,d}$ of example 2.5 is actually a stack.

Both prestacks and stacks over k form 2-categories: A 1-morphism $\Phi : \mathcal{M} \to \mathcal{M}'$ is given by a functor $\Phi(S) : \mathcal{M}(S) \to \mathcal{M}'(S)$ for each k-scheme S, together with an isomorphism of functors $f^* \circ \Phi(S) \cong \Phi(T) \circ f^*$ for each k-morphism $f : T \to S$ satisfying appropriate compatibility conditions; a 2-morphism $\tau : \Phi_1 \Rightarrow \Phi_2$ is given by a natural transformation $\tau(S) : \Phi_1(S) \Rightarrow \Phi_2(S)$ for each k-scheme S satisfying appropriate compatibility conditions.

Recall that for every presheaf, one has an associated sheaf (obtained by the sheafification process). For every prestack, one similarly has an associated stack, obtained by an analogous process called stackification.

3 Algebraic stacks

A sheaf in the sense of definition 1.3 is not yet a very geometric object; it only yields a geometric object (namely a fine moduli scheme) if it is representable. Similarly, a stack is not yet a very geometric object; it only is so if it is "algebraic", i. e. if it satisfies some further condition which we explain next. There are two variants of this condition, one due to Deligne-Mumford [3] and one due to M. Artin [1].

Definition 3.1. *A* groupoid scheme *over k is a groupoid object in the category* $\mathsf{Schemes}/k$ *of schemes over k.*

More explicitly, a groupoid scheme over k consists of two k-schemes X, R and five k-morphisms

$$e : X \to R, \quad s : R \to X, \quad t : R \to X, \quad \circ : R \times_{s,X,t} R \to R \quad \text{and} \quad i : R \to R,$$

such that for each k-scheme S, the sets $X(S)$ and $R(S)$ of k-morphisms $S \to X$ and $S \to R$ together with the induced maps

$$e_* : X(S) \longrightarrow R(S), \qquad s_* : R(S) \longrightarrow X(S), \qquad t_* : R(S) \longrightarrow X(S),$$
$$\circ_* : R(S) \times_{s_*, X(S), t_*} R(S) \longrightarrow R(S) \qquad \text{and} \qquad i_* : R(S) \longrightarrow R(S)$$

form a (small) groupoid, in which $X(S)$ is the set of objects, $R(S)$ is the set of morphisms, e_* sends each object to its identity automorphism, s_* sends each morphism to its source, t_* to its target, $\circ_*$ sends each composable pair of morphisms to their composition, and i_* sends each morphism to its inverse.

Note that a groupoid scheme $(X, R, e, s, t, \circ, i)$ over k induces a prestack $\mathcal{M}$ over k, by sending each k-scheme S to the groupoid $\mathcal{M}(S)$ with objects $X(S)$ and morphisms $R(S)$ that has just been described.

Example 3.2. Let a group scheme G over k act on a scheme X over k. In analogy to example 2.3, one has a quotient groupoid scheme $(X, R := G \times_k X, e, s, t, \circ, i)$ over k where $s : G \times_k X \to X$ is the second projection, $t : G \times_k X \to X$ is the group action, and $\circ : G \times_k G \times_k X \to G \times_k X$ is given by the group multiplication.

Definition 3.3. *Let a group scheme G over k act on a scheme X over k. The* stack quotient X/G *is the stackification of the prestack $\mathcal{M}$ induced by the groupoid scheme $(X, G \times_k X, \ldots)$ described in example 3.2.*

Remark 3.4. The stack quotient X/G contains information not only about the orbit sets, but also about the stabilizer groups, cf. example 2.3.

Example 3.5. Let $X = \mathrm{Spec}(k)$ be just one point, and let the group scheme G over k act trivially on X. The resulting stack quotient $\mathrm{B}G := X/G$ is called the *classifying stack* associated to G. It is easy to check that $\mathrm{B}G(S)$ is precisely the groupoid of locally trivial principal G-bundles over S for every k-scheme S.

Definition 3.6. *A stack $\mathcal{M}$ over k is* algebraic *in the sense of Artin (resp. in the sense of Deligne-Mumford) if $\mathcal{M}$ is (1-isomorphic to) the stackification of the prestack induced by a groupoid scheme $(X, R, e, s, t, \circ, i)$ over k in which the morphisms $s, t : R \to X$ are both smooth (resp. étale).*

A stack $\mathcal{M}$ that is algebraic in the sense of Artin (resp. of Deligne-Mumford) is usually called an Artin stack (resp. Deligne-Mumford stack).

Example 3.7. Let a group scheme G over k act on a scheme X over k.

i) Suppose that G is smooth over k. Then X/G is an Artin stack. In particular, the classifying stack $\mathrm{B}G$ is an Artin stack in this case.

ii) Suppose that G is étale over k. Then X/G is a Deligne-Mumford stack. In particular, the classifying stack $\mathrm{B}G$ is a Deligne-Mumford stack in this case.

Remark 3.8. One can guess how to apply geometric notions to a quotient stack X/G: the natural thing to do is to apply the same notions to the scheme X in a G-equivariant way. This actually works, no matter how bad the action of G on X is (maybe huge stabilizers, non-closed orbits, etc.). Thus the geometry of the quotient stack X/G is really just the G-equivariant geometry of X, even if this would not at all be the case for any kind of orbit space X/G because the action is bad.

Remark 3.9. The above way of applying geometric notions to quotient stacks X/G can often be generalized to stacks given by a groupoid scheme $(X, R, \ldots)$ in a rather formal way. In some sense, this is one reason why *algebraic* stacks are quite geometric objects, even if general stacks aren't.

Proposition 3.10. *The stack $\mathcal{B}un_{r,d}$ is algebraic in the sense of Artin.*

Sketch of the proof. One can define what an open covering of a stack is. A stack is algebraic if it admits an open covering by algebraic stacks. Choose a very ample line bundle $\mathcal{O}(1)$ on C, and let

$$P(n) := d + r(1 - g + n \deg \mathcal{O}(1))$$

be the common Hilbert polynomial of all vector bundles E on C with rank r and degree d. An appropriate open subscheme in one of Grothendieck's Quot-schemes

$$X_n \subseteq \mathrm{Quot}_P(\mathcal{O}(-n)^{P(n)})$$

is a fine moduli scheme of pairs (E, B) consisting of a vector bundles E on C with

- $\mathrm{rank}(E) = r$ and $\deg(E) = d$,
- $E(n) := E \otimes \mathcal{O}(n)$ is generated by its global section,
- the Zariski sheaf cohomology $\mathrm{H}^1(C, E(n))$ vanishes,

and a basis B of $\mathrm{H}^0(E(n))$. $G_n := \mathrm{GL}_{P(n)}$ acts on X_n by changing this basis B. Because the above conditions on E are open conditions, X_n/G_n is an open substack of $\mathcal{B}un_{r,d}$. Since every vector bundle E on C with rank r and degree d satisfies them for $n \gg 0$, we obtain an open covering

$$\mathcal{B}un_{r,d} = \bigcup_n X_n/G_n$$

which shows that $\mathcal{B}un_{r,d}$ is an Artin stack because the X_n/G_n are. □

Remark 3.11. $\mathcal{B}un_{r,d}$ is not a Deligne-Mumford stack, because vector bundles have automorphism groups of positive dimension.

Example 3.12. The moduli stack $\mathcal{M}_g$ of smooth projective curves over k with fixed genus $g \geq 2$ is given by the following groupoid for each k-scheme S:

- The objects of $\mathcal{M}_g(S)$ are the smooth projective morphisms $\pi : \mathcal{C} \to S$ all of whose geometric fibers are connected curves of genus g.
- The morphisms in $\mathcal{M}_g(S)$ from $\pi : \mathcal{C} \to S$ to $\pi' : \mathcal{C}' \to S$ are the isomorphisms $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ of schemes over S.

$\mathcal{M}_g$ is known to be a Deligne-Mumford stack [3] (and hence also an Artin stack).

Remark 3.13. An Artin stack $\mathcal{M}$ given by a groupoid scheme $(X, R, e, s, t, \circ, i)$ is called *smooth* if X is smooth. The *dimension* of $\mathcal{M}$ is by definition the dimension of X minus the relative dimension of R over X.

For example, the stack quotient X/G of a k-scheme X modulo a smooth group scheme G over k is smooth if and only if X is, and $\dim(X/G) = \dim(X) - \dim(G)$. In particular, $\mathrm{B}G$ is smooth of dimension $-\dim(G)$.

Remark 3.14. $\mathcal{B}un_{r,d}$ is known to be smooth of dimension $r^2(g-1)$, one less than the dimension of the coarse moduli scheme $\mathfrak{Bun}_{r,d}^{\mathrm{stab}}$. This difference comes from the one-dimensional (scalar) automorphism groups of stable vector bundles which $\mathfrak{Bun}_{r,d}^{\mathrm{stab}}$ does not see, but which the stack $\mathcal{B}un_{r,d}$ does take into account.

Remark 3.15. The stack $\mathcal{M}_g$ is known to be smooth of dimension $3g-3$.

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