

EINSTEIN–HERMITIAN CONNECTION ON TWISTED HIGGS BUNDLES

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ABSTRACT. Let X be a smooth projective variety over \mathbb{C} . We prove that a twisted Higgs vector bundle (\mathcal{E}, θ) on X admits an Einstein–Hermitian connection if and only if (\mathcal{E}, θ) is polystable. A similar result for twisted vector bundles (no Higgs fields) was proved in [10]. Our approach is simpler.

RÉSUMÉ. **Connexions d’Einstein–Hermite sur les fibrés de Higgs tordus.** Soit X une variété projective lisse sur \mathbb{C} . Nous démontrons qu’un fibré de Higgs tordu (\mathcal{E}, θ) sur X possède une connexion d’Einstein–Hermite si et seulement si (\mathcal{E}, θ) est polystable. Un résultat analogue pour les fibrés vectoriels (dépourvus d’un champ de Higgs) a été démontré dans [10]. Notre approche est plus simple.

1. INTRODUCTION

Donaldson and Uhlenbeck–Yau proved that a vector bundle on a complex projective manifold admits an Einstein–Hermitian connection if and only if it is polystable [3], [9]. A generalization of Einstein–Hermitian connections for Higgs bundles was formulated by Hitchin (for curves) and Simpson (higher dimensions). They proved that a Higgs bundle (\mathcal{E}, θ) admits an Einstein–Hermitian connection if and only if it is polystable [4], [8].

Our aim here is to establish a similar result for twisted sheaves on a smooth complex projective variety. Let X be an irreducible smooth projective variety over \mathbb{C} . A *twisted vector bundle* on X is a pair $(\mathcal{X}, \mathcal{E})$, where

$$\mathcal{X} \longrightarrow X$$

is a gerbe banded by μ_n (the n -th roots of unity) for some n , and \mathcal{E} is a vector bundle over \mathcal{X} ; see [7], [6], [5], [11] for twisted bundles. A twisted Higgs bundle on X is a twisted vector bundle together with a Higgs field on it.

We prove that a twisted Higgs bundle on X admits an Einstein–Hermitian connection if and only if it is polystable (see Theorem 3.1).

Let G be a connected reductive linear algebraic group defined over \mathbb{C} . Theorem 3.1 generalizes to twisted Higgs principal G -bundles (this is explained at the end).

In [10], Wang proved a similar result for twisted vector bundles without Higgs structure.

2. TWISTED HIGGS BUNDLES

The base field will be \mathbb{C} . For any positive integer n , by μ_n we will denote the finite subgroup of \mathbb{C}^* consisting of the n -th roots of 1.

Let X be an irreducible smooth projective variety. Let

$$(2.1) \quad f : \mathcal{X} \longrightarrow X$$

be a gerbe banded by μ_n . The cotangent bundle of \mathcal{X} will be denoted by $\Omega_{\mathcal{X}}^1$. For any nonnegative integer i , let $\Omega_{\mathcal{X}}^i := \bigwedge^i \Omega_{\mathcal{X}}^1$ be the i -th exterior power.

Let

$$\mathcal{E} \longrightarrow \mathcal{X}$$

be a vector bundle. Let $End(\mathcal{E}) := \mathcal{E} \otimes \mathcal{E}^*$ be the endomorphism bundle. The associative algebra structure of $End(\mathcal{E})$ and the exterior algebra structure of $\bigoplus_{i \geq 0} \Omega_{\mathcal{X}}^i$ together define an algebra structure on $End(\mathcal{E}) \otimes (\bigoplus_{i \geq 0} \Omega_{\mathcal{X}}^i)$.

A *Higgs field* on \mathcal{E} is a section θ of $End(\mathcal{E}) \otimes \Omega_{\mathcal{X}}^1$ such that the section $\theta \wedge \theta$ of $End(\mathcal{E}) \otimes \Omega_{\mathcal{X}}^2$ vanishes identically.

A *Higgs bundle* on \mathcal{X} is a pair (\mathcal{E}, θ) , where \mathcal{E} is a vector bundle on \mathcal{X} , and θ is a Higgs field on \mathcal{E} . A Higgs bundle on \mathcal{X} will be called a *twisted Higgs bundle* on X . Given a Higgs bundle (\mathcal{E}, θ) on \mathcal{X} , a coherent subsheaf \mathcal{F} of \mathcal{E} will be called a *Higgs subsheaf* if $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_{\mathcal{X}}^1$.

Let G be a complex linear algebraic group. A *Higgs G -bundle* on X is a principal G -bundle $E_G \rightarrow X$ and a section $\beta \in H^0(X, ad(E_G) \otimes \Omega_X^1)$ such that $\beta \wedge \beta = 0$, where $ad(E_G)$ is the adjoint vector bundle.

Fix a very ample line bundle L over X . The *degree* of a torsionfree coherent sheaf \mathcal{F} on \mathcal{X} will be defined to be $\text{degree}((\det \mathcal{F})^{\otimes n})/n^2 \in \mathbb{Q}$. Note that $(\det \mathcal{F})^{\otimes n}$ descends to a line bundle on X ; its degree is computed using L . Fix a Kähler form ω_X on X representing $c_1(L)$. Since the morphism f in (2.1) is étale, the pullback

$$(2.2) \quad \omega_{\mathcal{X}} := f^* \omega_X$$

is a Kähler form on \mathcal{X} . A Higgs bundle (\mathcal{E}, θ) is called *stable* (respectively, *semistable*) if for every Higgs subsheaf \mathcal{F} with $1 \leq \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$, the inequality

$$\frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})} < \frac{\text{degree}(\mathcal{E})}{\text{rank}(\mathcal{E})} \quad (\text{respectively, } \frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})} \leq \frac{\text{degree}(\mathcal{E})}{\text{rank}(\mathcal{E})})$$

holds. A semistable Higgs bundle is called *polystable* if it is a direct sum of stable Higgs bundles.

For any vector bundle $\mathcal{E} \rightarrow \mathcal{X}$, we have a decomposition $\mathcal{E} = \bigoplus_{\chi \in \mu_n^*} \mathcal{E}_{\chi}$. Henceforth, we will consider vector bundles \mathcal{E} with $\mathcal{E}_{\chi} \neq 0$ for at most one character χ .

Define the homomorphism

$$(2.3) \quad \rho : GL(r, \mathbb{C}) \longrightarrow PGL(r, \mathbb{C}) \times \mathbb{G}_m =: H$$

by sending A to the class of A and to $(\det A)^n$. Given a vector bundle $\mathcal{E} \rightarrow \mathcal{X}$, the extension of its structure group along ρ defines a principal H -bundle $\mathcal{E}_H \rightarrow \mathcal{X}$. Since the inertia μ_n acts trivially on \mathcal{E}_H , it descends to a principal H -bundle $E_H \rightarrow X$. A Higgs field θ on \mathcal{E} induces a Higgs field θ_H on \mathcal{E}_H . This Higgs field θ_H on \mathcal{E}_H descends to a Higgs field on E_H , which we again denote by θ_H .

The definitions of Higgs (semi)stable and polystable principal bundles are recalled in [1, p. 551], [2].

Lemma 2.1. *A Higgs bundle (\mathcal{E}, θ) on \mathcal{X} is polystable if and only if the induced Higgs H -bundle (E_H, θ_H) on X is polystable.*

Proof. The central isogeny ρ in (2.3) produces a bijection of parabolic subgroups. For any parabolic subgroup $P \subset \mathrm{GL}(r, \mathbb{C})$, there is a natural bijective correspondence between the reductions of structure group of the principal $\mathrm{GL}(r, \mathbb{C})$ -bundle \mathcal{E} to P over any open subset $f^{-1}(U)$ and the reductions of structure group of the principal H -bundle E_H to $\rho(P)$ over U . This bijection proves the lemma. \square

3. EINSTEIN–HERMITIAN CONNECTION ON POLYSTABLE TWISTED HIGGS BUNDLES

A *Hermitian structure* on a vector bundle \mathcal{E} on \mathcal{X} is a smooth inner product on the fibers which is invariant under the action of μ_n on the fibers of \mathcal{E} . A Hermitian structure on \mathcal{E} produces a C^∞ complex connection on \mathcal{E} . Let (\mathcal{E}, θ) be a Higgs bundle. An *Einstein–Hermitian connection* on (\mathcal{E}, θ) is a Hermitian structure on \mathcal{E} such that corresponding connection ∇ on \mathcal{E} has the following property:

$$\Lambda_{\omega_{\mathcal{X}}}(\mathrm{Curv}(\nabla) + [\theta, \theta^*]) = c \cdot \mathrm{Id}_{\mathcal{E}},$$

for some constant scalar c , where $\Lambda_{\omega_{\mathcal{X}}}$ is the adjoint of multiplication by the Kähler form $\omega_{\mathcal{X}}$ (see (2.2)), $\mathrm{Curv}(\nabla)$ is the curvature of ∇ , and θ^* is the adjoint of θ constructed using the Hermitian form on \mathcal{E} .

Theorem 3.1. *Let (\mathcal{E}, θ) be a twisted Higgs bundle on X . Then (\mathcal{E}, θ) is polystable if and only if it admits an Einstein–Hermitian connection.*

Proof. Let (\mathcal{E}, θ) be a Higgs bundle on \mathcal{X} . First assume that (\mathcal{E}, θ) is polystable. From Lemma 2.1 we know that the induced Higgs H -bundle (E_H, θ_H) on X is polystable. A polystable Higgs H -bundle on X admits an Einstein–Hermitian connection [8], [1]. Since (E_H, θ_H) is the descent of $(\mathcal{E}_H, \theta_H)$, an Einstein–Hermitian connection on (E_H, θ_H) produces an Einstein–Hermitian connection on $(\mathcal{E}_H, \theta_H)$. A connection on \mathcal{E}_H defines connection on \mathcal{E} because the homomorphism of Lie algebras

$$\mathrm{Lie}(\mathrm{GL}(r, \mathbb{C})) \longrightarrow \mathrm{Lie}(H)$$

induced by the homomorphism ρ in (2.3) is an isomorphism. The connection on (\mathcal{E}, θ) induced by an Einstein–Hermitian connection on $(\mathcal{E}_H, \theta_H)$ is clearly Einstein–Hermitian.

Conversely, an Einstein–Hermitian connection on (\mathcal{E}, θ) induces an Einstein–Hermitian connection on the associated Higgs H -bundle $(\mathcal{E}_H, \theta_H)$, which, in turn, induces an Einstein–Hermitian connection on the descended Higgs H -bundle (E_H, θ_H) . Therefore, the Higgs H -bundle (E_H, θ_H) is polystable. Hence from Lemma 2.1 we conclude that the Higgs bundle (\mathcal{E}, θ) is polystable. \square

Let G be a connected reductive linear algebraic group defined over \mathbb{C} . Let Z be the center of G ; define $G' := [G, G]$.

The above theorem holds for principal Higgs G -bundles on \mathcal{X} . The proof is the same, but, instead of the homomorphism (2.3), we use the homomorphism

$$\rho : G \longrightarrow H := G/Z \times (G/G') : g \mapsto (p(g), q(g)^n) ,$$

where $p : G \longrightarrow G/Z$ and $q : G \longrightarrow G/G'$ are the natural projections. Note that $G/G' \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$.

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