

SOME MODULI STACKS OF SYMPLECTIC BUNDLES ON A CURVE ARE RATIONAL

INDRANIL BISWAS AND NORBERT HOFFMANN

ABSTRACT. Let C be a smooth projective curve of genus $g \geq 2$ over a field k . Given a line bundle L on C , let $\mathcal{S}ympl_{2n,L}$ be the moduli stack of vector bundles E of rank $2n$ on C endowed with a nowhere degenerate symplectic form $b : E \otimes E \rightarrow L$ up to scalars. We prove that this stack is birational to $B\mathbb{G}_m \times \mathbb{A}^s$ for some s if $\deg(E) = n \cdot \deg(L)$ is odd and C admits a rational point $P \in C(k)$ as well as a line bundle ξ of degree 0 with $\xi^{\otimes 2} \not\cong \mathcal{O}_C$. It follows that the corresponding coarse moduli scheme of Ramanathan-stable symplectic bundles is rational in this case.

1. INTRODUCTION

Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k . Soon after the coarse moduli schemes of vector bundles E over C were constructed in the sixties, the question of their rationality (in the fixed determinant case) was addressed. It is relatively easy to prove that these moduli schemes are unirational. Newstead proved that if the rank and degree are mutually coprime, then the moduli scheme is rational under a numerical condition [6]. King and Schofield showed that the assumption that the rank and degree are mutually coprime is enough to ensure that the moduli scheme is rational [5].

This coprime case is also the case where a Poincaré family of vector bundles parameterized by the coarse moduli scheme exists. A deeper relation between rationality and the existence of Poincaré families appears in the proof of King and Schofield: Arguing by induction on the rank, they use not only the birational type of the coarse moduli scheme for lower rank, but also the obstruction class against the existence of Poincaré families on it.

We address the same rationality question for moduli spaces of vector bundles with symplectic structure. Instead of coarse moduli schemes, we prefer to study moduli stacks, with the aim of showing that they are birational to $B\mathbb{G}_m \times \mathbb{A}^s$ for some s . The latter means in more classical terms that the corresponding coarse moduli scheme is rational, and that some open subscheme of it admits a Poincaré family.

Here are the moduli stacks that we work with: Given a positive integer n and a line bundle L on C , we consider the moduli stack $\mathcal{S}ympl_{2n,L}$ of rank $2n$ vector bundles E on C , equipped with a nondegenerate symplectic form $b : E \otimes E \rightarrow L$; as isomorphisms between two such bundles

$$(E, b : E \otimes E \rightarrow L) \quad \text{and} \quad (E', b' : E' \otimes E' \rightarrow L)$$

we allow all vector bundle isomorphisms between E and E' that commute with b and b' up to an automorphism of L . Such symplectic bundles (E, b) can also be viewed as principal bundles under a well-known reductive group Gp_{2n} which contains the symplectic group Sp_{2n} as a normal subgroup with \mathbb{G}_m as the quotient; see Section 2 for the details.

Our main result, Theorem 5.2, states that the stack $\mathcal{S}ympl_{2n,L}$ is birational to $B\mathbb{G}_m \times \mathbb{A}^s$ for some s if n and the degree of L are both odd. The latter condition ensures the existence of a Poincaré family on some open subscheme of the coarse moduli scheme; in that sense, it is analogous to the condition on vector bundles that their rank and degree be coprime.

For that theorem, we do not have to assume that the ground field k is algebraically closed; instead, it suffices for us that the curve C admits a rational point $P \in C(k)$ and also a line bundle ξ of degree 0 with $\xi^{\otimes 2} \not\cong \mathcal{O}_C$.

2000 *Mathematics Subject Classification.* 14H60, 14A20.

Key words and phrases. Symplectic bundle, moduli stack, rationality.

The idea of the proof is to find a *canonical* reduction of the structure group for every sufficiently general Gp_{2n} -bundle. More precisely, we show that every sufficiently general symplectic bundle (E, b) admits a canonical line subbundle of E . This allows us to reconstruct E from bundles of lower rank and some appropriate extension data; we manage to parameterise all these rationally.

2. SYMPLECTIC BUNDLES ON A CURVE

Let k be an arbitrary field. Given vector spaces V and L over k with $\dim(L) = 1$, we call a bilinear form

$$b : V \otimes V \longrightarrow L$$

alternating if $b(v, v) = 0$ for all $v \in V$. This implies that $b(v, w) + b(w, v) = 0$ for all $v, w \in V$. These two conditions are equivalent if the characteristic of k is different from 2.

The *adjoint* of b is the linear map

$$b^\# : V \longrightarrow \mathrm{Hom}(V, L), \quad v \mapsto b(_, v) : V \longrightarrow L.$$

The form b is called *nondegenerate* if $b^\#$ is an isomorphism. If b is nondegenerate, then the alternating form b is called a *symplectic form*. If b is symplectic, then $\dim(V)$ is finite and even, say $\dim(V) = 2n$. In this case, the triple (V, L, b) is isomorphic to k^{2n} equipped with the standard symplectic form, so the automorphism group of the triple (V, L, b) is isomorphic to the group

$$\mathrm{Gp}_{2n}(k) := \{(A, \lambda) \in \mathrm{GL}_{2n}(k) \times k^* \mid A^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A = \lambda \cdot \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$$

where I_n denotes the $n \times n$ unit matrix. Varying k , we obtain a reductive algebraic group Gp_{2n} . It occurs in a canonical short exact sequence

$$1 \longrightarrow \mathrm{Sp}_{2n} \longrightarrow \mathrm{Gp}_{2n} \xrightarrow{\mathrm{pr}_2} \mathbb{G}_m \longrightarrow 1$$

where Sp_{2n} denotes the usual symplectic group and $\mathrm{pr}_2(A, \lambda) := \lambda$. Using the standard fact that $\mathrm{Sp}_{2n} \subseteq \mathrm{SL}_{2n}$, it is easy to check that

$$\det(A) = \lambda^n \text{ for all } (A, \lambda) \in \mathrm{Gp}_{2n}(k). \quad (1)$$

Now let C be a smooth, geometrically irreducible, projective curve of genus $g \geq 2$ over k with a rational point $P \in C(k)$. We denote by k_P the coherent skyscraper sheaf supported at P with stalk k . A *vector bundle* E on C is a locally free coherent sheaf; a *subbundle* of E is a coherent subsheaf with torsion-free quotient. Let

$$\eta_P : H^0(E) \longrightarrow E_P$$

be the canonical evaluation map to the fibre E_P of E at P .

We consider (twisted) *symplectic bundles* E on C . Here ‘twisted’ means that we replace the 1-dimensional vector space L above by a line bundle on C ; slightly abusing notation, we use L for denoting this line bundle as well. So a twisted symplectic bundle on C consists more precisely of a rank $2n$ vector bundle E on C , a line bundle L on C and a nowhere degenerate symplectic form

$$b : E \otimes E \longrightarrow L.$$

Such symplectic bundles correspond to principal Gp_{2n} -bundles on C , as follows: Given a principal Gp_{2n} -bundle, the associated vector bundles E and L can be constructed by means of the canonical representations

$$\mathrm{pr}_1 : \mathrm{Gp}_{2n} \longrightarrow \mathrm{GL}_{2n}, \quad (A, \lambda) \mapsto A \quad \text{and} \quad \mathrm{pr}_2 : \mathrm{Gp}_{2n} \longrightarrow \mathbb{G}_m, \quad (A, \lambda) \mapsto \lambda.$$

Conversely, given a rank $2n$ symplectic bundle $(E, b : E \otimes E \longrightarrow L)$, the pair (E, L) determines a principal $(\mathrm{GL}_{2n} \times \mathbb{G}_m)$ -bundle, and b determines a reduction of its structure group to Gp_{2n} . These two constructions are inverses of each other. Eq. (1) implies that the symplectic form

$$b : E \otimes E \longrightarrow L$$

induces an isomorphism

$$b^{\otimes n} : \det(E) \xrightarrow{\sim} L^{\otimes n}.$$

Fix a line bundle L on C , and fix a positive integer n . We denote by

$$\mathrm{Sympl}_{2n, L} \quad (2)$$

the moduli stack of symplectic bundles $(E, b : E \otimes E \longrightarrow L)$ on C of the above type with $\text{rk}(E) = 2n$; it is more precisely given by the following groupoid $\mathcal{S}ympl_{2n,L}(S)$ for every k -scheme S :

- Each object consists of a vector bundle \mathcal{E} of rank $2n$ on $C \times_k S$, a line bundle \mathcal{L} on $C \times_k S$ which is locally in S isomorphic to the pullback of L from C , and a nowhere degenerate symplectic form $b : \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{L}$.
- Each morphism from $(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{L})$ to $(\mathcal{E}', b' : \mathcal{E}' \otimes \mathcal{E}' \longrightarrow \mathcal{L}')$ consists of vector bundle isomorphisms $\mathcal{E} \longrightarrow \mathcal{E}'$ and $\mathcal{L} \longrightarrow \mathcal{L}'$ that intertwine b and b' .

This stack is known to be algebraic (in the sense of Artin) and locally of finite type over k . By standard deformation theory, it is smooth of dimension $n(2n+1)(g-1)-1$ over k . We will show that $\mathcal{S}ympl_{2n,L}$ is also irreducible; see Corollary 3.11. We denote by

$$\mathbb{S}ympl_{2n,L}$$

the corresponding coarse moduli scheme of Ramanathan-stable symplectic bundles $(E, b : E \otimes E \longrightarrow L)$ as above. It is a normal quasi-projective variety of dimension $n(2n+1)(g-1)$ over k ; cf. [7] and [1].

Every invertible function $f \in \Gamma(S, \mathcal{O}_S^*)$ induces an automorphism of every object

$$(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{L})$$

in $\mathcal{S}ympl_{2n,L}(S)$, acting by multiplication with f on \mathcal{E} and by multiplication with f^2 on \mathcal{L} . This defines a canonical group homomorphism

$$\Gamma(S, \mathcal{O}_S^*) \longrightarrow \text{Aut}(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \longrightarrow \mathcal{L}). \quad (3)$$

It will be necessary to keep track of these scalar automorphisms systematically. Some terminology for that purpose is introduced in [4]; for the convenience of the reader, we repeat the definitions here.

Definition 2.1. A \mathbb{G}_m -stack $\mathcal{M} = (\mathcal{M}, \iota)$ over k consists of an algebraic stack \mathcal{M} over k together with a group homomorphism $\iota(\mathcal{E}) : \Gamma(S, \mathcal{O}_S^*) \rightarrow \text{Aut}_{\mathcal{M}(S)}(\mathcal{E})$ for each k -scheme S and each object \mathcal{E} of the groupoid $\mathcal{M}(S)$ such that the diagrams

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ & \searrow \iota(\mathcal{E}') & \downarrow \alpha \mapsto \varphi \alpha \varphi^{-1} \\ & & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}') \end{array} \quad \text{and} \quad \begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ f^* \downarrow & & \downarrow f^* \\ \Gamma(T, \mathcal{O}_T^*) & \xrightarrow{\iota(f^*\mathcal{E})} & \text{Aut}_{\mathcal{M}(T)}(f^*\mathcal{E}) \end{array}$$

commute for each morphism $\varphi : \mathcal{E} \rightarrow \mathcal{E}'$ in $\mathcal{M}(S)$ and each k -morphism $f : T \rightarrow S$.

For example, the above group homomorphisms (3) turn $\mathcal{S}ympl_{2n,L}$ into a \mathbb{G}_m -stack.

Definition 2.2. A \mathbb{G}_m -stack (\mathcal{M}, ι) is a \mathbb{G}_m -gerbe if $\iota(\mathcal{E}) : \Gamma(S, \mathcal{O}_S^*) \rightarrow \text{Aut}_{\mathcal{M}(S)}(\mathcal{E})$ is an isomorphism for every k -scheme S and every object \mathcal{E} of $\mathcal{M}(S)$.

For example, the open substack in $\mathcal{S}ympl_{2n,L}$ of symplectic bundles admitting only scalar automorphisms is a \mathbb{G}_m -gerbe. This open substack is known to be non-empty for all $n \geq 1$.

Definition 2.3. Let (\mathcal{M}, ι) and (\mathcal{M}', ι') be \mathbb{G}_m -stacks over k . A 1-morphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ has weight $w \in \mathbb{Z}$ if the diagram

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ (\cdot)^w \downarrow & & \downarrow \Phi(S) \\ \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota'(\Phi(\mathcal{E}))} & \text{Aut}_{\mathcal{M}'(S)}(\Phi(\mathcal{E})) \end{array}$$

commutes for every k -scheme S and every object \mathcal{E} of the groupoid $\mathcal{M}(S)$.

3. REDUCTION OF STRUCTURE GROUP

Lemma 3.1. *Let L and $I \subseteq H \subseteq V$ be finite dimensional k -vector spaces with $\dim(L) = \dim(I) = 1$ and $\dim(H) = \dim(V) - 1$. Then there is a canonical bijection between the following two collections:*

- *nondegenerate symplectic forms $b : V \otimes V \rightarrow L$ such that $I^\perp = H$, and*
- *nondegenerate symplectic forms $c : H/I \otimes H/I \rightarrow L$ together with an isomorphism of short exact sequences*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H/I & \longrightarrow & V/I & \longrightarrow & V/H \longrightarrow 0 \\ & & \downarrow c^\# & & \downarrow f & & \downarrow \bar{f} \\ 0 & \longrightarrow & \text{Hom}(H/I, L) & \longrightarrow & \text{Hom}(H, L) & \longrightarrow & \text{Hom}(I, L) \longrightarrow 0 \end{array}$$

in which $c^\#$ is the adjoint of c .

Proof. One direction is easy: Given the form b , its restriction to $H \otimes H$ induces a nondegenerate symplectic form c on H/I , and the required isomorphism $(c^\#, f, \bar{f})$ of short exact sequences is induced by the adjoint $b^\# : V \rightarrow \text{Hom}(V, L)$ of b .

For the converse direction, assume that a pairing c and an isomorphism of short exact sequences $(c^\#, f, \bar{f})$ as above are given. It is easy to see that the diagram of canonical linear maps

$$\begin{array}{ccc} H \otimes H & \longrightarrow & \Lambda^2 H \\ \downarrow & & \downarrow \\ V \otimes H & \longrightarrow & \Lambda^2 V \end{array}$$

is cocartesian; hence the linear maps $\Lambda^2 H \rightarrow L$ and $V \otimes H \rightarrow L$ given by c and f are both induced by a unique linear map $\Lambda^2 V \rightarrow L$. The latter defines a symplectic form b with the required properties. \square

Definition 3.2. For a given line bundle I on C , we denote by

$$\text{Sub}_{1,I}(\mathcal{E}^{\text{univ}}) \xrightarrow{\Phi_I} \text{Sympl}_{2n,L}$$

the moduli stack of rank $2n$ symplectic bundles $(E, b : E \otimes E \rightarrow L)$ (see (2)), together with a vector subbundle of E isomorphic to I .

More precisely, this moduli stack $\text{Sub}_{1,I}(\mathcal{E}^{\text{univ}})$ is given by the following groupoid $\text{Sub}_{1,I}(\mathcal{E}^{\text{univ}})(S)$ for each k -scheme S :

- Its objects consist of an object $(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{L})$ in $\text{Sympl}_{2n,L}(S)$ and a subbundle $\mathcal{I} \subseteq \mathcal{E}$ that is locally in S isomorphic to the pullback of I .
- Morphisms from $(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{L}, \mathcal{I} \subseteq \mathcal{E})$ to $(\mathcal{E}', b' : \mathcal{E}' \otimes \mathcal{E}' \rightarrow \mathcal{L}', \mathcal{I}' \subseteq \mathcal{E}')$ consist of three vector bundle isomorphisms

$$\mathcal{E} \rightarrow \mathcal{E}', \quad \mathcal{L} \rightarrow \mathcal{L}' \quad \text{and} \quad \mathcal{I} \rightarrow \mathcal{I}'$$

that commute with b and b' and with the inclusions $\mathcal{I} \subseteq \mathcal{E}$ and $\mathcal{I}' \subseteq \mathcal{E}'$.

Forgetting the subbundle \mathcal{I} defines the canonical 1-morphism Φ_I above.

Lemma 3.3. *$\text{Sub}_{1,I}(\mathcal{E}^{\text{univ}})$ is an algebraic stack locally of finite type over k .*

Proof. The 1-morphism Φ_I is of finite type and representable, namely by appropriate locally closed subschemes of relative Quot-schemes. The lemma thus follows from the corresponding statement about $\text{Sympl}_{2n,L}$. \square

We consider $\text{Sub}_{1,I}(\mathcal{E}^{\text{univ}})$ as a \mathbb{G}_m -stack in the canonical way that makes Φ_I a morphism of weight 1.

Definition 3.4. For a line bundle I on C , we denote by

$$\text{Ext}(\mathcal{F}^{\text{univ}}, I) \longrightarrow \text{Sympl}_{2n-2,L} \tag{4}$$

the moduli stack of rank $2n - 2$ symplectic bundles $(F, c : F \otimes F \rightarrow L)$, together with a vector bundle extension of F by I

$$0 \rightarrow I \rightarrow H \rightarrow F \rightarrow 0. \quad (5)$$

More precisely, this moduli stack $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ is given by the following groupoid $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)(S)$ for each k -scheme S :

- Its objects consist of an object $(\mathcal{F}, c : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{L})$ in $\text{Sympl}_{2n-2, L}(S)$ together with an exact sequence of vector bundles on $C \times_k S$

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{H} \rightarrow \mathcal{F} \rightarrow 0$$

such that \mathcal{I} is locally in S isomorphic to the pullback of I .

- Its morphisms consist of four vector bundle isomorphisms

$$\mathcal{F} \rightarrow \mathcal{F}', \mathcal{L} \rightarrow \mathcal{L}', \mathcal{I} \rightarrow \mathcal{I}' \text{ and } \mathcal{H} \rightarrow \mathcal{H}'$$

that commute with all the given maps.

Forgetting the extension by \mathcal{I} defines the canonical 1-morphism in (4).

Lemma 3.5. (i) $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ is an algebraic stack locally of finite type over k .

(ii) The morphism (4) in Definition 3.4 is smooth, surjective, and all its fibres are irreducible.

Proof. (i) Put $L' := I \otimes L^{\otimes(n-1)}$, and let $\mathcal{B}un_{2n-1, L'}$ denote the moduli stack of vector bundles H of rank $2n - 1$ over C with $\det(H) \cong L'$. We have a canonical 1-morphism

$$\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) \rightarrow \mathcal{B}un_{2n-1, L'}$$

that sends the exact sequence in (5) to the vector bundle H . This morphism is of finite type and representable. It is represented by appropriate locally closed subschemes of iterated relative Quot-schemes that parameterise the quotients $H \twoheadrightarrow F$ and $\Lambda^2 F \twoheadrightarrow L$. Since $\mathcal{B}un_{2n-1, L'}$ is known to be algebraic and locally of finite type over k , the same follows for $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$.

(ii) It is a direct consequence of [4, Lemma 1.10]. \square

We consider $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ as a \mathbb{G}_m -stack in the canonical way that makes (4) a morphism of weight 1.

Definition 3.6. For a line bundle I on C , the canonical 1-morphism Ψ_I

$$\begin{array}{ccc} \text{Sub}_{1, I}(\mathcal{E}^{\text{univ}}) & \xrightarrow{\Psi_I} & \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) \\ \downarrow & & \downarrow \\ \text{Sympl}_{2n, L} & & \text{Sympl}_{2n-2, L} \end{array}$$

sends each triple $(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{L}, \mathcal{I} \subseteq \mathcal{E})$ to the vector bundle $\mathcal{F} := \mathcal{I}^\perp / \mathcal{I}$, equipped with the symplectic form $c : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{L}$ induced by b , and the vector bundle extension

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{H} := \mathcal{I}^\perp \rightarrow \mathcal{F} \rightarrow 0.$$

Our next aim is to prove that Ψ_I is smooth and surjective with irreducible fibres. For that purpose, we relate it to the following stacks of lifted vector bundle extensions.

Definition 3.7. Let A be a finitely generated k -algebra. Given an exact sequence

$$0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$$

and a morphism $f : \tilde{K} \rightarrow K$ of vector bundles on $C \times_k \text{Spec}(A)$, we denote by

$$\mathcal{E}xt(Q, \tilde{K} \xrightarrow{f} K) / E$$

the moduli stack over A of lifted vector bundle extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{K} & \longrightarrow & \tilde{E} & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q \longrightarrow 0. \end{array}$$

More precisely, $\mathcal{E}xt(Q, \tilde{K} \xrightarrow{f} K)/E(S)$ denotes the following groupoid for each A -scheme $\pi : S \rightarrow \text{Spec}(A)$:

- Its objects consist of a vector bundle $\tilde{\mathcal{E}}$ on $C \times_k S$ and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^* \tilde{K} & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & \pi^* Q \longrightarrow 0 \\ & & \pi^* f \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \pi^* K & \longrightarrow & \pi^* E & \longrightarrow & \pi^* Q \longrightarrow 0. \end{array}$$

- Its morphisms are the vector bundle isomorphisms $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}'$ that commute with all the given maps.

By definition, we have the cartesian square of canonical 1-morphisms

$$\begin{array}{ccc} \mathcal{E}xt(Q, \tilde{K} \xrightarrow{f} K)/E & \longrightarrow & \mathcal{E}xt(Q, \tilde{K}) \\ \downarrow & & \downarrow f_* \\ \text{Spec}(A) & \xrightarrow{c_E} & \mathcal{E}xt(Q, K) \end{array} \quad (6)$$

in which $\mathcal{E}xt(Q, \tilde{K})$ and $\mathcal{E}xt(Q, K)$ are moduli stacks over A of vector bundle extensions (cf. [4, Example 1.9]), and c_E is the classifying morphism of the given extension E . Using [4, Lemma 1.10], this implies in particular that the stack $\mathcal{E}xt(Q, \tilde{K} \rightarrow K)/E$ is algebraic and that the stack $\mathcal{E}xt(Q, \tilde{K} \rightarrow K)/E$ is of finite type over k .

Lemma 3.8. *Suppose that the homomorphism of vector bundles*

$$f : \tilde{K} \rightarrow K$$

is surjective. Then the structural morphism

$$\mathcal{E}xt(Q, \tilde{K} \xrightarrow{f} K)/E \rightarrow \text{Spec}(A)$$

is smooth and surjective with all fibres irreducible.

Proof. It suffices to show that the 1-morphism f_* in (6) is smooth and surjective with irreducible fibres. Using [2], we can represent the morphism

$$f_* : \text{RHom}(Q, K) \rightarrow \text{RHom}(Q, \tilde{K})$$

in the derived category of finitely generated A -modules by a chain morphism

$$\begin{array}{ccc} V^0 & \xrightarrow{f^0} & \tilde{V}^0 \\ \delta \downarrow & & \downarrow \tilde{\delta} \\ V^1 & \xrightarrow{f^1} & \tilde{V}^1 \end{array}$$

of length one complexes V^\bullet and \tilde{V}^\bullet that consist of vector bundles V^0, V^1 and \tilde{V}^0, \tilde{V}^1 on $\text{Spec}(A)$. The proof of [4, Lemma 1.10] yields 1-isomorphisms

$$V^1/V^0 \xrightarrow{\sim} \mathcal{E}xt(Q, K) \quad \text{and} \quad \tilde{V}^1/\tilde{V}^0 \xrightarrow{\sim} \mathcal{E}xt(Q, \tilde{K})$$

where V^1/V^0 and \tilde{V}^1/\tilde{V}^0 are the Picard stacks over $\text{Spec}(A)$ associated to V^\bullet and \tilde{V}^\bullet [3, Exp. XVIII, 1.4]; by their construction, the diagram

$$\begin{array}{ccc} V^1/V^0 & \xrightarrow{\sim} & \mathcal{E}xt(Q, K) \\ (f^1, f^0) \downarrow & & \downarrow f_* \\ \tilde{V}^1/\tilde{V}^0 & \xrightarrow{\sim} & \mathcal{E}xt(Q, \tilde{K}) \end{array}$$

commutes up to a 2-isomorphism. Therefore, it remains to show that (f^1, f^0) is smooth and surjective with irreducible fibres.

The diagram of Picard stacks

$$\begin{array}{ccc} (V^1 \oplus \tilde{V}^0)/V^0 & \longrightarrow & V^1/V^0 \\ (f^1, -\tilde{\delta}) \downarrow & & \downarrow (f^1, f^0) \\ \tilde{V}^1 & \longrightarrow & \tilde{V}^1/\tilde{V}^0 \end{array}$$

is easily checked to be cartesian. Since the canonical morphisms

$$V^1 \oplus \tilde{V}^0 \longrightarrow (V^1 \oplus \tilde{V}^0)/V^0 \quad \text{and} \quad \tilde{V}^1 \longrightarrow \tilde{V}^1/\tilde{V}^0$$

are smooth and surjective with irreducible fibres, it suffices to prove the same for

$$(f^1, -\tilde{\delta}) : V^1 \oplus \tilde{V}^0 \longrightarrow \tilde{V}^1.$$

This is now simply a morphism of vector bundles, so we just have to show that it is surjective. For that we first note that the Nakayama's lemma allows us to assume that the ground ring A is a field, say our base field k . In this case, the cokernel of $(f^1, -\tilde{\delta})$, by the choices of V^\bullet and \tilde{V}^\bullet , is isomorphic to the cokernel of the k -linear map

$$f_* : \text{Ext}_{\mathcal{O}_C}^1(Q, K) \longrightarrow \text{Ext}_{\mathcal{O}_C}^1(Q, \tilde{K});$$

hence this cokernel vanishes if f is surjective because C is a smooth curve. \square

Corollary 3.9. *The canonical 1-morphism Ψ_I is smooth, surjective, and all its fibres are irreducible.*

Proof. Let A be a finitely generated k -algebra, and let

$$c_{\mathcal{H}} : \text{Spec}(A) \longrightarrow \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$$

be the classifying morphism of an object

$$(\mathcal{F}, \quad c : \mathcal{F} \otimes \mathcal{F} \longrightarrow \mathcal{L}, \quad 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{H} \xrightarrow{p} \mathcal{F} \longrightarrow 0) \quad (7)$$

in $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)(\text{Spec}(A))$. From Lemma 3.1 we know that the objects in $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})(\text{Spec}(A))$ over (7) correspond to lifted vector bundle extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{I}^{\text{dual}} \otimes \mathcal{L} \longrightarrow 0 \\ & & \downarrow c^{\# \text{op}} & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{F}^{\text{dual}} \otimes \mathcal{L} & \longrightarrow & \mathcal{H}^{\text{dual}} \otimes \mathcal{L} & \longrightarrow & \mathcal{I}^{\text{dual}} \otimes \mathcal{L} \longrightarrow 0; \end{array}$$

furthermore, the analogous statement holds after any base change $\pi : S \longrightarrow \text{Spec}(A)$. This means that the diagram

$$\begin{array}{ccc} \mathcal{E}xt(\mathcal{I}^{\text{dual}} \otimes \mathcal{L}, \mathcal{H} \xrightarrow{c^{\# \text{op}}} \mathcal{F}^{\text{dual}} \otimes \mathcal{L}) / \mathcal{H}^{\text{dual}} \otimes \mathcal{L} & \longrightarrow & \mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}}) \\ \downarrow & & \downarrow \Psi_I \\ S & \xrightarrow{c_{\mathcal{H}}} & \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) \end{array}$$

is cartesian. Thus the corollary follows from Lemma 3.8. \square

Corollary 3.10. *The stacks $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ and $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})$ are both smooth.*

Proof. This follows from the combination of Lemma 3.5(ii), Corollary 3.9, and the smoothness of the stack $\text{Sympl}_{2n-2,L}$. \square

Corollary 3.11. *The stacks $\text{Sympl}_{2n,L}$, $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})$ and $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ are all irreducible. In particular, they are all non-empty.*

Proof. We argue by induction on n . For $n = 0$, $\text{Sympl}_{0,L} \cong \text{Spec}(k)$ is irreducible.

For the induction step, let us assume $n \geq 1$ and that $\text{Sympl}_{2n-2,L}$ is non-empty and irreducible. According to Lemma 3.5 and Corollary 3.9, first $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ over $\text{Sympl}_{2n-2,L}$ and then $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})$ over $\text{Sympl}_{2n,L}$ are non-empty and irreducible as well, for every line bundle I . Hence $\text{Sympl}_{2n,L}$ is non-empty.

Let $[E_1]$ and $[E_2]$ be an arbitrary pair of points in $\mathcal{S}ympl_{2n,L}$ corresponding to the symplectic bundles E_1 and E_2 , respectively. It is easy to see that both E_1 and E_2 have a line subbundle isomorphic to I if $\deg(I) \ll 0$ is sufficiently negative. In this case, both points $[E_1]$ and $[E_2]$ are in the image of the irreducible stack $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})$ and hence they are in the same component of $\mathcal{S}ympl_{2n,L}$. This proves that $\mathcal{S}ympl_{2n,L}$ is indeed irreducible. Therefore, the proof is complete by induction. \square

4. AN AUXILIARY RATIONALITY RESULT

Let \mathcal{M} be an irreducible algebraic stack over k , endowed with the structure of a \mathbb{G}_m -stack. We say that \mathcal{M} is *rational* as a \mathbb{G}_m -stack if it is birational to $\text{B}\mathbb{G}_m \times \mathbb{A}^s$ for some s . Here birational means that the two stacks contain non-empty open substacks which are 1-isomorphic. We say that \mathcal{M} is *unirational* if it admits a dominant 1-morphism from a dense open subscheme of \mathbb{A}^s for some s .

For every vector bundle \mathcal{V} on \mathcal{M} , we denote by $\text{P}\mathcal{V} = \text{Gr}_1(\mathcal{V})$ the projective bundle of lines in the fibres of \mathcal{V} , and by $\mathbb{P}\mathcal{V} = \text{Gr}_{\text{rk}(\mathcal{V})-1}(\mathcal{V})$ the projective bundle of hyperplanes in the fibres of \mathcal{V} (cf. [4, Section 4]). The birational type of such projective bundles will in general depend on the action of the scalar automorphisms in \mathcal{M} on the fibres of \mathcal{V} . This action is encoded in the notion of weight [5, p. 526]; for the convenience of the reader, we repeat here the general definition [4, Definition 2.6].

Definition 4.1. Let (\mathcal{M}, ι) be a \mathbb{G}_m -stack over k . A vector bundle \mathcal{V} on \mathcal{M} has weight $w \in \mathbb{Z}$ if the diagram

$$\begin{array}{ccc} \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\iota(\mathcal{E})} & \text{Aut}_{\mathcal{M}(S)}(\mathcal{E}) \\ (-)^w \downarrow & & \downarrow \mathcal{V}(S) \\ \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{\text{id}_{\mathcal{V}(\mathcal{E})}} & \text{Aut}_{\mathcal{O}_S}(\mathcal{V}(\mathcal{E})) \end{array}$$

commutes for every k -scheme S and every object \mathcal{E} of the groupoid $\mathcal{M}(S)$.

If the \mathbb{G}_m -stack \mathcal{M} is rational and the vector bundle \mathcal{V} has some weight $w \in \mathbb{Z}$ in this sense, then both projective bundles $\text{P}\mathcal{V}$ and $\mathbb{P}\mathcal{V}$ are again rational \mathbb{G}_m -stacks; this can be seen as follows:

We may assume $\mathcal{M} = \text{B}\mathbb{G}_m \times \mathbb{A}^s$ without loss of generality. Pulling back the tautological line bundle of weight 1 on $\text{B}\mathbb{G}_m$, we obtain a line bundle of weight 1 on \mathcal{M} . Since tensoring \mathcal{V} with a line bundle does not change $\text{P}\mathcal{V}$ or $\mathbb{P}\mathcal{V}$, this reduces us to the case $w = 0$. Then the vector bundle \mathcal{V} and hence also the projective bundles $\text{P}\mathcal{V}$ and $\mathbb{P}\mathcal{V}$ are trivial over some open substack $\emptyset \neq \mathcal{U} \subseteq \mathcal{M}$ due to [4, Corollary 3.8]. This implies that $\text{P}\mathcal{V}$ and $\mathbb{P}\mathcal{V}$ are indeed both rational as \mathbb{G}_m -stacks.

Lemma 4.2. *Given line bundles L_1 and L_2 on C , there is an extension*

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0 \tag{8}$$

such that the connecting homomorphism δ in its long exact cohomology sequence

$$0 \longrightarrow H^0(L_1) \longrightarrow H^0(E) \longrightarrow H^0(L_2) \xrightarrow{\delta} H^1(L_1) \longrightarrow H^1(E) \longrightarrow H^1(L_2) \longrightarrow 0$$

has maximal rank, more precisely $\text{rk}(\delta) = \min\{\dim H^0(L_2), \dim H^1(L_1)\}$.

Proof. Serre duality allows us to assume

$$\dim H^0(L_2) \leq \dim H^1(L_1) \tag{9}$$

without loss of generality; we then have to show that δ is injective.

All such extensions (8) are classified by the affine space $\text{Ext}^1(L_2, L_1)$; those with injective connecting homomorphism δ form an open subscheme $U \subseteq \text{Ext}^1(L_2, L_1)$. We will prove $U(k) \neq \emptyset$ by estimating the dimension of the complement.

If δ is not injective, then some section $s \in H^0(L_2)$ can be lifted to E ; this means that the extension class $[E] \in \text{Ext}^1(L_2, L_1)$ is annihilated by the homomorphism

$$s^* : \text{Ext}^1(L_2, L_1) \longrightarrow \text{Ext}^1(\mathcal{O}, L_1) \cong H^1(L_1). \tag{10}$$

Now s^* is surjective because its cokernel embeds into $\text{Ext}^2(L_2/\mathcal{O}, L_1) = 0$, so

$$\text{codim}(\ker(s^*) \subseteq \text{Ext}^1(L_2, L_1)) = \dim H^1(L_1). \quad (11)$$

Such sections s are parameterized by the projective space $\text{PH}^0(L_2)$, whose dimension is smaller than the codimension in (11) since we have assumed (9). Therefore,

$$\dim(\text{Ext}^1(L_2, L_1) \setminus U) < \dim \text{Ext}^1(L_2, L_1)$$

and consequently $U \neq \emptyset$. If k is infinite, then the non-empty open subscheme U of an affine space automatically contains a k -rational point, and we are done.

So suppose that k is finite with q elements. Since (10) is a surjective k -linear map, the cardinalities of these vector spaces then satisfy

$$\frac{\#\ker(s^*)}{\#\text{Ext}^1(L_2, L_1)} = \frac{1}{q^d} \quad \text{with} \quad d := \dim H^1(L_1).$$

On the other hand, the number of such sections s up to k^* is

$$\#\text{PH}^0(L_2) \leq \#\text{PH}^1(L_1) = \frac{q^d - 1}{q - 1} < q^d.$$

Hence $\text{Ext}^1(L_2, L_1) \setminus U$ contains less k -rational points than $\text{Ext}^1(L_2, L_1)$; consequently, $U(k) \neq \emptyset$ holds for finite fields k as well. \square

Corollary 4.3. *Let $n \geq 1$ be given, and let L and I be line bundles on C .*

(i) *If $2 \deg(I) < \deg(L) + 2 - 2g$, then there is a rank $2n$ symplectic bundle*

$$(E, b : E \otimes E \longrightarrow L)$$

on C which admits a vector subbundle isomorphic to I and satisfies

$$\text{Ext}_{\mathcal{O}_C}^1(I, E) = 0.$$

(ii) *If $2 \deg(I) < \deg(L) + 2g - 2$, then there is a rank $2n$ symplectic bundle*

$$(F, c : F \otimes F \longrightarrow L)$$

on C which satisfies

$$\text{Hom}_{\mathcal{O}_C}(F, I) = 0 \quad \text{and} \quad \text{Ext}_{\mathcal{O}_C}^1(F, I) \neq 0$$

as well as

$$\text{Hom}_{\mathcal{O}_C}(I^{\text{dual}} \otimes L, F) = 0 \quad \text{and} \quad \text{Ext}_{\mathcal{O}_C}^1(I^{\text{dual}} \otimes L, F) \neq 0.$$

Proof. It suffices to treat the special case of $n = 1$, since the general case immediately follows from it by taking the fibrewise orthogonal direct sum of n copies.

(i) By assumption, the line bundle $I' := I^{\otimes -2} \otimes L$ has degree $\geq 2g - 1$, so $H^1(I') = 0$ by Clifford's theorem, and also we have $\dim H^0(I') \geq g = \dim H^1(\mathcal{O})$ by Riemann-Roch. Thus, according to Lemma 4.2, there is a vector bundle extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E' \longrightarrow I' \longrightarrow 0$$

whose connecting homomorphism $\delta : H^0(I') \longrightarrow H^1(\mathcal{O})$ is surjective; this implies that $H^1(E') = H^1(I') = 0$. Now $E := E' \otimes I$, equipped with the symplectic form b given by $\det(E) \cong I' \otimes I^{\otimes 2} \cong L$, has the required properties.

(ii) The two Ext-groups in question are nonzero for every rank 2 vector bundle F with $\det(F) \cong L$ due to Riemann-Roch. The two Hom-groups in question are isomorphic via $c^\# : F \longrightarrow F^{\text{dual}} \otimes L$, so it suffices to construct one such symplectic bundle F with $\text{Hom}(I^{\text{dual}} \otimes L, F) = 0$.

By assumption, the line bundle $L' := I^{\otimes 2} \otimes L^{\text{dual}} \otimes \omega_C^{\text{dual}}$ has negative degree, so $H^0(L') = 0$, and $\dim H^1(L') \geq g = \dim H^0(\omega_C)$ by Riemann-Roch. Thus, according to Lemma 4.2, there is a vector bundle extension

$$0 \longrightarrow L' \longrightarrow F' \longrightarrow \omega_C \longrightarrow 0$$

whose connecting homomorphism $\delta : H^0(\omega_C) \rightarrow H^1(L')$ is injective; this implies $H^0(F') = H^0(L') = 0$. Now $F := F' \otimes I^{\text{dual}} \otimes L$, equipped with the symplectic form c given by $\det(F) \cong L' \otimes \omega_C \otimes (I^{\text{dual}} \otimes L)^{\otimes 2} \cong L$, has the required properties. \square

Proposition 4.4. *Let $n \geq 1$ be given, and let L be a line bundle over our curve C .*

- (i) *If I is a line bundle on C with $2 \deg(I) < \deg(L) + 2g - 2$, then the algebraic stack $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ over $\text{Sympl}_{2n-2, L}$ is rational as a \mathbb{G}_m -stack.*
- (ii) *If I is a line bundle on C with $2 \deg(I) < \deg(L)$, then the algebraic stack $\text{Sub}_{1, I}(\mathcal{E}^{\text{univ}})$ over $\text{Sympl}_{2n, L}$ is rational as a \mathbb{G}_m -stack.*
- (iii) *If \mathcal{V} is a vector bundle on some non-empty open substack $\mathcal{U} \subseteq \text{Sympl}_{2n, L}$ with odd weight w and rank*

$$\text{rk}(\mathcal{V}) \geq \begin{cases} n & \text{if } \deg(L) \text{ is odd,} \\ 2n & \text{if } \deg(L) \text{ is even,} \end{cases}$$

then the projective bundle $P\mathcal{V}$ over $\text{Sympl}_{2n, L}$ is rational as a \mathbb{G}_m -stack.

Proof. We first show that (i) implies (ii) and (ii) implies (iii) for any fixed $n \geq 1$.

(i) \Rightarrow (ii): We consider the strictly commutative diagram of 1-morphisms

$$\begin{array}{ccc} \text{Sub}_{1, I}(\mathcal{E}^{\text{univ}}) & \supseteq & \Psi_I^{-1}(\mathcal{U}_1) \xrightarrow{\Gamma_1} \text{PW} \subseteq \text{PExt}(I^{\text{dual}} \otimes L, \mathcal{H}^{\text{univ}}) \\ \Psi_I \downarrow & & \Psi_I \downarrow \\ \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) & \supseteq & \mathcal{U}_1 \end{array}$$

in which

- $\mathcal{U}_1 \subseteq \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ is the open substack of all rank $2n - 2$ symplectic bundles $(F, c : F \otimes F \rightarrow L)$ together with a *nontrivial* vector bundle extension

$$0 \rightarrow I \rightarrow H \xrightarrow{p} F \rightarrow 0 \tag{12}$$

with the property that $\text{Hom}_{\mathcal{O}_C}(I^{\text{dual}} \otimes L, H) = 0$,

- $\text{Ext}(I^{\text{dual}} \otimes L, \mathcal{H}^{\text{univ}})$ denotes the vector bundle of weight 1 on \mathcal{U}_1 whose fibre over such an extension (12) is the vector space

$$\text{Ext}_{\mathcal{O}_C}^1(I^{\text{dual}} \otimes L, H),$$

the isomorphisms in $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ acting only on the second variable H ,

- $\mathcal{W} \subseteq \text{Ext}(I^{\text{dual}} \otimes L, \mathcal{H}^{\text{univ}})$ is the subbundle whose fibre over such an extension (12) is the inverse image, under the canonical surjection

$$(c^\# \circ p)_* : \text{Ext}_{\mathcal{O}_C}^1(I^{\text{dual}} \otimes L, H) \rightarrow \text{Ext}_{\mathcal{O}_C}^1(I^{\text{dual}} \otimes L, F^{\text{dual}} \otimes L),$$

of the line spanned by the extension class of $H^{\text{dual}} \otimes L$, and

- Γ_1 is defined by sending each triple

$$E, \quad b : E \otimes E \rightarrow L, \quad I \subseteq E$$

to the class of the extension

$$0 \rightarrow H := I^\perp \rightarrow E \xrightarrow{b^\#} I^{\text{dual}} \otimes L \rightarrow 0.$$

It is a straightforward consequence of Lemma 3.1 that Γ_1 is a 1-isomorphism onto PW . Corollary 4.3(ii) asserts that $\mathcal{U}_1 \neq \emptyset$; hence $\Psi_I^{-1}(\mathcal{U}_1) \neq \emptyset$ according to Corollary 3.9. This shows that $\text{Sub}_{1, I}(\mathcal{E}^{\text{univ}})$ is birational to PW over \mathcal{U}_1 . Since we assume (i), the \mathbb{G}_m -stack \mathcal{U}_1 is rational here, so PW is rational as well; thus (ii) follows.

(ii) \Rightarrow (iii): Assigning to each object $(\mathcal{E}, b : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{L})$ of $\text{Sympl}_{2n, L}(S)$ the restriction of \mathcal{L} to the point $P \in C(k)$ defines a line bundle of weight 2 over $\text{Sympl}_{2n, L}$. Since tensoring the given vector bundle \mathcal{V} with a line bundle does not change $P\mathcal{V}$, it suffices to consider one particular odd weight w in the proposition, say $w = 1$.

We choose a line bundle I on C with degree

$$\deg(I) = \begin{cases} (\deg(L) + 1)/2 - g & \text{if } \deg(L) \text{ is odd,} \\ \deg(L)/2 - g & \text{if } \deg(L) \text{ is even,} \end{cases} \quad (13)$$

and consider the strictly commutative diagram of 1-morphisms

$$\begin{array}{ccc} \mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}}) & \supseteq & \Phi_I^{-1}(\mathcal{U}_2) \xrightarrow{\Gamma_2} \text{P}\mathcal{H}om(I, \mathcal{E}^{\text{univ}}) \\ \Phi_I \downarrow & & \downarrow \Phi_I \swarrow \\ \mathcal{S}ympl_{2n,L} & \supseteq & \mathcal{U}_2 \end{array}$$

in which

- $\mathcal{U}_2 \subseteq \mathcal{S}ympl_{2n,L}$ is the open locus of all bundles E with $\text{Ext}_{\mathcal{O}_C}^1(I, E) = 0$,
- $\text{Hom}(I, \mathcal{E}^{\text{univ}})$ denotes the vector bundle of weight 1 on \mathcal{U}_2 whose fibre over such a symplectic bundle E is the vector space $\text{Hom}_{\mathcal{O}_C}(I, E)$,
- Γ_2 sends each object $(\mathcal{E}, b, \mathcal{I} \subseteq \mathcal{E})$ to the sheaf of all morphisms from the pullback of I to \mathcal{E} that factor through the subbundle $\mathcal{I} \subseteq \mathcal{E}$.

It is straightforward to verify that Γ_2 is an open immersion, more precisely a 1-isomorphism onto the open locus of all nonzero morphisms $I \rightarrow E$ up to k^* whose cokernel is torsion-free. Corollary 4.3(i) asserts that $\Phi_I^{-1}(\mathcal{U}_2) \neq \emptyset$; hence the stack $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})$ is birational to the stack $\text{P}\mathcal{H}om(I, \mathcal{E}^{\text{univ}})$. Since we assume (ii), the former is rational as a \mathbb{G}_m -stack; thus the latter is so as well.

Our assumption on $\text{rk}(\mathcal{V})$ together with our choice in (13) and the Riemann-Roch theorem ensure

$$\text{rk}(\text{Hom}(I, \mathcal{E}^{\text{univ}})) \leq \text{rk}(\mathcal{V}).$$

Since both $\text{Hom}(I, \mathcal{E}^{\text{univ}})$ and \mathcal{V} are weight 1 vector bundles on open substacks of $\mathcal{S}ympl_{2n,L}$, the former is a direct summand of the latter on some possibly smaller open substack of $\mathcal{S}ympl_{2n,L}$ due to [4, Lemma 3.10(iv) and Lemma 3.10(v)]. The rationality of $\text{P}\mathcal{H}om(I, \mathcal{E}^{\text{univ}})$ thus implies the rationality of $\text{P}\mathcal{V}$ according to [4, Lemma 4.5(i)]. This shows that (ii) indeed implies (iii).

Now we can prove the proposition by induction on n . For $n = 1$, we have $\mathcal{S}ympl_{2n-2,L} \cong \text{Spec}(k)$, and over it $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) \cong \text{B}\mathbb{G}_m$, so (i) holds trivially. For the induction step, we consider the strictly commutative diagram of 1-morphisms

$$\begin{array}{ccc} \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) & \supseteq & \mathcal{M} \xrightarrow{\Gamma_3} \text{P}\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I) \\ \downarrow & & \downarrow \swarrow \\ \mathcal{S}ympl_{2n-2,L} & \supseteq & \mathcal{U}_3 \end{array}$$

in which

- $\mathcal{U}_3 \subseteq \mathcal{S}ympl_{2n-2,L}$ is the open substack of all F with $\text{Hom}_{\mathcal{O}_C}(F, I) = 0$,
- $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ denotes the vector bundle of weight -1 on \mathcal{U}_3 whose fibre over such a symplectic bundle F is the vector space $\text{Ext}_{\mathcal{O}_C}^1(F, I)$,
- $\mathcal{M} \subseteq \mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ is the open substack in the inverse image of \mathcal{U}_3 where the extension $0 \rightarrow I \rightarrow H \rightarrow F \rightarrow 0$ does not split, and
- Γ_3 sends every such nonsplit extension to the extension class of H .

Here it is straightforward to verify that Γ_3 is a 1-isomorphism. Corollary 4.3(ii) asserts $\mathcal{M} \neq \emptyset$; hence $\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ and $\text{P}\mathcal{E}xt(\mathcal{F}^{\text{univ}}, I)$ over $\mathcal{S}ympl_{2n-2,L}$ are birational. By part (iii) of the induction hypothesis, the latter is rational as a \mathbb{G}_m -stack; hence the former is so as well, which suffices to complete the induction. \square

Corollary 4.5. *$\mathcal{S}ympl_{2n,L}$ is unirational.*

Proof. It is easy to see that the canonical 1-morphism

$$\Phi_I : \mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}}) \rightarrow \mathcal{S}ympl_{2n,L}$$

is dominant for every line bundle I on C of sufficiently negative degree $\deg(I) \ll 0$. According to part (ii) of Proposition 4.4, the stack $\mathcal{S}ub_{1,I}(\mathcal{E}^{\text{univ}})$ is in particular unirational; hence $\mathcal{S}ympl_{2n,L}$ is so as well. \square

5. RATIONALITY OF THE MODULI STACK

The main result is proved in this section. We start with a lemma.

Lemma 5.1. *Suppose that $k = \bar{k}$ is algebraically closed. Let $n \geq 2$ be even, and let L be a line bundle on C with*

$$\deg(L) = 2g - 1 \quad \text{and} \quad H^1(C, L(-P)) = 0$$

for our chosen point $P \in C(k)$. Then there is a symplectic bundle of rank $2n$

$$(E, b : E \otimes E \longrightarrow L)$$

on C with the following properties:

- (i) $\dim H^0(E) = n$ and $H^1(E) = 0$.
- (ii) The canonical morphism of vector bundles on C

$$\mathcal{O}_C \otimes_k H^0(E) \longrightarrow E \tag{14}$$

is injective with torsion-free cokernel.

- (iii) The induced alternating pairing on the n -dimensional vector space $H^0(E)$

$$H^0(E) \otimes H^0(E) \xrightarrow{\eta_P \otimes \eta_P} E_P \otimes E_P \xrightarrow{b_P} L_P \tag{15}$$

is nondegenerate.

Proof. It suffices to treat the special case of $n = 2$, since the general case immediately follows from it by taking the fibrewise orthogonal direct sum of $n/2$ copies.

The Riemann-Roch theorem implies $\dim H^0(L) = g = \dim H^1(\mathcal{O}_C)$. Thus, according to Lemma 4.2, there is a vector bundle extension

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{i} F \xrightarrow{p} L \longrightarrow 0$$

with $H^0(F) = H^0(\mathcal{O}_C) \cong k$ and $H^1(F) = H^1(L) = 0$. The alternating form

$$c := \det : F \otimes F \longrightarrow L$$

turns F into a symplectic bundle on C . The fibrewise orthogonal direct sum

$$E := F \perp F$$

satisfies (i) and (ii) for $n = 2$, but does not satisfy (iii).

Let $\mathcal{U} \subseteq \mathcal{S}ympl_{4,L}$ be the open substack defined by (i) and (ii); let $\mathcal{U}' \subseteq \mathcal{U}$ be the open substack where (iii) holds as well. We just saw $\mathcal{U} \neq \emptyset$.

Suppose $\mathcal{U}' = \emptyset$. For every point $[E]$ in \mathcal{U} , the alternating pairing (15) on $H^0(E) \cong k^2$ is then degenerate; in other words, this pairing on k^2 is zero. Since \mathcal{U} is reduced, it follows that (15) vanishes identically on \mathcal{U} . In particular, it vanishes for every infinitesimal deformation \mathcal{E} of $E = F \perp F$, meaning for every object \mathcal{E} of $\mathcal{U}(k[\epsilon])$ with $\epsilon^2 = 0$ that satisfy the condition that its reduction modulo ϵ is isomorphic to this E . But we construct below a deformation \mathcal{E} of $E = F \perp F$ for which the pairing (15) does not vanish identically. This contradiction will show $\mathcal{U}' \neq \emptyset$, proving the lemma.

Let $\check{F} \subseteq F$ be the inverse image of the coherent subsheaf $L(-P) \subseteq L$; then

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{i} \check{F} \xrightarrow{\check{p}} L(-P) \longrightarrow 0$$

is exact as well. In particular, $H^0(\check{F}) \subseteq H^0(F)$ contains the nonzero section of F ; hence $H^0(\check{F}) = H^0(F)$. Since $\deg(\check{F}) = \deg(F) - 1$, Riemann-Roch implies that $\dim H^1(\check{F}) = \dim H^1(F) + 1 = 1$.

We claim that the composed map

$$H^0(F) \otimes \text{Ext}^1(F, \mathcal{O}_C) \xrightarrow{\cup} H^1(\mathcal{O}_C) \xrightarrow{i_*} H^1(\check{F}) \tag{16}$$

is nonzero. In fact, the second map i_* is surjective since $H^1(L(-P)) = 0$ by hypothesis, and the first map \cup is also surjective because of the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_C) \otimes \text{Ext}^1(F, \mathcal{O}_C) & \xrightarrow{\text{id} \otimes i^*} & H^0(\mathcal{O}_C) \otimes \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \\ \downarrow i_* \otimes \text{id} & & \downarrow \cup \\ H^0(F) \otimes \text{Ext}^1(F, \mathcal{O}_C) & \xrightarrow{\cup} & H^1(\mathcal{O}_C) \end{array}$$

in which i_* is an isomorphism by the choice of F . The other vertical map \cup is obviously an isomorphism as well, and i^* is surjective due to $\text{Ext}^2(L, \mathcal{O}_C) = 0$.

Having shown that (16) is really nonzero, we can choose a class

$$\alpha \in \text{Ext}^1(F, \mathcal{O}_C)$$

such that the composed map

$$k \cong H^0(F) \xrightarrow{\cup \alpha} H^1(\mathcal{O}_C) \xrightarrow{i^*} H^1(\check{F}) \cong k \quad (17)$$

is nonzero. We will use this class α to construct the required deformation \mathcal{E} of E .

Let $G \subseteq \text{Gp}_4$ be the closed subgroup given by

$$G(k) := \{(A, \lambda) \in \text{Gp}_4(k) \mid A \cdot e_1 = e_1\},$$

where $e_1 \in k^4$ is the first standard basis vector. The trivial line subbundle

$$\mathcal{O}_C \cong 0 \oplus \mathcal{O}_C \subseteq F \oplus F = E \quad (18)$$

defines a reduction of structure group to G for our Gp_4 -bundle $E = F \perp F$. Let

$$\text{ad}_G(E) \subseteq \text{ad}_{\text{Gp}_4}(E) \subseteq \text{End}(F \oplus F)$$

be the corresponding adjoint bundles. We consider the bundle morphism

$$\text{Hom}(F, \mathcal{O}_C) \longrightarrow \text{End}(F \oplus F) \quad (19)$$

which sends a local morphism $f : F \longrightarrow \mathcal{O}_C$ to the local endomorphism

$$\begin{pmatrix} 0 & f^t \circ p \\ i \circ f & 0 \end{pmatrix} : F \oplus F \longrightarrow F \oplus F, \quad (20)$$

where the local morphism $f^t : L \longrightarrow F$ is the adjoint of f , defined by the formula $c(v, f^t(\ell)) = f(v) \cdot \ell$ for $\ell \in L$ and $v \in F$. It is easy to check that (20) is indeed a local section of the subbundle $\text{ad}_G(E) \subseteq \text{End}(F \oplus F)$, so (19) restricts to a bundle morphism

$$\text{Hom}(F, \mathcal{O}_C) \longrightarrow \text{ad}_G(E)$$

and hence it induces a linear map

$$\text{Ext}^1(F, \mathcal{O}_C) \longrightarrow H^1(C, \text{ad}_G(E)).$$

By standard deformation theory, the image of α in $H^1(C, \text{ad}_G(E))$ corresponds to an infinitesimal deformation \mathcal{E} of E as a G -bundle and hence in particular as a Gp_4 -bundle. It remains to check that the deformed symplectic pairing (15) on the free $k[\varepsilon]$ -module $H^0(\mathcal{E})$ of rank 2 is nonzero.

Let $\check{E} := F \oplus \check{F} \subseteq F \oplus F = E$ be the subsheaf of all sections of E which are orthogonal at P to the distinguished trivial line subbundle (18). Our construction of \mathcal{E} as a deformed G -bundle implies that \mathcal{E} still contains a distinguished trivial line subbundle. So taking all sections orthogonal at P to that defines a subsheaf $\check{\mathcal{E}} \subseteq \mathcal{E}$ which is a deformation of $\check{E} \subseteq E$. We have to show the inequality

$$H^0(\check{\mathcal{E}}) \neq H^0(\mathcal{E}),$$

for it implies that there is a section of \mathcal{E} which is at P not orthogonal to our distinguished section, so the symplectic pairing in question is nonzero.

Since $\check{\mathcal{E}}$ is a deformation of \check{E} , we have an exact sequence of \mathcal{O}_C -modules

$$0 \longrightarrow \check{E} \xrightarrow{\epsilon} \check{\mathcal{E}} \longrightarrow \check{E} \longrightarrow 0$$

and consequently a long exact sequence of k -vector spaces

$$0 \longrightarrow H^0(\check{E}) \longrightarrow H^0(\check{\mathcal{E}}) \longrightarrow H^0(\check{E}) \xrightarrow{\delta} H^1(\check{E}) \longrightarrow \dots$$

By construction, one of the four matrix entries of δ is the map (17), so $\delta \neq 0$ and

$$\dim_k H^0(\check{\mathcal{E}}) < \dim_k H^0(\check{E}) + \dim_k H^0(\check{E}) = 2 + 2 = 4.$$

On the other hand, we have $\dim_k H^0(\mathcal{E}) = 2 \dim_k H^0(E) = 4$. This follows for example from the fact that $H^1(E) = 0$. This proves that $H^0(\check{\mathcal{E}}) \neq H^0(\mathcal{E})$, as required. \square

Theorem 5.2. *Let $n \geq 1$ be odd, and let L be a line bundle of odd degree over our curve C . Suppose that there is a line bundle I on C with*

$$\deg(L) + 2 \deg(I) = 2g - 1 \quad \text{and} \quad L \otimes I^{\otimes 2} \not\cong \omega_C(P)$$

for some rational point $P \in C(k)$. Then the \mathbb{G}_m -stack $\mathcal{S}ympl_{2n,L}$ is rational.

Proof. Sending each symplectic bundle $(E, b : E \otimes E \rightarrow L)$ to the symplectic bundle

$$(E \otimes I, \quad b \otimes \text{id} : (E \otimes I) \otimes (E \otimes I) \rightarrow L \otimes I^{\otimes 2})$$

we construct a 1-isomorphism $\mathcal{S}ympl_{2n,L} \xrightarrow{\sim} \mathcal{S}ympl_{2n,L \otimes I^{\otimes 2}}$. This reduces us to the case $I = \mathcal{O}_C$; in other words, we may assume without loss of generality that

$$\deg(L) = 2g - 1 \quad \text{and} \quad L \not\cong \omega_C(P).$$

According to Clifford's theorem, these imply that $H^1(C, L(-P)) = 0$.

We consider the moduli stack

$$\check{\mathcal{S}ympl}_{2n,L,P}$$

of vector bundles \check{E} on C of rank $2n$ endowed with an alternating bilinear form

$$\check{b} : \check{E} \otimes \check{E} \rightarrow L$$

which is nondegenerate except for a 2-dimensional radical in the fibre \check{E}_P . More precisely, $\check{\mathcal{S}ympl}_{2n,L,P}(S)$ is the following groupoid for each k -scheme S :

- Each object consists of a rank $2n$ vector bundle $\check{\mathcal{E}}$ on $C \times_k S$, a line bundle \mathcal{L} on $C \times_k S$ locally in S isomorphic to the pullback of L from C , and an alternating form

$$\check{b} : \check{\mathcal{E}} \otimes \check{\mathcal{E}} \rightarrow \mathcal{L}$$

whose adjoint $b^\# : \check{\mathcal{E}} \rightarrow \check{\mathcal{E}}^{\text{dual}} \otimes \mathcal{L}$ is injective with $\text{coker}(b^\#)$ locally, in S , isomorphic to the pullback of k_P^2 from C .

- Each morphism from $(\check{\mathcal{E}}, \check{b} : \check{\mathcal{E}} \otimes \check{\mathcal{E}} \rightarrow \mathcal{L})$ to $(\check{\mathcal{E}}', \check{b}' : \check{\mathcal{E}}' \otimes \check{\mathcal{E}}' \rightarrow \mathcal{L}')$ is a pair of vector bundle isomorphisms $\check{\mathcal{E}} \rightarrow \check{\mathcal{E}}'$ and $\mathcal{L} \rightarrow \mathcal{L}'$ that intertwine \check{b} and \check{b}' .

Over this stack $\check{\mathcal{S}ympl}_{2n,L,P}$, let

$$\check{\Phi} : \mathcal{S}ub_{1,\mathcal{O}}(\check{\mathcal{E}}^{\text{univ}}) \rightarrow \check{\mathcal{S}ympl}_{2n,L,P} \tag{21}$$

be the moduli stack of such bundles (\check{E}, \check{b}) together with a trivial line subbundle of \check{E} with the property that its fibre at P is in the radical of \check{E}_P . More precisely, the stack $\mathcal{S}ub_{1,\mathcal{O}}(\check{\mathcal{E}}^{\text{univ}})$ is given by the following groupoid $\mathcal{S}ub_{1,\mathcal{O}}(\check{\mathcal{E}}^{\text{univ}})(S)$ for each k -scheme S :

- Its objects consist of an object $(\check{\mathcal{E}}, \check{b} : \check{\mathcal{E}} \otimes \check{\mathcal{E}} \rightarrow \mathcal{L})$ in $\check{\mathcal{S}ympl}_{2n,L,P}(S)$ and a line subbundle $\mathcal{I} \subseteq \check{\mathcal{E}}$, which is locally in S isomorphic to \mathcal{O} , and for which the restriction

$$\check{b}_P : \mathcal{I}_P \otimes \check{\mathcal{E}}_P \rightarrow \mathcal{L}_P$$

to $\{P\} \times S$ vanishes identically.

- Morphisms from $(\check{\mathcal{E}}, \check{b} : \check{\mathcal{E}} \otimes \check{\mathcal{E}} \rightarrow \mathcal{L}, \mathcal{I} \subseteq \check{\mathcal{E}})$ to $(\check{\mathcal{E}}', \check{b}' : \check{\mathcal{E}}' \otimes \check{\mathcal{E}}' \rightarrow \mathcal{L}', \mathcal{I}' \subseteq \check{\mathcal{E}}')$ consist of three vector bundle isomorphisms

$$\check{\mathcal{E}} \rightarrow \check{\mathcal{E}}', \quad \mathcal{L} \rightarrow \mathcal{L}' \quad \text{and} \quad \mathcal{I} \rightarrow \mathcal{I}'$$

that commute with \check{b} and \check{b}' and with the inclusions $\mathcal{I} \subseteq \check{\mathcal{E}}$ and $\mathcal{I}' \subseteq \check{\mathcal{E}}'$.

These objects have obvious scalar automorphisms; they turn $\check{Sympl}_{2n,L,P}$ and $Sub_{1,\mathcal{O}}(\check{\mathcal{E}}^{\text{univ}})$ into \mathbb{G}_m -stacks. Forgetting the subbundle \mathcal{I} defines the 1-morphism $\check{\Phi}$ in (21); it is a morphism of weight 1.

We construct a diagram of stacks and 1-morphisms over k

$$\begin{array}{ccccccc}
 \mathcal{M} & \longrightarrow & Sub_{1,\mathcal{O}}(\mathcal{E}^{\text{univ}}) & \xrightarrow{\Pi_1} & Sub_{1,\mathcal{O}}(\check{\mathcal{E}}^{\text{univ}}) & \xrightarrow{\Pi_2} & Sub_{1,\mathcal{O}(P)}(\check{\mathcal{E}}^{\text{univ}}) \\
 \Phi|_{\mathcal{M}} \downarrow & & \downarrow \Phi & & \downarrow \check{\Phi} & & \downarrow \check{\Phi} \\
 \mathcal{U} & \subseteq & Sympl_{2n,L} & & \check{Sympl}_{2n,L,P} & & Sympl_{2n,L} \supseteq \check{\mathcal{U}}
 \end{array} \tag{22}$$

as follows:

- $Sub_{1,\mathcal{O}}(\mathcal{E}^{\text{univ}})$ is the moduli stack introduced in Definition 3.2 that parameterises symplectic bundles $(E, b : E \otimes E \rightarrow L)$ together with a trivial line subbundle $\mathcal{O} \subseteq E$.
- $Sub_{1,\mathcal{O}(P)}(\check{\mathcal{E}}^{\text{univ}})$ is the moduli stack introduced in Definition 3.2 that parameterises symplectic bundles $(\check{E}, \check{b} : \check{E} \otimes \check{E} \rightarrow L)$ together with a subbundle $\mathcal{O}(P) \subseteq \check{E}$.
- $\Phi := \Phi_{\mathcal{O}}$ and $\check{\Phi} := \Phi_{\mathcal{O}(P)}$ are the forgetful 1-morphisms of Definition 3.2.
- Π_1 sends each triple $(E, b : E \otimes E \rightarrow L, i : \mathcal{O} \subseteq E)$ to the kernel $\check{E} \subseteq E$ of the sheaf homomorphism

$$E \xrightarrow{b^\#} Hom(E, L) \xrightarrow{i^*} L \rightarrow L_P := L \otimes_{\mathcal{O}_C} k_P,$$

together with the restriction

$$\check{b} : \check{E} \otimes \check{E} \rightarrow L$$

of b and the same subbundle $\mathcal{O} \subseteq \check{E} \subseteq E$ (which is indeed contained in \check{E} because b is alternating).

- Π_2 sends each triple $(\check{E}, \check{b} : \check{E} \otimes \check{E} \rightarrow L, i : \mathcal{O} \subseteq \check{E})$ to the pushout $\mathcal{O}(P) \subseteq \check{E}$ of i along the sheaf monomorphism $\mathcal{O} \hookrightarrow \mathcal{O}(P)$, together with the alternating form $\check{b} : \check{E} \otimes \check{E} \rightarrow L$ that coincides with \check{b} outside P .

(A straightforward computation over the local ring $\mathcal{O}_{C,P}$ shows that there is precisely one such form \check{b} and that it is nondegenerate.)

- $\mathcal{U} \subseteq Sympl_{2n,L}$ is the open substack of symplectic bundles (E, b) with

$$\dim H^0(E) = n \quad \text{and} \quad H^1(E) = 0 \tag{23}$$

for which the radical $\text{rad } H^0(E)$ of the pairing

$$H^0(E) \otimes H^0(E) \xrightarrow{\eta_P \otimes \eta_P} E_P \otimes E_P \xrightarrow{b_P} L_P \tag{24}$$

is 1-dimensional, and any $0 \neq s \in \text{rad } H^0(E)$ is everywhere on C nonzero.

(Grothendieck's theory of cohomology and base change, [2], shows that (23) is an open condition. Assuming it, $H^0(E)$ has odd dimension, so any alternating form on it is degenerate; hence $\dim \text{rad } H^0(E) = 1$ is then an open condition as well. This proves that \mathcal{U} is indeed an open substack.)

- $\mathcal{M} \hookrightarrow \Phi^{-1}(\mathcal{U})$ is the closed substack of triples $(E, b : E \otimes E \rightarrow L, \mathcal{O} \subseteq E)$ for which the restriction of (24) to

$$H^0(\mathcal{O}) \otimes H^0(E) \subseteq H^0(E) \otimes H^0(E)$$

vanishes.

- $\check{\mathcal{U}} \subseteq Sympl_{2n,L}$ is the open substack of symplectic bundles (\check{E}, \check{b}) with

$$\dim H^0(\check{E}) = n \quad \text{and} \quad H^1(\check{E}) = 0$$

for which the evaluation map $\tilde{\eta}_P : H^0(\check{E}) \rightarrow \check{E}_P$ has rank $\geq n - 1$.

By construction, Φ restricts to a 1-isomorphism $\Phi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{U}$; its inverse endows each symplectic bundle (E, b) with the image $\mathcal{O} \subseteq E$ of any $0 \neq s \in \text{rad } H^0(E)$. Thus it suffices to show that \mathcal{M} is non-empty and rational as a \mathbb{G}_m -stack.

Lemma 5.3. *Even the intersection $\mathcal{M} \cap \Pi_1^{-1} \Pi_2^{-1} \check{\Phi}^{-1}(\check{\mathcal{U}})$ is non-empty.*

Proof. We may assume without loss of generality that $k = \bar{k}$ is algebraically closed. Let F be a general vector bundle extension

$$0 \longrightarrow \mathcal{O} \longrightarrow F \longrightarrow L \longrightarrow 0,$$

and let the vector bundle extension

$$0 \longrightarrow \mathcal{O}(P) \longrightarrow \tilde{F} \longrightarrow L(-P) \longrightarrow 0$$

be the image of $[F]$ under the natural surjection

$$\mathrm{Ext}^1(L, \mathcal{O}) \rightarrow \mathrm{Ext}^1(L(-P), \mathcal{O}(P)).$$

According to Riemann-Roch and our assumptions on L , we have

$$\dim H^0(L) = g = \dim H^1(\mathcal{O}) \quad \text{and} \quad \dim H^0(L(-P)) = g - 1 = \dim H^1(\mathcal{O}(P));$$

using Lemma 4.2, it follows that

$$H^0(F) \cong k \cong H^0(\tilde{F}) \quad \text{and} \quad H^1(F) = 0 = H^1(\tilde{F}).$$

Now let $(E', b' : E' \otimes E' \longrightarrow L)$ be a symplectic bundle of rank $2n - 2$ with the properties given in Lemma 5.1, and let (E, b) be the fibrewise orthogonal direct sum of E' and F . Then the tuple

$$(E, \quad b, \quad \mathcal{O} \cong 0 \oplus \mathcal{O} \subseteq E' \oplus F = E)$$

defines a point in \mathcal{M} ; its image under $\tilde{\Phi} \circ \Pi_2 \circ \Pi_1$ is by construction the symplectic bundle $\tilde{E} := E' \perp \tilde{F}$ and thus contained in $\tilde{\mathcal{U}}$. \square

According to Proposition 4.4(ii), the \mathbb{G}_m -stack $\mathrm{Sub}_{1, \mathcal{O}(P)}(\tilde{\mathcal{E}}^{\mathrm{univ}})$ is rational. To deduce from that the required rationality of \mathcal{M} , we study the fibres of the above 1-morphisms Π_1 and Π_2 via the strictly commutative diagram

$$\begin{array}{ccccc} \mathcal{M} \hookrightarrow \mathrm{Sub}_{1, \mathcal{O}}(\tilde{\mathcal{E}}^{\mathrm{univ}}) & \xrightarrow{\Gamma_1} & \mathbb{P}(\mathrm{rad} \check{\mathcal{E}}_P^{\mathrm{univ}}) & & \\ \Pi_1 \downarrow & \swarrow & \searrow & & \\ \mathrm{Sub}_{1, \mathcal{O}}(\check{\mathcal{E}}^{\mathrm{univ}}) & \xrightarrow{\Gamma_2} & \mathbb{P}(\tilde{\mathcal{E}}_P^{\mathrm{univ}}) & \longleftarrow & \mathcal{M} = \mathbb{P}\left(\frac{\tilde{\mathcal{E}}_P^{\mathrm{univ}}}{\mathrm{im} H^0(\check{\mathcal{E}}^{\mathrm{univ}})}\right) \\ & \searrow \Pi_2 & \downarrow & & \downarrow \\ & & \mathrm{Sub}_{1, \mathcal{O}(P)}(\tilde{\mathcal{E}}^{\mathrm{univ}}) & \supseteq & \tilde{\Phi}^{-1}(\tilde{\mathcal{U}}) \end{array}$$

constructed as follows:

- $\mathrm{rad} \check{\mathcal{E}}_P^{\mathrm{univ}}$ is the vector bundle of rank 2 and weight 1 on $\mathrm{Sub}_{1, \mathcal{O}}(\check{\mathcal{E}}^{\mathrm{univ}})$ whose fibre over any triple $(\check{E}, \check{b} : \check{E} \otimes \check{E} \longrightarrow L, \mathcal{O} \subseteq \check{E})$ is the radical $\mathrm{rad} \check{E}_P$ of the alternating pairing $\check{b}_P : \check{E}_P \otimes \check{E}_P \longrightarrow L_P$ on the fibre \check{E}_P .
- Γ_1 sends a triple $(E, b : E \otimes E \longrightarrow L, \mathcal{O} \subseteq E)$ to its image $\check{E} \subseteq E$ under Π_1 and the image of the k -linear map $E(-P)_P \longrightarrow \mathrm{rad} \check{E}_P \subseteq \check{E}_P$ induced by the inclusion $E(-P) \subseteq \check{E}$ as subsheaves of E .
- $\tilde{\mathcal{E}}_P^{\mathrm{univ}}$ is the vector bundle of rank $2n$ and weight 1 on $\mathrm{Sub}_{1, \mathcal{O}(P)}(\tilde{\mathcal{E}}^{\mathrm{univ}})$ whose fibre over any triple $(\tilde{E}, \tilde{b}, \mathcal{O}(P) \subseteq \tilde{E})$ is the fibre \tilde{E}_P of \tilde{E} at P .
- Γ_2 sends a triple $(\check{E}, \check{b}, \mathcal{O} \subseteq \check{E})$ to its image $\check{E} \subseteq \tilde{E}$ under Π_2 and the image of the induced k -linear map $\check{E}_P \longrightarrow \tilde{E}_P$.
- $\mathcal{M} \hookrightarrow \mathbb{P}(\tilde{\mathcal{E}}_P^{\mathrm{univ}})|_{\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})}$ is the closed substack of those hyperplanes in the fibres \tilde{E}_P which contain the values at P of all section $s \in H^0(\tilde{E})$.

The vector spaces $H^0(\tilde{E})$ are the fibres of a rank n vector bundle $H^0(\tilde{\mathcal{E}}^{\mathrm{univ}})$ on $\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})$ by cohomology and base change [2]. The evaluations $\tilde{\eta}_P : H^0(\tilde{E}) \longrightarrow \tilde{E}_P$ at P define a morphism of vector bundles on $\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})$

$$\tilde{\eta}_P^{\mathrm{univ}} : H^0(\tilde{\mathcal{E}}^{\mathrm{univ}}) \longrightarrow \tilde{\mathcal{E}}_P^{\mathrm{univ}}.$$

Every section $s \in H^0(\mathcal{O}(P))$ vanishes in the fibre $\mathcal{O}(P)_P$. If \tilde{E} admits a subbundle isomorphic to $\mathcal{O}(P)$, then the rank of $\tilde{\eta}_P$ can thus be at most $n - 1$. This and the definition of $\tilde{\mathcal{U}}$ show that $\tilde{\eta}_P^{\text{univ}}$ has constant rank $n - 1$ on $\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})$; thus its cokernel

$$\text{coker}(\tilde{\eta}_P^{\text{univ}}) = \frac{\tilde{\mathcal{E}}_P^{\text{univ}}}{\text{im } H^0(\tilde{\mathcal{E}}^{\text{univ}})}$$

is a vector bundle of weight 1 and rank $n + 1$ on $\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})$. The associated projective subbundle of $\mathbb{P}(\tilde{\mathcal{E}}_P^{\text{univ}})|_{\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})}$ is by construction the above closed substack $\tilde{\mathcal{M}}$.

Now Lemma 5.3 implies in particular that $\tilde{\Phi}^{-1}(\tilde{\mathcal{U}})$ is non-empty. It follows that the projective bundle $\tilde{\mathcal{M}}$ over it is also non-empty, algebraic, locally of finite type over k , smooth, irreducible, and rational as a \mathbb{G}_m -stack.

It is easy to check that Γ_1 and Γ_2 are open immersions. More precisely:

- Γ_1 is a 1-isomorphism onto the open substack of tuples $(\check{E}, \check{b}, \mathcal{O} \subseteq \check{E})$ together with a line in $\text{rad } \check{E}_P$ different from the fibre of $\mathcal{O} \subseteq \check{E}$. Its inverse can be described as follows:
Using the canonical isomorphism $\mathbb{P}\check{E}_P \cong \mathbb{P}\check{E}(P)_P$, such a line in \check{E}_P yields a line in $\check{E}(P)_P$. Let $E \subseteq \check{E}(P)$ be its inverse image under the sheaf surjection $\check{E}(P) \rightarrow \check{E}(P)_P$. Then it is easy to see that $\mathcal{O} \subseteq \check{E}$ is in fact a subbundle of E . A straightforward computation over the local ring $\mathcal{O}_{C,P}$ shows that there is precisely one alternating form b on E equal to \check{b} outside P , and that b is nondegenerate. Now the inverse of Γ_1 sends $(\check{E}, \check{b}, \mathcal{O} \subseteq \check{E})$ and the line in question to $(E, b, \mathcal{O} \subseteq E)$.
- Γ_2 is a 1-isomorphism onto the open substack of triples $(\tilde{E}, \tilde{b}, \mathcal{O}(P) \subseteq \tilde{E})$ together with a hyperplane in \tilde{E}_P which does not contain the fibre of the subbundle $\mathcal{O}(P) \subseteq \tilde{E}$. Its inverse sends such a hyperplane to its inverse image $\check{E} \subseteq \tilde{E}$ under the sheaf surjection $\tilde{E} \rightarrow \tilde{E}_P$, equipped with the restricted form $\check{b} := \tilde{b}|_{\check{E} \otimes \check{E}}$ and the subbundle $\mathcal{O} \cong \mathcal{O}(P) \cap \check{E} \subseteq \check{E}$.

This implies in particular that the stack $\text{Sub}_{1,\mathcal{O}}(\tilde{\mathcal{E}}^{\text{univ}})$ is algebraic, locally of finite type over k , smooth and irreducible.

Inside the open substack

$$\Phi^{-1}(\mathcal{U}) \cap \Pi_1^{-1} \Pi_2^{-1} \tilde{\Phi}^{-1}(\tilde{\mathcal{U}}) \subseteq \text{Sub}_{1,\mathcal{O}}(\tilde{\mathcal{E}}^{\text{univ}}), \quad (25)$$

(see (22)) we have the closed substack

$$\mathcal{M} \cap \Pi_1^{-1} \Pi_2^{-1} \tilde{\Phi}^{-1}(\tilde{\mathcal{U}}) \quad (26)$$

which has been defined by the vanishing of

$$b_P \circ (\eta_P \otimes \eta_P) : H^0(\mathcal{O}) \otimes H^0(E) \longrightarrow L_P.$$

We also have the closed substack

$$\Phi^{-1}(\mathcal{U}) \cap \Pi_1^{-1} \Gamma_2^{-1}(\tilde{\mathcal{M}}) \subseteq \Phi^{-1}(\mathcal{U}) \cap \Pi_1^{-1} \Pi_2^{-1} \tilde{\Phi}^{-1}(\tilde{\mathcal{U}}) \quad (27)$$

of all triples $(E, b, \mathcal{O} \subseteq E)$ such that their images $E \supseteq \check{E} \subseteq \tilde{E}$ under Π_1 and Π_2 satisfy the following closed condition:

$$\tilde{\eta}_P(s) \in \tilde{E}_P \text{ lies in the image of } \check{E}_P \longrightarrow \tilde{E}_P \text{ for all } s \in H^0(\tilde{E}).$$

Now these two closed conditions are equivalent: Since $\dim H^0(E) = n = \dim H^0(\tilde{E})$ everywhere on the open substack (25), both closed conditions are equivalent to the condition that $\dim H^0(\check{E}) = n$. Hence the closed substacks (26) and (27) coincide.

In particular, (27) is non-empty, because (26) is so according to Lemma 5.3. But Γ_1 yields an open immersion of (27) into the restriction of the projective bundle $\mathbb{P}(\text{rad } \tilde{\mathcal{E}}_P^{\text{univ}})$ to the rational \mathbb{G}_m -stack $\Gamma_2^{-1}(\tilde{\mathcal{M}})$. It follows that the \mathbb{G}_m -stack (27) is rational as well; hence the same holds for (26) and consequently also for \mathcal{M} . \square

Corollary 5.4. *Under the hypotheses of Theorem 5.2, the coarse moduli scheme $\text{Sympl}_{2n,L}$ is rational.*

Remark 5.5. Suppose that C admits a rational point $P \in C(k)$ and a line bundle ξ of degree 0 on C with $\xi^{\otimes 2} \not\cong \mathcal{O}_C$. Then there is, for every given line bundle L of odd degree on C , a line bundle I with the properties required in Theorem 5.2. Indeed, we may take either $I := \mathcal{O}_C(dP)$ or $I := \xi(dP)$, where $d := (2g - 1 - \deg L)/2 \in \mathbb{Z}$.

Remark 5.6. One can also vary the line bundle L , keeping only its degree fixed. More precisely, let

$$\mathit{Sympl}_{2n,d}$$

denote the moduli stack of all tuples $(E, b : E \otimes E \rightarrow L)$ in which E is a vector bundle of rank $2n$ on C , L is a line bundle of degree d on C , and b is a nowhere degenerate symplectic form. Let $\mathfrak{Sympl}_{2n,d}$ be the corresponding coarse moduli scheme of Ramanathan-stable symplectic bundles $(E, b : E \otimes E \rightarrow L)$, as constructed in [7] and [1]. Concerning their birational type, we have the following:

Corollary 5.7. *Suppose that n and d are both odd, and that the curve C over k has a rational point $P \in C(k)$. Then the stack $\mathit{Sympl}_{2n,d}$ is birational to*

$$\mathrm{BG}_m \times \mathbb{A}^s \times \mathrm{Pic}^d(C) \quad \text{for some } s.$$

Proof. Forgetting E and b defines a canonical 1-morphism

$$\mathit{Sympl}_{2n,d} \longrightarrow \mathrm{Pic}^d(C)$$

to the Picard *scheme* $\mathrm{Pic}^d(C)$. The fibres of this morphism are the moduli stacks $\mathit{Sympl}_{2n,L}$ studied above. In particular, its generic fibre coincides with the moduli stack $\mathit{Sympl}_{2n,L^{\mathrm{generic}}}$ over the function field $K := k(\mathrm{Pic}^d(C))$, where the line bundle L^{generic} on $C_K := C \times_k K$ is the generic fibre of a Poincaré family on $C \times_k \mathrm{Pic}^d(C)$. Now C_K has a rational point because C has; due to Remark 5.5, it thus suffices to construct a line bundle ξ of degree 0 on C_K with $\xi^{\otimes 2} \not\cong \mathcal{O}_{C_K}$.

We take for ξ the generic fibre of a Poincaré family on $C \times_k \mathrm{Pic}^0(C)$. Then $\xi^{\otimes 2} \not\cong \mathcal{O}_{C_K}$ holds indeed, because the endomorphism of the abelian variety $\mathrm{Pic}^0(C)$ that sends each line bundle on C to its square is nonconstant. \square

Remark 5.8. Corollary 5.7 automatically implies the following: $\mathfrak{Sympl}_{2n,d}$ is birational to $\mathbb{A}^s \times \mathrm{Pic}^d(C)$ for some s if $n \cdot d$ is odd and $C(k)$ is non-empty.

REFERENCES

- [1] T.L. Gomez, A. Langer, A.H.W. Schmitt, I. Sols, Moduli Spaces for Principal Bundles in Arbitrary Characteristic, Adv. Math. (2008), doi:10.1016/j.aim.2008.05.015.
- [2] A. Grothendieck, EGA III: Étude cohomologique des faisceaux cohérents, Publ. Math. Inst. Hautes Études Sci. 11 (1961); Publ. Math. Inst. Hautes Études Sci. 17 (1963).
- [3] A. Grothendieck, et al., SGA IV: Théorie des topos et cohomologie étale des schémas, Lecture Notes in Math., vols. 269, 270, 305, Springer-Verlag, Berlin, 1972/73.
- [4] N. Hoffmann, Rationality and Poincaré families for vector bundles with extra structure on a curve, Int. Math. Res. Not. 2007 (2007), article ID rnm010.
- [5] A. King, A. Schofield, Rationality of moduli of vector bundles on curves, Indag. Math. (N.S.) 10 (1999) 519–535.
- [6] P.E. Newstead, Rationality of moduli spaces of stable bundles, Math. Ann. 215 (1975) 251–268.
- [7] A. Ramanathan, Moduli for principal bundles over algebraic curves, Proc. Indian Acad. Sci. Math. Sci. 106 (3) (1996) 301–328, Proc. Indian Acad. Sci. Math. Sci. 106 (4) (1996) 421–449.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA
E-mail address: indranil@math.tifr.res.in

MATHEMATISCHES INSTITUT DER GEORG-AUGUST-UNIVERSITÄT, BUNSENSTRASSE 3–5, 37073 GÖTTINGEN, GERMANY
E-mail address: hoffmann@uni-math.gwdg.de