Journal of Physics: Conference Series

The Belavin-Drinfeld theorem on non-degenerate solutions of the classical Yang-Baxter equation

To cite this article: Lisa Kierans and Bernd Kreussler 2012 *J. Phys.: Conf. Ser.* **346** 012011

Journal of Physics: Conference Series **346** (2012) 012011 doi:10.1088/1742-6596/346/1/012011

The Belavin-Drinfeld theorem on non-degenerate solutions of the classical Yang-Baxter equation

Lisa Kierans, Bernd Kreussler
Mary Immaculate College, South Circular Road, Limerick, Ireland
E-mail: `bernd.kreussler@mic.ul.ie`

Abstract. We give a coordinate free proof of Belavin and Drinfeld's Theorem about the classification of non-degenerate solutions of the classical Yang-Baxter equation. The equivalence of different characterisations of non-degeneracy is also shown in such a way.

1. Introduction

What is now called the (Quantum) Yang-Baxter Equation was introduced around 1970 independently by C. N. Yang and R. J. Baxter. The Classical Yang-Baxter Equation (CYBE), see Section 3 equation (3.1), can be derived from it by considering the linear part of a solution which is a power series in an auxiliary parameter. It now plays an important role in the study of Lie bialgebras and quantum groups. More recently, A. Polishchuk [10] discovered an interesting connection between solutions of CYBE and structural properties of derived categories of coherent sheaves on projective curves of arithmetic genus one. Further work on this connection was done by I. Burban and B. Kreussler [6] which includes some explicit examples and a relative version of Polishchuk's construction which can be used to explain degeneration of solutions.

Despite the developments in the past three decades, the results obtained by A. Belavin and V. Drinfeld in [2] are still fundamental for the study of solutions of CYBE. In this note we focus on non-degenerate solutions of CYBE and the trichotomy of solutions: rational, trigonometric and elliptic. Our contribution is a reformulation of the technical details of the proofs in [2] in a coordinate free way. We think that the key steps of the proofs become more transparent this way. We also filled in some details which we felt were a bit sketchy originally.

2. Preliminaries

All Lie algebras considered in this article will be finite dimensional complex Lie algebras. Let $\mathfrak{g}$ be a semi-simple finite dimensional complex Lie algebra. If $\beta : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$ is a non-degenerate (symmetric invariant) bilinear form, we denote by $\varphi^{\beta} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ the corresponding isomorphism, given by $\varphi^{\beta}(a) = \beta(a, \cdot)$. A bilinear form β is called *invariant* or *associative* if for all $a, b, c \in \mathfrak{g}$ we have $\beta(a, [b, c]) = \beta([a, b], c)$. If $\beta = \kappa$ is the Killing form, which is non-degenerate by Cartan's Theorem and symmetric and invariant, we simply write $\varphi = \varphi^{\kappa}$.

The isomorphisms $\mathbb{1}_{\mathfrak{g}} \otimes \varphi : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}^*$ and $\mathbb{1}_{\mathfrak{g}} \otimes \varphi \otimes \varphi : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$ give us isomorphisms of vector spaces

$$\varphi_1 : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathsf{Hom}(\mathfrak{g}, \mathfrak{g}) \quad \varphi_2 : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathsf{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$$

Journal of Physics: Conference Series **346** (2012) 012011 doi:10.1088/1742-6596/346/1/012011

which are explicitly given by

$$\varphi_1(a\otimes b)(x)=\kappa(b,x)\cdot a \quad \varphi_2(a\otimes b\otimes c)(x\otimes y)=\kappa(b,x)\kappa(c,y)\cdot a.$$

We call the tensor $\Omega\in\mathfrak{g}\otimes\mathfrak{g}$, which is characterised by $\varphi_1(\Omega)=\mathbb{1}_{\mathfrak{g}}$, the *Casimir element.* If $I_1,\dots,I_n$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form, then $\Omega=\sum_{\mu=1}^n I_\mu\otimes I_\mu$.

These definitions work for any non-degenerate symmetric invariant bilinear form β. If $\beta\neq\kappa$, we write $\varphi_1^\beta,\varphi_2^\beta$ and Ω^β instead of φ_1,φ_2 and Ω respectively. For $\lambda\in\mathbb{C}^*$ we then have $\varphi_1^{\lambda\beta}=\lambda\varphi_1^\beta,\varphi_2^{\lambda\beta}=\lambda^2\varphi_2^\beta$ and $\lambda\Omega^{\lambda\beta}=\Omega^\beta$. As we will later focus on simple Lie algebras, this shows that it is sufficient to work with κ, because on a simple Lie algebra over $\mathbb{C}$ any non-degenerate symmetric invariant bilinear form is a multiple of κ.

Example 2.1. The Lie algebra $\mathfrak{sl}(2)$, whose elements are the 2×2 matrices of trace zero, is simple. The standard basis elements

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

enjoy the identities $[e,f]=h,[h,e]=2e,[h,f]=-2f$. The trace form with respect to the natural representation is given by $\beta(a,b)=\mathrm{tr}(ab)$ for $a,b\in\mathfrak{sl}(2)$. It is related to the Killing form $\kappa(a,b)=\mathrm{tr}\big(\mathrm{ad}(a)\,\mathrm{ad}(b)\big)$ by $\kappa=4\beta$. Hence, $\Omega=\frac{1}{4}\Omega^\beta$ and a straightforward calculation shows $4\Omega=\Omega^\beta=\frac{1}{2}h\otimes h+e\otimes f+f\otimes e$.

Lemma 2.2. *If $\mathfrak{g}$ is simple, $f\in\mathsf{Hom}(\mathfrak{g},\mathfrak{g})$ and $[\mathrm{ad}(x),f]=0$ for all $x\in\mathfrak{g}$, then $f\in\mathbb{C}\cdot\mathbb{1}_{\mathfrak{g}}$.*

Proof. This follows from Schur's Lemma, because the assumption that $[\mathrm{ad}(x),f]=0$ for all $x\in\mathfrak{g}$ is equivalent to $f:\mathfrak{g}\longrightarrow\mathfrak{g}$ being a homomorphism of $\mathfrak{g}$-modules. □

Lemma 2.3. *If $\mathfrak{g}$ is a simple Lie algebra and $\psi:\mathfrak{g}\longrightarrow\mathfrak{g}$ a homomorphism of Lie algebras, then* $\det(\psi)\in\{-1,0,1\}$.

Proof. Because $\mathfrak{g}$ is simple, $\ker(\psi)$ can only be 0 or $\mathfrak{g}$. Hence, $\det(\psi)=0$ iff $\psi=0$. Assume now $\psi\neq 0$, then ψ is an automorphism of $\mathfrak{g}$. We show that then $\kappa(\psi(x),\psi(y))=\kappa(x,y)$ for all $x,y\in\mathfrak{g}$. To do so, we rewrite $\psi[x,y]=[\psi(x),\psi(y)]$ as $\psi\circ\mathrm{ad}(x)=\mathrm{ad}\big(\psi(x)\big)\circ\psi$. This implies $\psi\circ\mathrm{ad}(x)\circ\psi^{-1}=\mathrm{ad}\big(\psi(x)\big)$ from which we obtain $\psi\circ\mathrm{ad}(x)\circ\mathrm{ad}(y)\circ\psi^{-1}=\mathrm{ad}\big(\psi(x)\big)\circ\mathrm{ad}\big(\psi(y)\big)$ for all $x,y\in\mathfrak{g}$. Because $\mathrm{tr}(MCM^{-1})=\mathrm{tr}(C)$, this implies $\kappa(\psi(x),\psi(y))=\kappa(x,y)$ for all $x,y\in\mathfrak{g}$. This means that $\psi^*=\psi^{-1}$, hence $\det(\psi)^2=1$ and the claim follows. □

Lemma 2.4. *Let $f,g\in\mathsf{Hom}(\mathfrak{g},\mathfrak{g})$ and denote by g^* the adjoint of g with respect to the Killing form. Then, for all $a,b\in\mathfrak{g}$,*

$$\varphi_1\big(f(a)\otimes g(b)\big)=f\circ\varphi_1(a\otimes b)\circ g^*.$$

This means that the following diagram is commutative

$$\begin{array}{ccc}\mathfrak{g}\otimes\mathfrak{g} & \xrightarrow{\varphi_1} & \mathsf{Hom}(\mathfrak{g},\mathfrak{g})\\ {\scriptstyle f\otimes g}\big\downarrow & & \big\downarrow{\scriptstyle f\circ\_\_\circ g^*}\\ \mathfrak{g}\otimes\mathfrak{g} & \xrightarrow{\varphi_1} & \mathsf{Hom}(\mathfrak{g},\mathfrak{g}).\end{array}$$

Proof. For $a,b,x\in\mathfrak{g}$ we have $\varphi_1\big(f(a)\otimes g(b)\big)(x)=\kappa(g(b),x)f(a)=f\big(\kappa(g(b),x)a\big)=f\big(\kappa(b,g^*(x))a\big)=f\big(\varphi_1(a\otimes b)(g^*(x))\big)$ where we have used the symmetry of κ. □

Corollary 2.5. *For each* $A \in \mathfrak{g} \otimes \mathfrak{g}$ *we have* $(\varphi_1(A) \otimes \mathbb{1}_{\mathfrak{g}})(\Omega) = A$.

Proof. Because $\varphi_1(\Omega) = \mathbb{1}_{\mathfrak{g}}$, we have $\varphi_1(\varphi_1(A) \otimes \mathbb{1}_{\mathfrak{g}})(\Omega) = \varphi_1(A) \circ \varphi_1(\Omega) = \varphi_1(A)$. The claim follows as φ_1 is an isomorphism. □

Lemma 2.6. *Let* $\tau : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ *denote the swapping map* $\tau(a \otimes b) = b \otimes a$ *and* $A \in \mathfrak{g} \otimes \mathfrak{g}$. *Then* $\varphi_1(\tau A) = \varphi_1(A)^*$ *(the adjoint with respect to the Killing form).*

Proof. It is sufficient to consider $A = a \otimes b$. Using the symmetry of the Killing form, we obtain $\kappa\big(\varphi_1(a \otimes b)(x), y\big) = \kappa\big(\kappa(b, x)a, y\big) = \kappa(b, x)\kappa(a, y) = \kappa\big(x, \kappa(a, y)b\big) = \kappa\big(x, \varphi_1(b \otimes a)(y)\big)$. □

A tensor $A \in \mathfrak{g} \otimes \mathfrak{g}$ is called *unitary* if $\tau A = -A$ as this is equivalent to $\varphi_1(A)^* = -\varphi_1(A)$.

Definition 2.7. A meromorphic function $r : U \to \mathfrak{g} \otimes \mathfrak{g}$, defined on an open disc $U \subset \mathbb{C}$ centred at the origin, is called *unitary* iff for all $u \in U$ at which r is holomorphic, $\tau r(u) = -r(-u)$.

A calculation similar to the one in the proof of Lemma 2.4 gives the following.

Lemma 2.8. *Let* $f, g, h \in \mathsf{Hom}(\mathfrak{g}, \mathfrak{g})$ *and denote by* g^*, h^* *the adjoint of* g, h *with respect to the Killing form. Then, for all* $a, b, c \in \mathfrak{g}$,

$$\varphi_2\big(f(a) \otimes g(b) \otimes h(c)\big) = f \circ \varphi_2(a \otimes b \otimes c) \circ (g^* \otimes h^*).$$

This means that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\varphi_2} & \mathsf{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \\ {\scriptstyle f\otimes g\otimes h}\Big\downarrow & & \Big\downarrow{\scriptstyle f\circ \_\circ(g^*\otimes h^*)} \\ \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\varphi_2} & \mathsf{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}). \end{array}$$

Corollary 2.9. *For each* $A \in \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ *we have* $\big(\varphi_2(A)\otimes\mathbb{1}_{\mathfrak{g}}\otimes\mathbb{1}_{\mathfrak{g}}\big)(\Omega') = A$, *where* $\Omega' = \tau_{23}(\Omega\otimes\Omega)$ *and* $\tau_{23} : \mathfrak{g}^{\otimes 4} \longrightarrow \mathfrak{g}^{\otimes 4}$ *is the swapping map* $\tau_{23}(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = a_1 \otimes a_2 \otimes b_1 \otimes b_2$.

Proof. If we set $\varphi_2' = (\varphi_1\otimes\varphi_1)\circ\tau_{23}$, then $\varphi_2'(\Omega') = \mathbb{1}_{\mathfrak{g}}\otimes\mathbb{1}_{\mathfrak{g}} = \mathbb{1}_{\mathfrak{g}\otimes\mathfrak{g}}$. A straightforward calculation shows $\varphi_2 \circ \big(\varphi_2(A) \otimes \mathbb{1}_{\mathfrak{g}} \otimes \mathbb{1}_{\mathfrak{g}}\big) = \varphi_2(A) \circ \varphi_2'$, from which we obtain the claim, because φ_2 is an isomorphism. □

Definition 2.10. We call $A \in \mathfrak{g} \otimes \mathfrak{g}$ *non-degenerate* iff $\det(\varphi_1(A)) \neq 0$.

For any $A \in \mathfrak{g} \otimes \mathfrak{g}$ we define $V_A = \operatorname{im}(\varphi_1(A)) \subset \mathfrak{g}$. In general, this is a subspace only, and we have

$$A \text{ is non-degenerate} \iff V_A = \mathfrak{g}.$$

If $V \subset \mathfrak{g}$ is a subspace and $A \in V \otimes \mathfrak{g}$, then it is clear from the definition of φ_1 that $V_A \subset V$. On the other hand, Corollary 2.5 implies $A \in V_A \otimes \mathfrak{g}$. Therefore, we have

$$A \in V \otimes \mathfrak{g} \iff V_A \subset V. \tag{2.1}$$

Similarly, using Corollary 2.9, we obtain for each subspace $V \subset \mathfrak{g}$ and $A \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$

$$A \in V \otimes \mathfrak{g} \otimes \mathfrak{g} \iff \operatorname{im}(\varphi_2(A)) \subset V. \tag{2.2}$$

Definition 2.11. A meromorphic function $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$, defined on an open subset $U \subset \mathbb{C}^n$, is called *non-degenerate* iff there exists $u_0 \in U$ such that the tensor $r(u_0)$ is non-degenerate, i.e. $V_{r(u_0)} = \mathfrak{g}$.

Remark 2.12. If $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is non-degenerate, then the meromorphic function $u \mapsto \det(\varphi_1(r(u)))$ is not identically zero, hence vanishes only on a subset of codimension one in U. This means that a non-degenerate function r will have $r(u)$ non-degenerate for all u in an open dense subset of U, provided that U is connected.

Definition 2.13. If $\mathcal{A}$ is an associative unital $\mathbb{C}$-algebra, we define linear maps $\phi_{ij} : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ for all $i, j \in \{1, 2, 3\}$ with $i > j$ by $\phi_{12}(a \otimes b) = a \otimes b \otimes 1$, $\phi_{13}(a \otimes b) = a \otimes 1 \otimes b$ and $\phi_{23}(a \otimes b) = 1 \otimes a \otimes b$. We shall often write $(a \otimes b)^{ij}$ instead of $\phi_{ij}(a \otimes b)$.

Any associative algebra $\mathcal{A}$ becomes a Lie algebra if we define a Lie-bracket by

$$[A, B] = A \cdot B - B \cdot A.$$

Any Lie algebra $\mathfrak{g}$ is a subalgebra of its universal enveloping algebra $U(\mathfrak{g})$, which is an associative unital algebra. Using the Lie bracket on $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})$, for $a, b, c, d \in \mathfrak{g}$, we obtain the following expressions in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$

$$\begin{aligned}
[(a \otimes b)^{12}, (c \otimes d)^{13}] &= [a, c] \otimes b \otimes d, \\
[(a \otimes b)^{12}, (c \otimes d)^{23}] &= a \otimes [b, c] \otimes d, \\
[(a \otimes b)^{13}, (c \otimes d)^{23}] &= a \otimes c \otimes [b, d].
\end{aligned}$$

The Classical Yang-Baxter Equation involves terms of this type only, see Section 3.

Proposition 2.14. *For $A, B \in \mathfrak{g} \otimes \mathfrak{g}$, $x, y \in \mathfrak{g}$ we have*

(a) $\varphi_2\big([A^{12}, B^{13}]\big)(x \otimes y) = \big[\varphi_1(A)(x), \varphi_1(B)(y)\big]$

(b) $\varphi_2\big([A^{12}, B^{23}]\big)(x \otimes y) = -\varphi_1(A)\big([x, \varphi_1(B)(y)]\big)$

(c) $\varphi_2\big([A^{13}, B^{23}]\big)(x \otimes y) = \varphi_1(A)\big([\varphi_1(\tau B)(x), y]\big)$

(d) $\varphi_1\big([A, y \otimes 1]\big)(x) = \big[\varphi_1(A)(x), y\big]$

(e) $\varphi_1\big([A, 1 \otimes y]\big)(x) = \varphi_1(A)\big([y, x]\big)$

Proof. It is sufficient to consider $A = a \otimes b$ and $B = c \otimes d$ with $a, b, c, d \in \mathfrak{g}$. Using the definitions, we obtain $\varphi_2\big([A^{12}, B^{13}]\big)(x \otimes y) = \varphi_2\big([a, c] \otimes b \otimes d\big)(x \otimes y) = \kappa(b, x)\kappa(d, y) \cdot [a, c] = \big[\kappa(b, x)a, \kappa(d, y)c\big] = \big[\varphi_1(A)(x), \varphi_1(B)(y)\big]$.
To show (b), we use the invariance of the Killing form:

$$\begin{aligned}
&\varphi_2\big([A^{12}, B^{23}]\big)(x \otimes y) = \varphi_2\big(a \otimes [b, c] \otimes d\big)(x \otimes y) = \kappa([b, c], x)\kappa(d, y) \cdot a \\
&= -\kappa(b, [x, c])\kappa(d, y) \cdot a = -\kappa(b, [x, \kappa(d, y)c]) \cdot a = -\varphi_1(A)\big([x, \varphi_1(B)(y)]\big).
\end{aligned}$$

For equation (c) we use the invariance again

$$\begin{aligned}
&\varphi_2\big([A^{13}, B^{23}]\big)(x \otimes y) = \varphi_2\big(a \otimes c \otimes [b, d]\big)(x \otimes y) = \kappa(c, x)\kappa([b, d], y) \cdot a \\
&= \kappa(c, x)\kappa(b, [d, y]) \cdot a = \kappa(b, [\kappa(c, x)d, y]) \cdot a = \varphi_1(A)\big([\varphi_1(\tau B)(x), y]\big).
\end{aligned}$$

Finally, (d) follows directly from the definition and (e) with the aid of the invariance:

$$\begin{aligned}
&\varphi_1\big([a \otimes b, y \otimes 1]\big)(x) = \varphi_1\big([a, y] \otimes b\big)(x) = \kappa(b, x)[a, y] = [\kappa(b, x)a, y] = \big[\varphi_1(a \otimes b)(x), y\big] \\
&\varphi_1\big([a \otimes b, 1 \otimes y]\big)(x) = \varphi_1\big(a \otimes [b, y]\big)(x) = \kappa([b, y], x)a = \kappa(b, [y, x])a = \varphi_1(a \otimes b)\big([y, x]\big).
\end{aligned}$$

□

Corollary 2.15. *Let $\mathfrak{g}$ be a simple Lie algebra and $A \in \mathfrak{g} \otimes \mathfrak{g}$.*

(a) $[\Omega^{12}, A^{13} + A^{23}] = 0$ *and* $[A^{12} + A^{13}, \Omega^{23}] = 0$
(b) $[\Omega^{12}, \Omega^{13}] = -[\Omega^{12}, \Omega^{23}] = [\Omega^{13}, \Omega^{23}]$
(c) $\varphi_2\big([\Omega^{12}, \Omega^{13}]\big)(x \otimes y) = [x, y]$ *for all* $x, y \in \mathfrak{g}$
(d) $[A^{12}, \Omega^{23}] = 0$ *implies* $A = 0$
(e) $[\Omega^{12} - \Omega^{23}, A^{13}] = 0$ *implies* $A = 0$

Proof. For $x, y \in \mathfrak{g}$ we obtain from Proposition 2.14

$$\begin{aligned}\varphi_2\big([\Omega^{12}, A^{13} + A^{23}]\big)(x \otimes y) &= \big[\varphi_1(\Omega)(x), \varphi_1(A)(y)\big] - \varphi_1(\Omega)\big([x, \varphi_1(A)(y)]\big) \\ &= \big[x, \varphi_1(A)(y)\big] - \big[x, \varphi_1(A)(y)\big] = 0,\end{aligned}$$

hence $\big[\Omega^{12}, A^{13} + A^{23}\big] = 0$. The proof of the second equality in (a) is similar, but uses $\tau\Omega = \Omega$, which follows from Lemma 2.6. With $A = \Omega$ in (a) we obtain (b). Part (c) is a direct consequence of Proposition 2.14 (a) and $\varphi_1(\Omega) = \mathbb{1}_{\mathfrak{g}}$.

If $\big[A^{12}, \Omega^{23}\big] = 0$, we obtain from Proposition 2.14 (b) that $\varphi_1(A)[x, y] = 0$ for all $x, y \in \mathfrak{g}$. As $\mathfrak{g}$ is simple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and so we conclude $\varphi_1(A) = 0$, hence $A = 0$ and (d) is shown.

Finally, using Proposition 2.14, we translate $[\Omega^{12} - \Omega^{23}, A^{13}] = 0$ into

$$[x, \varphi_1(A)(y)] + \varphi_1(A)[x, y] = 0 \tag{2.3}$$

for all $x, y \in \mathfrak{g}$, which can be rewritten as $\mathrm{ad}(x) \circ \varphi_1(A) + \varphi_1(A) \circ \mathrm{ad}(x) = 0$ for all $x \in \mathfrak{g}$. If we apply $\mathrm{ad}(y) \circ \_\_ + \_\_ \circ \mathrm{ad}(y)$ to this identity, we obtain for all $x, y \in \mathfrak{g}$

$$\begin{aligned}\mathrm{ad}(y) \circ \mathrm{ad}(x) \circ \varphi_1(A) + \varphi_1(A) \circ \mathrm{ad}(x) \circ \mathrm{ad}(y) + \mathrm{ad}(x) \circ \varphi_1(A) \circ \mathrm{ad}(y)& \\ + \mathrm{ad}(y) \circ \varphi_1(A) \circ \mathrm{ad}(x) = 0.&\end{aligned}$$

Taking the difference between this expression and the one obtained from it by swapping x and y gives $\mathrm{ad}\big([x, y]\big) \circ \varphi_1(A) - \varphi_1(A) \circ \mathrm{ad}\big([x, y]\big) = 0$. As $\mathfrak{g}$ is simple, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and so $[\mathrm{ad}(x), \varphi_1(A)] = 0$ for all $x \in \mathfrak{g}$. Lemma 2.2 implies now that $\varphi_1(A) = \lambda \mathbb{1}_{\mathfrak{g}}$ for some $\lambda \in \mathbb{C}$. Using this in (2.3), we obtain $2\lambda[x, y] = 0$ for all $x, y \in \mathfrak{g}$, hence $\lambda = 0$ and so also $A = 0$. □

Lemma 2.16. *If* $A \in \mathfrak{g} \otimes \mathfrak{g}$ *is non-degenerate, then* $[A^{12}, A^{13}] \neq 0$.

Proof. If $\big[A^{12}, A^{13}\big] = 0$, Proposition 2.14 (a) implies $\big[\varphi_1(A)(x), \varphi_1(A)(y)\big] = 0$ for all $x, y \in \mathfrak{g}$. This means $[X, Y] = 0$ for all $X, Y \in V_A$. As A is non-degenerate, we have $V_A = \mathfrak{g}$. But $\mathfrak{g}$ is not abelian, hence $\big[A^{12}, A^{13}\big]$ cannot vanish. □

Lemma 2.17. *Let* $A, B \in \mathfrak{g} \otimes \mathfrak{g}$ *and* $\mathfrak{a} = \{x \in \mathfrak{g} \mid [x, v] \in V_B$ *for all* $v \in V_B\}$*, then*

(a) $\mathfrak{a} \subset \mathfrak{g}$ *is a Lie subalgebra and* $\mathfrak{a} = \mathfrak{g}$ *if and only if* V_B *is an ideal in* $\mathfrak{g}$*;*
(b) If $\big[B^{12}, A^{13}\big] \in V_B \otimes \mathfrak{g} \otimes \mathfrak{g}$*, then* $A \in \mathfrak{a} \otimes \mathfrak{g}$*;*
(c) If $\big[A^{12}, B^{23}\big] \in \mathfrak{g} \otimes V_B \otimes \mathfrak{g}$*, then* $A \in \mathfrak{g} \otimes \mathfrak{a}$.

Proof. The proof of (a) is a consequence of the Jacobi identity and the definitions. If $\big[B^{12}, A^{13}\big] \in V_B \otimes \mathfrak{g} \otimes \mathfrak{g}$, the image of $\varphi_2\big(\big[B^{12}, A^{13}\big]\big)$ is contained in V_B, see (2.2). Proposition 2.14 (a) implies now that $\big[\varphi_1(B)(x), \varphi_1(A)(y)\big] \in V_B$ for all $x, y \in \mathfrak{g}$. As $V_B = \mathrm{im}\,(\varphi_1(B))$ this means that $\varphi_1(A)(y) \in \mathfrak{a}$ for all $y \in \mathfrak{g}$, i.e. $V_A \subset \mathfrak{a}$. This implies $A \in \mathfrak{a} \otimes \mathfrak{g}$, see (2.1), hence we have shown (b). To prove (c) we use the swapping operator $\tau_{12} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$, given by $\tau_{12}(a \otimes b \otimes c) = b \otimes a \otimes c$. Because $\tau_{12}\big[A^{12}, B^{23}\big] = [(\tau A)^{12}, B^{13}]$, the assumption implies $[(\tau A)^{12}, B^{13}] \in V_B \otimes \mathfrak{g} \otimes \mathfrak{g}$. Like before, this implies $\big[\varphi_1(\tau A)(x), \varphi_1(B)(y)\big] \in V_B$ for all $x, y \in \mathfrak{g}$, from which we get $\varphi_1(\tau A)(x) \in \mathfrak{a}$ for all $x \in \mathfrak{g}$. Therefore, $\tau A \in \mathfrak{a} \otimes \mathfrak{g}$, i.e. $A \in \mathfrak{g} \otimes \mathfrak{a}$. □

Journal of Physics: Conference Series **346** (2012) 012011 doi:10.1088/1742-6596/346/1/012011

Lemma 2.18. *Let $A, B \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\left[B^{12}, A^{13} + A^{23}\right] = 0$, then $A \in \mathfrak{a} \otimes \mathfrak{g}$, where $\mathfrak{a} = \{x \in \mathfrak{g} \mid [\mathrm{ad}(x), \varphi_1(B)] = 0\} \subset \mathfrak{g}$, which is a Lie subalgebra.*

Proof. Because $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathsf{End}(\mathfrak{g})$ is a homomorphism of Lie algebras and centralisers are subalgebras, $\mathfrak{a} \subset \mathfrak{g}$ is a Lie subalgebra. If $\left[B^{12}, A^{13} + A^{23}\right] = 0$, we obtain from Proposition 2.14 for all $x, y \in \mathfrak{g}$

$$\begin{aligned} 0 &= \varphi_2\big(\left[B^{12}, A^{13} + A^{23}\right]\big)(x \otimes y) = \left[\varphi_1(B)(x), \varphi_1(A)(y)\right] - \varphi_1(B)\left[x, \varphi_1(A)(y)\right] \\ &= \varphi_1(B)\left[\varphi_1(A)(y), x\right] - \left[\varphi_1(A)(y), \varphi_1(B)(x)\right] \\ &= \Big(\varphi_1(B) \circ \mathrm{ad}\big(\varphi_1(A)(y)\big) - \mathrm{ad}\big(\varphi_1(A)(y)\big) \circ \varphi_1(B)\Big)(x), \end{aligned}$$

hence $\left[\varphi_1(B), \mathrm{ad}\big(\varphi_1(A)(y)\big)\right] = 0$ and so $\varphi_1(A)(y) \in \mathfrak{a}$ for all $y \in \mathfrak{g}$. Because of (2.1) this implies $A \in \mathfrak{a} \otimes \mathfrak{g}$. □

3. The Classical Yang Baxter Equation

The Classical Yang Baxter Equation, referred to as (CYBE), is the equation

$$[r^{12}(u), r^{13}(u+v)] + [r^{12}(u), r^{23}(v)] + [r^{13}(u+v), r^{23}(v)] = 0 \tag{3.1}$$

for $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ being a meromorphic function defined on an open neighbourhood $U \subset \mathbb{C}$ of the origin. In this equation, it is assumed that $u, v, u+v \in U$. This equation makes also sense if U is a neighbourhood of the origin in $\mathbb{C}^n$, but should not be confused with the classical Yang Baxter equation with two spectral parameters if $n = 2$.

Throughout this section, $\mathfrak{g}$ denotes a finite-dimensional simple complex Lie algebra.

Lemma 3.1. *If $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a non-degenerate holomorphic solution of (CYBE), defined on an open disc $U \subset \mathbb{C}$ centred at the origin, then $r(0)$ is a non-degenerate tensor.*

Proof. There exists $u_0 \in U$ such that $r(u_0)$ is non-degenerate. Let us denote $T_u = \varphi_1(r(u))$ and $\psi_u = T_u \circ T_{u_0}^{-1}$ for all $u \in U$. We show now that ψ_u is a homomorphism of Lie algebras.

Let $v = 0$ in (3.1) and apply φ_2. Using Proposition 2.14, this produces

$$[T_u(x), T_u(y)] = T_u\big([x, T_0(y)] - [T_0^*(x), y]\big).$$

Setting $u = u_0$ and applying ψ_u makes this into

$$\psi_u[T_{u_0}(x), T_{u_0}(y)] = T_u\big([x, T_0(y)] - [T_0^*(x), y]\big).$$

Both equations have the same right hand side. Therefore, with $X = T_{u_0}(x)$ and $Y = T_{u_0}(y)$ we obtain $\psi_u[X, Y] = [\psi_u(X), \psi_u(Y)]$. Because T_{u_0} is an isomorphism, this equation shows that ψ is a homomorphism of Lie algebras. Because $\det(\psi_u)$ is a continuous function on U which can, by Lemma 2.3, only take the three values $-1, 0, 1$, it must be constant. Therefore $\det(\psi_0) = \det(\psi_{u_0}) = 1$ and T_0 is invertible. This shows that $r(0)$ is non-degenerate. □

3.1. Constant Solutions

Let $\mathfrak{g}$ be a simple Lie algebra. The aim of this subsection is to show that the Classical Yang-Baxter Equation (3.1) does not have non-degenerate constant solutions. This result will be needed in the proof of Theorem 3.6.

A constant solution of (3.1) is given by a tensor $r_0 \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies

$$\left[r_0^{12}, r_0^{13}\right] + \left[r_0^{12}, r_0^{23}\right] + \left[r_0^{13}, r_0^{23}\right] = 0. \tag{3.2}$$

Within this subsection we shall frequently use the abbreviation $T = \varphi_1(r_0)$. A tensor r_0 is unitary iff $\tau r_0 = -r_0$, or equivalently, if the adjoint with respect to the Killing form on $\mathfrak{g}$ satisfies $T^* = -T$, see Lemma 2.6.

Lemma 3.2. *Let $r_0 \in \mathfrak{g}\otimes\mathfrak{g}$ and $T = \varphi_1(r_0)$.*

(a) Equation (3.2) is equivalent to

$$[T(x),T(y)] - T\Big([x,T(y)] - [T^*(x),y]\Big) = 0 \qquad \forall\, x,y \in \mathfrak{g}. \tag{3.3}$$

(b) For $\lambda \in \mathbb{C}$, the equation $r_0 + \tau r_0 = \lambda\Omega$ is equivalent to $T + T^* = \lambda \mathbb{1}_{\mathfrak{g}}$.

(c) If $\lambda \in \mathbb{C}$ and $r_0 + \tau r_0 = \lambda\Omega$, then equation (3.2) is equivalent to

$$[T(x),T(y)] = T\Big([x,T(y)] + [T(x),y] - \lambda[x,y]\Big) \qquad \forall\, x,y \in \mathfrak{g}. \tag{3.4}$$

Proof. From Proposition 2.14 we obtain

$$\varphi_2\big([r_0^{12}, r_0^{13}]\big)(x\otimes y) = [T(x),T(y)]\,, \quad \varphi_2\big([r_0^{12}, r_0^{23}]\big)(x\otimes y) = -T\,[x,T(y)]$$
$$\text{and} \quad \varphi_2\big([r_0^{13}, r_0^{23}]\big)(x\otimes y) = T\,[T^*(x),y] \quad \text{which gives (a).}$$

Part (b) follows from Lemma 2.6, and (c) is clear from (a) and (b). □

We need the following result, a proof of which can be found in [4, Theorem 9.2].

Theorem 3.3. *If $\psi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an automorphism of the simple Lie algebra $\mathfrak{g}$, then there exists a non-zero element $x \in \mathfrak{g}$ such that $\psi(x) = x$, i.e. $\det(\psi - \mathbb{1}_{\mathfrak{g}}) = 0$.*

Lemma 3.4. *If $r_0 \in \mathfrak{g}\otimes\mathfrak{g}$ is a non-degenerate solution of (3.2), then it is unitary.*

Proof. With notation as above, we have equation (3.3) for all $x,y\in\mathfrak{g}$. Interchanging x and y and using the skew-symmetry of the Lie-bracket, we obtain from it

$$-[T(x),T(y)] - T\Big(-[T(x),y] + [x,T^*(y)]\Big) = 0. \tag{3.5}$$

Adding (3.3) and (3.5) and using that T is an isomorphism by assumption, we obtain $\big[(T+T^*)(x),y\big] = \big[x,(T+T^*)(y)\big]$ for all $x,y\in\mathfrak{g}$. This equation implies, using the associativity of the Killing form and that $(T+T^*)$ is self-adjoint,

$$\begin{aligned}\kappa\big(x,(T+T^*)[y,z]\big) &= \kappa\big((T+T^*)(x),[y,z]\big) = \kappa\big(\big[(T+T^*)(x),y\big],z\big)\\ &= \kappa\big(\big[x,(T+T^*)(y)\big],z\big) = \kappa\big(x,\big[(T+T^*)(y),z\big]\big)\end{aligned}$$

for each $z\in\mathfrak{g}$. As κ is non-degenerate, we obtain $(T+T^*)[y,z] = \big[(T+T^*)(y),z\big]$ for all $y,z\in\mathfrak{g}$, i.e. $(T+T^*)\circ \mathrm{ad}(z) = \mathrm{ad}(z)\circ(T+T^*)$ for all $z\in\mathfrak{g}$. As $\mathfrak{g}$ is simple, Lemma 2.2 implies now that $(T+T^*) = \lambda\cdot\mathbb{1}_{\mathfrak{g}}$ for some $\lambda\in\mathbb{C}$.

We have to show that $\lambda = 0$. For a proof by contradiction, we assume $\lambda \neq 0$. Because we could then replace r_0 by r_0/λ, we may even assume $\lambda = 1$. In this case, subtracting $T[T(x),T(y)]$ from both sides of (3.4) gives

$$(T-\mathbb{1}_{\mathfrak{g}})\big[T(x),T(y)\big] = T\big[(T-\mathbb{1}_{\mathfrak{g}})(x),(T-\mathbb{1}_{\mathfrak{g}})(y)\big]. \tag{3.6}$$

As T was assumed to be invertible, $T^* = \mathbb{1}_{\mathfrak{g}} - T$ is invertible as well. Let $\psi = T\circ(T-\mathbb{1}_{\mathfrak{g}})^{-1} = (T-\mathbb{1}_{\mathfrak{g}})^{-1}\circ T$ and apply $(T-\mathbb{1}_{\mathfrak{g}})^{-1}$ to (3.6). This gives

$$\big[T(x),T(y)\big] = \psi\big[(T-\mathbb{1}_{\mathfrak{g}})(x),(T-\mathbb{1}_{\mathfrak{g}})(y)\big].$$

If we replace x,y with $(T-\mathbb{1}_{\mathfrak{g}})^{-1}(x)$ and $(T-\mathbb{1}_{\mathfrak{g}})^{-1}(y)$ respectively, we arrive at $[\psi(x),\psi(y)] = \psi[x,y]$, i.e. ψ is an automorphism of Lie algebras. We have $\psi - \mathbb{1}_{\mathfrak{g}} = T\circ(T-\mathbb{1}_{\mathfrak{g}})^{-1} - (T-\mathbb{1}_{\mathfrak{g}})\circ(T-\mathbb{1}_{\mathfrak{g}})^{-1} = (T-\mathbb{1}_{\mathfrak{g}})^{-1}$, hence $\det(\psi-\mathbb{1}_{\mathfrak{g}})\neq 0$ in contradiction to Theorem 3.3. This proves that $\lambda = 0$ and so T is unitary. □

Proposition 3.5. *There are no non-degenerate solutions of* (3.2).

Proof. Assume r_0 is a non-degenerate solutions of (3.2) and let $T = \varphi_1(r_0)$ as before and $S = T^{-1}$. By Lemma 3.4, T is unitary. Therefore, from Lemma 3.2 (c) with $\lambda = 0$ we have $[T(x), T(y)] = T[x, T(y)] + T[T(x), y]$. Applying S and replacing x, y with $S(x)$ and $S(y)$ respectively, gives

$$S[x, y] = [S(x), y] + [x, S(y)],$$

that is S is a derivation of $\mathfrak{g}$. For semi-simple $\mathfrak{g}$ each derivation is equal to $\mathrm{ad}(a)$ for some $a \in \mathfrak{g}$. But $\mathrm{ad}(a)$ is never an isomorphism, so it cannot be equal to S. This contradiction shows that there cannot be a non-degenerate solutions r_0 of (3.2). □

3.2. Characterisation of non-degeneracy

In the proof of the theorem below we use the following identity, which holds for arbitrary holomorphic functions $f, g : U \longrightarrow \mathfrak{g}$ defined on an open set $U \subset \mathbb{C}$ and which follows from the product rule for derivatives:

$$\frac{d}{du}[f(u), g(u)] = [f'(u), g(u)] + [f(u), g'(u)]. \tag{3.7}$$

Theorem 3.6 (Belavin-Drinfeld)**.** *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, $U \subset \mathbb{C}$ an open disc with centre 0 and $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ a meromorphic solution of CYBE. Then the following are equivalent.*

(A) The solution r is non-degenerate.

(B) The function r has at least one pole in U and there does not exist a proper Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that $r(u) \in \mathfrak{a} \otimes \mathfrak{a}$ for all u in a small neighbourhood of $0 \in U$.

(C) All poles of r are simple and the residue of r at 0 is equal to $\lambda\Omega$ for some $\lambda \in \mathbb{C}^*$.

(D) The function r has a simple pole at 0 with residue equal to $\lambda\Omega$ for some $\lambda \in \mathbb{C}^*$.

Proof. (A)$\Longrightarrow$(B) If r is a non-degenerate holomorphic solution of (3.1), Lemma 3.1 implies that $r(0)$ is a non-degenerate constant solution of (3.1). This contradicts Proposition 3.5, hence r must have at least one pole in U.

Assume $\mathfrak{a} \subset \mathfrak{g}$ is a Lie subalgebra such that there exists a small neighbourhood U' of 0 such that $r(u) \in \mathfrak{a} \otimes \mathfrak{a}$ for all $u \in U'$ at which r is holomorphic. For such u, $V_{r(u)} \subset \mathfrak{a}$ and so $r(u)$ can only be non-degenerate if $\mathfrak{a} = \mathfrak{g}$. As r was assumed to be non-degenerate, $r(u)$ must be non-degenerate for $u \in U'$ except at finitely many points, see Remark 2.12. Hence, $\mathfrak{a} = \mathfrak{g}$.

(B)$\Longrightarrow$(C) Let $\gamma \in U$ be a pole of r and $k \geq 1$ the order of this pole. Let $\theta_\gamma = \lim_{u \to \gamma}(u-\gamma)^k r(u)$ be the leading coefficient of the Laurent series of r at γ. If we multiply (3.1) by $(v-\gamma)^k$ and let v tend to γ, we obtain

$$[r^{12}(u) + r^{13}(u+\gamma), \theta_\gamma^{23}] = 0 \tag{3.8}$$

for all $u \in U$ with $u + \gamma \in U$. Similarly, after multiplication by $(u-\gamma)^k$ we obtain

$$[\theta_\gamma^{12}, r^{13}(v+\gamma) + r^{23}(v)] = 0 \tag{3.9}$$

for all $v \in U$ for which $v + \gamma \in U$. Recall that $V_{\theta_\gamma} = \mathrm{im}\,(\varphi_1(\theta_\gamma)) \subset \mathfrak{g}$ is equal to $\mathfrak{g}$ if and only if θ_γ is a non-degenerate element of $\mathfrak{g} \otimes \mathfrak{g}$.

Note that $[\theta_\gamma^{12}, r^{23}(v)] \in V_{\theta_\gamma} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and $[r^{13}(u+\gamma), \theta_\gamma^{23}] \in \mathfrak{g} \otimes V_{\theta_\gamma} \otimes \mathfrak{g}$, because $\theta_\gamma \in V_{\theta_\gamma} \otimes \mathfrak{g}$. From (3.8) and (3.9) we therefore get $[\theta_\gamma^{12}, r^{13}(v+\gamma)] \in V_{\theta_\gamma} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and $[r^{12}(u), \theta_\gamma^{23}] \in \mathfrak{g} \otimes V_{\theta_\gamma} \otimes \mathfrak{g}$. If $\mathfrak{a} = \{x \in \mathfrak{g} \mid [x, v] \in V \text{ for all } v \in V\}$, Lemma 2.17 implies that $r(u) \in \mathfrak{g} \otimes \mathfrak{a}$ and $r(v+\gamma) \in \mathfrak{a} \otimes \mathfrak{g}$ for all $u \in U$ for which $u+\gamma \in U$ and for all $v \in U$ with $v+\gamma \in U$. Hence, $r(u) \in \mathfrak{a} \otimes \mathfrak{g} \cap \mathfrak{g} \otimes \mathfrak{a} = \mathfrak{a} \otimes \mathfrak{a}$

whenever $u-\gamma, u, u+\gamma \in U$. As U is an open disc with centre 0 and $\gamma \in U$, this condition is satisfied for u in a small neighbourhood of 0. According to our assumption this implies $\mathfrak{a} = \mathfrak{g}$, hence $V_{\theta_\gamma} \subset \mathfrak{g}$ is an ideal. As $\mathfrak{g}$ is simple and $\theta_\gamma \neq 0$ by definition, we get $V_{\theta_\gamma} = \mathfrak{g}$, i.e. θ_γ is non-degenerate.

By Lemma 2.16 we obtain $[\theta_\gamma^{12}, \theta_\gamma^{13}] \neq 0$. If r does not have a pole of order at least k at the origin, multiplying (3.9) by v^k and letting v tend to 0 gives us $[\theta_\gamma^{12}, \theta_\gamma^{13}] = 0$. Hence, r has a pole of order l at the origin and $l \geq k \geq 1$. Therefore, we may set $\gamma = 0$ above and see that $\theta = \theta_0 = \lim_{u\to 0} u^l r(u)$ is non-degenerate and $[\theta^{12}, \theta^{13}] \neq 0$.

Assume $l \geq 2$, then multiplying (3.1) by u^l gives the equation

$$[u^l r^{12}(u), r^{13}(u+v) + r^{23}(v)] + u^l [r^{13}(u+v), r^{23}(v)] = 0 \tag{3.10}$$

in which all ingredients, including $u^l r^{12}(u)$, are holomorphic functions of u in a small neighbourhood of 0, provided that $v \in U$ is fixed and close to 0. Using (3.7) and $l \geq 2$, the derivative of (3.10) with respect to u at $u = 0$ evaluates as

$$[\theta^{12}, \big(r'(v)\big)^{13}] + [A^{12}, r^{13}(v) + r^{23}(v)] = 0$$

where A denotes the derivative of $u^l r(u)$ at 0. This equation holds for all $v \neq 0$ in a small neighbourhood of the origin. Note that the leading term of $r'(v)$ in its Laurent expansion is $-l\theta/v^{l+1}$. If we multiply the above equation by v^{l+1} and let v tend to 0 we obtain $[\theta^{12}, -l\theta^{13}] = 0$, in contradiction to $[\theta^{12}, \theta^{13}] \neq 0$. Therefore, $l = 1$ and so also $k = 1$. This proves the first part of (B).

To find the residue of the simple pole at the origin, we consider (3.9) with $\gamma = 0$ and the equation obtained from (3.8) with $\gamma = 0$ by applying the cyclic permutation operator $\tau_{231}(a \otimes b \otimes c) = b \otimes c \otimes a$, i.e.

$$[\theta^{12}, r^{13}(u) + r^{23}(u)] = 0 \qquad \text{and} \qquad [\theta^{12}, (\tau r)^{13}(u) + (\tau r)^{23}(u)] = 0. \tag{3.11}$$

If we define $\mathfrak{a} = \{x \in \mathfrak{g} \mid [\mathrm{ad}(x), \varphi_1(\theta)] = 0\}$, by Lemma 2.18, the equations (3.11) imply $r(u), \tau r(u) \in \mathfrak{a} \otimes \mathfrak{g}$, hence $r(u) \in \mathfrak{a} \otimes \mathfrak{g} \cap \mathfrak{g} \otimes \mathfrak{a} = \mathfrak{a} \otimes \mathfrak{a}$ for all $u \in U$ where r is holomorphic. By assumption this can only happen if $\mathfrak{a} = \mathfrak{g}$ and so $\varphi_1(\theta) = \lambda \mathbb{1}_{\mathfrak{g}}$ by Lemma 2.2 for some $\lambda \in \mathbb{C}$. Hence, $\theta = \lambda\Omega$ with $\lambda \in \mathbb{C}$ and $\lambda \neq 0$ as $\theta \neq 0$.

(C)$\Longrightarrow$(D) This is obvious.

(D)$\Longrightarrow$(A) By assumption, $\lim_{u\to 0} u \cdot r(u) = \lambda\Omega$, $\lambda \in \mathbb{C}^*$, hence $\lim_{u\to 0} u \cdot \varphi_1(r(u)) = \lambda \mathbb{1}_{\mathfrak{g}}$, and so $\lim_{u\to 0} \det(u \cdot \varphi_1(r(u)) \neq 0$. For $u \neq 0$ in a small neighbourhood of the origin we obtain $\det(\varphi_1(r(u)) \neq 0$, i.e. $r(u)$ is non-degenerate for such u. This means that r is non-degenerate. □

Proposition 3.7. *Let $\mathfrak{g}$ be simple and $U \subset \mathbb{C}^n$ an open neighbourhood of the origin. Then, all non-degenerate meromorphic solutions $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ of* (3.1) *are unitary, i.e. satisfy $\tau r(u) = -r(-u)$.*

Proof. If we replace u, v by $-u$ and $u+v$ respectively, we obtain from (3.1), after applying the swapping operator $\tau_{12}(a \otimes b \otimes c) = b \otimes a \otimes c$, the equation

$$\big[(\tau r)^{12}(-u), r^{23}(v)\big] + \big[(\tau r)^{12}(-u), r^{13}(u+v)\big] + \big[r^{23}(v), r^{13}(u+v)\big].$$

We choose v such that $r(v)$ is non-degenerate. If we add this to (3.1), we obtain

$$\Big[\big(r(u) + \tau r(-u)\big)^{12}, r^{23}(v) + r^{13}(u+v)\Big] = 0.$$

We replace v by tv with $t \in \mathbb{C}$, multiply the above expression by t and take the limit if t approaches 0. Because r, restricted to the one-dimensional subspace $\langle v \rangle \subset \mathbb{C}^n$, has (by Theorem 3.6) a simple pole at the origin with residue $\lambda\Omega \neq 0$, we obtain

$$\Big[\big(r(u) + \tau r(-u)\big)^{12}, \lambda\Omega^{23}\Big] = 0.$$

The claim follows now from Corollary 2.15 (d). □

4. The Classification Theorem of Belavin and Drinfeld

The purpose of this section is to prove the main classification result of Belavin and Drinfeld, Theorem 4.8. The proof presented here is a reformulation of the original proof in [2], with some details filled in. We have benefited from other versions of this proof in [4], [7] and [9]. Throughout this section, $\mathfrak{g}$ denotes a finite-dimensional simple complex Lie algebra.

Definition 4.1. Two meromorphic functions $r, s : \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ are called *equivalent* if there exists a holomorphic function $\psi : \mathbb{C} \longrightarrow \mathsf{Aut}(\mathfrak{g})$ such that, for all $u_1, u_2 \in \mathbb{C}$

$$s(u_1 - u_2) = \big(\psi(u_1) \otimes \psi(u_2)\big) r(u_1 - u_2).$$

It is easy to see that s is a solution of (3.1) if and only if r is so and that s is non-degenerate if and only if r is so. Note that, for general ψ, the right hand side may depend on (u_1, u_2) and not only on the difference $u_1 - u_2$.

For the proof of the main theorem, we need to introduce birational group laws and their relation to algebraic groups. This theory goes back to A. Weil [11]. Generalisations of his results can be found in [1] and [5] which are written in a language more familiar nowadays.

Definition 4.2 ([5], §5.1, Def. 1)**.** A *birational group law* on a scheme X over $\mathbb{C}$ is a rational map $P : X \times X \longrightarrow X$, written as $P(x, y) = x * y$, such that

(i) the rational maps $\Phi : X \times X \longrightarrow X \times X$, given by $\Phi(x, y) = (x, x * y)$ and $\Psi : X \times X \longrightarrow X \times X$, given by $\Psi(x, y) = (x * y, y)$ are birational;

(ii) P is associative, i.e. $(x * y) * z = x * (y * z)$ whenever both sides are defined.

The birational group law P is called *commutative* if $x * y = y * x$ whenever both sides are defined.

A typical example is a dense open subset of a group scheme. The relation to group schemes is clarified by Weil's Theorem which is given below. Because the neutral element and some inverses could be missing in X, the usual requirement that such elements exist is replaced by condition (i).

Theorem 4.3 (A. Weil [11], see also [5], §5.1, Thm. 5)**.** *Let (X, P) be a birational group law on a smooth separated scheme X of finite type over $\mathbb{C}$. Then there exists an algebraic group G, an open dense subscheme $X' \subset X$ and an open dense immersion $X' \subset G$ such that the restriction of the group law of G coincides with P on X'.*

If P was commutative then G will also be a commutative algebraic group, because commutativity on an open dense subset implies commutativity everywhere.

Proposition 4.4. *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, $U \subset \mathbb{C}^n$ an open neighbourhood of the origin which is convex and invariant under the involution $u \mapsto -u$. Let $\widetilde{U} = \{(u, v) \mid u + v \in U\} \subset U \times U$ and $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ be a non-degenerate meromorphic solution of CYBE.*
Then, there exist rational functions $P, Q : (\mathfrak{g} \otimes \mathfrak{g}) \times (\mathfrak{g} \otimes \mathfrak{g}) \longrightarrow (\mathfrak{g} \otimes \mathfrak{g})$ such that

$$r(u + v) = P\big(r(u), r(v)\big) \qquad \text{for all } (u, v) \in \widetilde{U} \text{ and}$$
$$r(u - v) = Q\big(r(u), r(v)\big) \qquad \text{for all } (u, -v) \in \widetilde{U}.$$

Proof. Let $Y_{u,v} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ denote the linear map which is given by $Y_{u,v}(A) = [r^{12}(u) - r^{23}(v), A^{13}]$. We first show that $Y_{u,v}$ is injective for (u, v) in an open subset of $\widetilde{U}$. To do so, we fix $u_0 \in U$ with $(u_0, u_0) \in \widetilde{U}$ such that $r(u_0)$ is non-degenerate and consider the family of linear maps tY_{tu_0,tu_0}, which depend holomorphically on $t \in \mathbb{C}$ in a small neighbourhood of 0. Without loss of generality, we may assume that the restriction of r to the one-dimensional subspace $\langle u_0 \rangle \subset \mathbb{C}^n$ has residue Ω at the origin. The limit of tY_{tu_0,tu_0} if t approaches 0 exists and is then equal to the linear map which sends $A \in \mathfrak{g} \otimes \mathfrak{g}$ to $\big[\Omega^{12} - \Omega^{23}, A^{13}\big]$. This map is injective by Corollary 2.15 (e). As injectivity is an open property, there exists $t_0 \neq 0$ with $(t_0u_0, t_0u_0) \in \widetilde{U}$ such that $t_0Y_{t_0u_0,t_0u_0}$ and so also $Y_{t_0u_0,t_0u_0}$ is injective. By the same reason, there exists an open neighbourhood $\widetilde{U}' \subset \widetilde{U}$ of (t_0u_0, t_0u_0) such that $Y_{u,v}$ is injective for all $(u, v) \in \widetilde{U}'$.

The assumption that r satisfies CYBE is equivalent to the equation

$$Y_{u,v}\big(r(u + v)\big) = [r^{23}(v), r^{12}(u)].$$

The injectivity of $Y_{u,v}$ implies that $r(u + v)$ can be obtained by applying to the right hand side of this equation the inverse of a certain square minor of a matrix representation of $Y_{u,v}$. Because the entries of the inverse of a square matrix are rational expressions of the entries of the original matrix, this shows that $r(u + v)$ depends rationally on $r(u)$ and $r(v)$, provided that $(u, v) \in \widetilde{U}'$. This shows that, there exists a rational function $P : (\mathfrak{g} \otimes \mathfrak{g}) \times (\mathfrak{g} \otimes \mathfrak{g}) \longrightarrow (\mathfrak{g} \otimes \mathfrak{g})$ such that $r(u + v) = P\big(r(u), r(v)\big)$ for all $(u, v) \in \widetilde{U}'$. Tracing this argument backwards, we see that the function r satisfies CYBE for $(u, v) \in \widetilde{U}'$ if and only if $r(u + v) = P\big(r(u), r(v)\big)$ for $(u, v) \in \widetilde{U}'$.

The meromorphic function $f(u, v) = r(u + v) - P\big(r(u), r(v)\big)$ is defined on $\widetilde{U}$ and identically zero on the open subset $\widetilde{U}'$. As U was assumed to be convex, $\widetilde{U}$ is connected and so f vanishes identically on $\widetilde{U}$. This proves $r(u + v) = P\big(r(u), r(v)\big)$ for all $(u, v) \in \widetilde{U}$.

From Proposition 3.7 we know that r is unitary, i.e. $r(-v) = -\tau r(v)$. As τ is linear, $Q(A, B) = P(A, -\tau B)$ is rational again. If $(u, -v) \in \widetilde{U}$, we have $u, v \in U$ and $r(u - v) = P\big(r(u), r(-v)\big) = P\big(r(u), -\tau r(v)\big) = Q\big(r(u), r(v)\big)$ as required. □

Corollary 4.5. *If $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra, $U \subset \mathbb{C}$ an open disc with centre 0 and $r : U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ a non-degenerate meromorphic solution of CYBE, then r extends to a meromorphic function $r : \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$.*

Proof. For any disc $D(0, \rho) \subset \mathbb{C}$ we say that r satisfies CYBE *on* $D(0, \rho)$, if (3.1) is satisfied for all $(u, v) \in \widetilde{D}(0, \rho) := \{(u, v) \mid u, v, u + v \in D(0, \rho)\} \subset \mathbb{C}^2$. Fix $\varepsilon > 0$ such that $D(0, \varepsilon) \subset U$, for example $D(0, \varepsilon) = U$. If $\rho > \varepsilon$ and r extends to a meromorphic function $r : D(0, \rho) \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$, then r satisfies CYBE on $D(0, \rho)$. This follows from the Identity Theorem because the meromorphic function, which is the left hand side of CYBE, vanishes identically on the open subset $\widetilde{D}(0, \varepsilon)$ of $\widetilde{D}(0, \rho)$, hence vanishes on $\widetilde{D}(0, \rho)$ as well.

Assume that r does not extend to a meromorphic function on $\mathbb{C}$. Then there exists a real number $\rho \geq \varepsilon$ such that r extends to $D(0, \rho)$ but not to $D(0, \rho + \varepsilon)$. For each fixed $v \in D(0, \varepsilon)$ we define a meromorphic function $s : D(v, \rho) \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ on the disc with centre v and radius ρ by $s(u) = P\big(r(u-v), r(v)\big)$, where P is the rational function from Proposition 4.4 for which we have $r(u) = P\big(r(u-v), r(v)\big)$ whenever $u, v, u-v \in D(0, \rho)$. Hence the two functions r and s coincide on $D(0, \rho)\cap D(v, \rho)$. Therefore, s defines a meromorphic extension of r to $D(0, \rho)\cup D(v, \rho)$. If we vary $v \in D(0, \varepsilon)$ we obtain an extension of r to $D(0, \rho + \varepsilon) = \bigcup_{v\in D(0,\varepsilon)} D(v, \rho)$ in contradiction to our assumption. Therefore r extends to the whole complex plane. □

Proposition 4.6. *The set of poles $\Gamma \subset \mathbb{C}$ of a meromorphic non-degenerate solution $r : \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ of* (3.1) *is a discrete subgroup and there exists a homomorphism $\Gamma \longrightarrow \mathsf{Aut}(\mathfrak{g})$ sending γ to A_γ such that $r(u + \gamma) = (A_\gamma \otimes \mathbb{1}_{\mathfrak{g}})r(u)$ for all $u \in \mathbb{C}$ and $\gamma \in \Gamma$.*

Proof. As the set of poles of a meromorphic function is always discrete, we only need to show that Γ is a subgroup. By Theorem 3.6, $0 \in \Gamma$ and all poles of r are simple. After rescaling, we are able to assume that the residue at the origin is equal to the Casimir element Ω. Let θ_γ denote the residue of r at $\gamma \in \Gamma$. Then $A_\gamma := \varphi_1(\theta_\gamma) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a non-zero linear map. As before, we use the abbreviation $T_u = \varphi_1(r(u))$. Because r is unitary by Proposition 3.7, we have $T_\gamma = -T^*_{-\gamma}$, hence $\gamma \in \Gamma$ implies $-\gamma \in \Gamma$. If we multiply (3.1) by $(u-\gamma)$ and take the limit if u approaches γ, we obtain the equation

$$\left[\theta_\gamma^{12}, r^{13}(v+\gamma) + r^{23}(v)\right] = 0. \tag{4.1}$$

After multiplying it by v and letting v tend to 0, we get $\left[\theta_\gamma^{12}, \theta_\gamma^{13} + \Omega^{23}\right] = 0$. Applying the map φ_2 and using Proposition 2.14, this equation is seen to be equivalent to $[A_\gamma(x), A_\gamma(y)] = A_\gamma[x,y]$ for all $x, y \in \mathfrak{g}$. This means that A_γ is a homomorphism of Lie algebras. It is even an isomorphism, as $\mathfrak{g}$ is simple and $A_\gamma \neq 0$. Now we apply φ_2 to (4.1) and use Proposition 2.14 to obtain

$$0 = [A_\gamma(x), T_{v+\gamma}(y)] - A_\gamma\,[x, T_v(y)] = [A_\gamma(x), T_{v+\gamma}(y)] - [A_\gamma(x), A_\gamma T_v(y)]$$

for all $x, y \in \mathfrak{g}$. Because A_γ is surjective, this implies $T_{v+\gamma} = A_\gamma \circ T_v$ for all $v \in \mathbb{C}$, which is, by Lemma 2.4, equivalent to the desired equality $r(v+\gamma) = (A_\gamma \otimes \mathbb{1}_\mathfrak{g})r(v)$. Hence, if $v \in \Gamma$ then $v + \gamma \in \Gamma$ and Γ is closed under addition.

Using $v \in \mathbb{C}$ with invertible T_v, we see that $A_{\gamma_1+\gamma_2} = A_{\gamma_1} \circ A_{\gamma_2}$ which concludes the proof. □

Lemma 4.7. *Let $R : \mathbb{C}^n \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ be a non-degenerate meromorphic function which satisfies* (3.1) *with $u, v \in \mathbb{C}^n$. Then there exists a vector space $V \subset \mathbb{C}^n$ of codimension* 1 *such that for all $\mathbf{e}, \mathbf{e}' \in \mathbb{C}^n \setminus \{0\}$ with $\mathbf{e}' - \mathbf{e} \in V$, the functions $r(u) = R(u\mathbf{e})$ and $r'(u) = R(u\mathbf{e}')$ are equivalent.*

Proof. We write $R(z) = Y(z)/f(z)$ with $Y : \mathbb{C}^n \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ and $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ holomorphic such that $S = \{z \in \mathbb{C}^n \mid f(z) = 0, Y(z) \neq 0\}$ is not empty. The set $\widetilde{S}$ of poles of R contains S, but might not coincide with it. We shall first show that there exist maps $\Phi : S \longrightarrow \mathsf{Aut}(\mathfrak{g})$ and $\lambda : S \longrightarrow \mathbb{C}^*$ such that for all $z \in \mathbb{C}^n, h \in S$

$$Y(h) = \lambda(h) \cdot (\Phi(h) \otimes \mathbb{1}_\mathfrak{g})\,(\Omega) \tag{4.2}$$
$$R(z+h) = \big(\Phi(h) \otimes \mathbb{1}_\mathfrak{g}\big) R(z). \tag{4.3}$$

As before, we use the abbreviation $T_z = \varphi_1(R(z))$. Fix $h \in S$, multiply CYBE (3.1) for R with variables u and $v = z$ by $f(u)$ and take the limit if u approaches h, to obtain $\left[Y^{12}(h), R^{13}(h+z) + R^{23}(z)\right] = 0$. By Proposition 2.14 this is equivalent to

$$\left[\varphi_1(Y(h))(x), T_{h+z}(y)\right] - \varphi_1(Y(h))\left[x, T_{z(y)}\right] = 0 \quad \text{for all } x, y \in \mathfrak{g}.$$

Using Proposition 2.14 (d), (e) this is seen to be equivalent to

$$\left[Y(h), T_{h+z}(y) \otimes 1 + 1 \otimes T_z(y)\right] = 0. \tag{4.4}$$

Consider now $\mathfrak{g} \oplus \mathfrak{g}$ as a subalgebra of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ via $(a, b) \mapsto a \otimes 1 + 1 \otimes b$. This embedding is indeed an algebra homomorphism since $[a \otimes 1, 1 \otimes b] = 0$. For fixed

$$z \in W = \left\{ z \;\middle|\; z, h+z \notin \widetilde{S} \text{ and } T_z, T_{h+z} \text{ are invertible} \right\} \subset \mathbb{C}^n$$

we let $\mathfrak{g}_0 \subset \mathfrak{g} \oplus \mathfrak{g}$ be the Lie subalgebra generated by $T_{h+z}(y) \otimes 1 + 1 \otimes T_z(y)$ for all $y \in \mathfrak{g}$. The two projections $\mathfrak{g} \oplus \mathfrak{g} \longrightarrow \mathfrak{g}$ induce Lie-algebra homomorphism $p_i : \mathfrak{g}_0 \longrightarrow \mathfrak{g}$, $i = 1, 2$ which are surjective because $z \in W$. As $\mathfrak{g}$ is simple, the kernel $\ker(p_2) = \mathfrak{g}_0 \cap (\mathfrak{g} \oplus 0) \subset \mathfrak{g} \oplus 0 \cong \mathfrak{g}$

is either $\mathfrak{g} \oplus 0$ or 0. If $\ker(p_2) = \mathfrak{g} \oplus 0$, the dimension formula implies $\mathfrak{g}_0 = \mathfrak{g} \oplus \mathfrak{g}$ and so, in particular, $y \otimes 1 \in \mathfrak{g}_0$. The definition of $\mathfrak{g}_0$ and equation (4.4) imply that $[Y(h), A] = 0$ for all $A \in \mathfrak{g}_0$, hence $[Y(h), y \otimes 1] = 0$ for all $y \in \mathfrak{g}$. But then, by Proposition 2.14 (d), $0 = \varphi_1\big([Y(h), y \otimes 1]\big)(x) = \big[\varphi_1(Y(h))(x), y\big]$ for all $x, y \in \mathfrak{g}$. As $\mathfrak{g}$ in not abelian, this is only possible if $Y(h) = 0$ in contradiction to $h \in S$. This shows that $\ker(p_2) = 0$ and $p_2 : \mathfrak{g}_0 \longrightarrow \mathfrak{g}$ is an isomorphism of Lie algebras. Let $\Phi(h, z) = p_1 \circ p_2^{-1} \in \mathsf{Aut}(\mathfrak{g})$, then each element of $\mathfrak{g}_0$ can be written as $(\Phi(h, z)(x), x) \in \mathfrak{g} \oplus \mathfrak{g}$. If we apply this to the generators of $\mathfrak{g}_0$ we obtain $T_{h+z} = \Phi(h, z) \circ T_z$, which is equivalent to

$$R(h + z) = \big(\Phi(h, z) \otimes \mathbb{1}_{\mathfrak{g}}\big) R(z). \tag{4.5}$$

If we plug this in (4.4), we get $\big[Y(h), (\Phi(h, z) \circ T_z)\,(y) \otimes 1 + 1 \otimes T_z(y)\big] = 0$. Because $\Phi(h, z)$ is a Lie-algebra isomorphism, this is equivalent to

$$\Big[\big(\Phi(h, z)^{-1} \otimes \mathbb{1}_{\mathfrak{g}}\big)\, Y(h), T_z(y) \otimes 1 + 1 \otimes T_z(y)\Big] = 0.$$

With the aid of Proposition 2.14 (d), (e) this translates into $[\varphi_1(B)(x), T_z(y)] - \varphi_1(B)\,[x, T_z(y)] = 0$ for all $x, y \in \mathfrak{g}$, where we have set $B = \big(\Phi(h, z)^{-1} \otimes \mathbb{1}_{\mathfrak{g}}\big)\, Y(h)$. As T_z is surjective, this is equivalent to $[\mathrm{ad}(y), \varphi_1(B)] = 0$ for all $y \in \mathfrak{g}$. Lemma 2.2 implies now $\varphi_1(B) \in \mathbb{C} \cdot \mathbb{1}_{\mathfrak{g}}$. As $Y(h) \neq 0$, this shows that there exists a non-zero number $\lambda(h, z)$ such that $\big(\Phi(h, z)^{-1} \otimes \mathbb{1}_{\mathfrak{g}}\big)\, Y(h) = \lambda(h, z)\Omega$, i.e.

$$Y(h) = \lambda(h, z) \cdot \left(\Phi(h, z) \otimes \mathbb{1}_{\mathfrak{g}}\right)(\Omega). \tag{4.6}$$

This is equivalent to $\varphi_1(Y(h)) = \lambda(h, z) \cdot \Phi(h, z)$. As the left hand side does not depend on z and there is only one multiple of the linear map $\varphi_1(Y(h))$ which is a Lie-algebra homomorphism, both $\lambda(h, z)$ and $\Phi(h, z)$ are determined by $Y(h)$ and do not depend on z. From (4.5) and (4.6) we see that the functions $\lambda(h) = \lambda(h, z)$ and $\Phi(h) = \Phi(h, z)$ satisfy (4.2) and (4.3) for all $z \in W$ and $h \in S$. As W is dense in $\mathbb{C}^n$ these two equations will be satisfied for all $z \in \mathbb{C}^n$.

Next, we show that there exists a vector space $V \subset \mathbb{C}^n$ of codimension 1 and a holomorphic homomorphism of groups $\Phi : V \longrightarrow \mathsf{Aut}(\mathfrak{g})$ such that for all $z \in \mathbb{C}^n, h \in V$

$$R(z + h) = \big(\Phi(h) \otimes \mathbb{1}_{\mathfrak{g}}\big) R(z) \tag{4.7}$$

$$R(z) = \big(\Phi(h) \otimes \Phi(h)\big) R(z). \tag{4.8}$$

To find the subspace V we denote by H the subgroup of $\mathbb{C}^n$ which is generated by S. Because of (4.3), which is equivalent to $T_{h+z} = \Phi(h) \circ T_z$, we can extend Φ to a homomorphism $\Phi : H \longrightarrow \mathsf{Aut}(\mathfrak{g})$ which satisfies (4.3) for all $z \in \mathbb{C}^n$ and $h \in H$. From $T_{h+z} = \Phi(h) \circ T_z$ we see that the set $\widetilde{S}$ of poles of R is invariant under H, hence $H \neq \mathbb{C}^n$. Because H contains the analytic subset S of codimension one in $\mathbb{C}^n$, this group contains a codimension one linear subspace $V \subset \mathbb{C}^n$.

To see that $\Phi : V \longrightarrow \mathsf{Aut}(\mathfrak{g})$ is holomorphic, we recall that R was assumed to be non-degenerate. This implies that for each $h_0 \in V$ the set which consist of those $z \in \mathbb{C}^n$ for which R is holomorphic and non-degenerate at $h_0 + z$, is open and dense in $\mathbb{C}^n$. Hence, for each $h_0 \in V$ there exists $z_0 \in \mathbb{C}^n$ such that R is holomorphic at z_0 and at $h_0 + z_0$ and T_{z_0} is an isomorphism. Because $\Phi(h) = T_{h+z_0} \circ T_{z_0}^{-1}$, we see now that Φ is holomorphic in a neighbourhood of $h_0 \in V$.

To show equation (4.8), we recall that R is unitary by Proposition 3.7. This means that $T_z^* = -T_z$. Equation (4.7) implies $T_{-z} = \Phi(h) \circ T_{-z-h}$ and $T_{z+h} = \Phi(h) \circ T_z$, hence

$$T_z = -T_{-z}^* = -T_{-z-h}^* \circ \Phi(h)^* = T_{z+h} \circ \Phi(h)^* = \Phi(h) \circ T_z \circ \Phi(h)^*,$$

which is equivalent to (4.8) by Lemma 2.4.

Finally, let $h = (u_1 - u_2)(\mathbf{e}' - \mathbf{e}) \in V$ for given $u_1, u_2 \in \mathbb{C}$. With $z = (u_1 - u_2)\mathbf{e}$, equation (4.7) implies $r'(u_1 - u_2) = R(h+z) = \big(\Phi(h) \otimes \mathbb{1}_{\mathfrak{g}}\big)R(z) = \big(\Phi\big((u_1-u_2)(\mathbf{e}'-\mathbf{e})\big) \otimes \mathbb{1}_{\mathfrak{g}}\big)r(u_1-u_2)$. From equation (4.8) we obtain $r(u_1 - u_2) = \big(\Phi\big(u_2(\mathbf{e}' - \mathbf{e})\big) \otimes \Phi\big(u_2(\mathbf{e}' - \mathbf{e})\big)\big)r(u_1 - u_2)$ and as Φ is a homomorphism, we have $\Phi\big((u_1 - u_2)(\mathbf{e}' - \mathbf{e})\big) = \Phi\big(u_1(\mathbf{e}' - \mathbf{e})\big) \circ \Phi\big(u_2(\mathbf{e}' - \mathbf{e})\big)^{-1}$. Therefore, $r'(u_1 - u_2) = \big(\Phi\big(u_1(\mathbf{e}' - \mathbf{e})\big) \otimes \Phi\big(u_2(\mathbf{e}' - \mathbf{e})\big)\big)r(u_1 - u_2)$, i.e. r and r' are equivalent. □

Theorem 4.8 (Belavin-Drinfeld)**.** *Let $r : \mathbb{C} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ be a non-degenerate meromorphic solution of* (3.1) *and $\Gamma \subset \mathbb{C}$ its lattice of poles. Then exactly one of the following three cases occurs.*

(a) If $\mathrm{rk}(\Gamma) = 2$, r is elliptic*, i.e. it has two periods that are independent over $\mathbb{R}$ and are contained in Γ.*

(b) If $\mathrm{rk}(\Gamma) = 1$, there exists a rational function $f : \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ and a constant $\lambda \in \mathbb{C}$ such that the function $u \mapsto f(e^{\lambda u})$ is equivalent to r. Such solutions are called trigonometric*.*

(c) If $\mathrm{rk}(\Gamma) = 0$, there exists a rational function $f : \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ which is equivalent to r. Such solutions f are called rational*.*

Proof. Let $X_0 = \{r(u) \mid u \in \mathbb{C} \setminus \Gamma\} \subset \mathfrak{g} \otimes \mathfrak{g}$ and denote by $X \subset \mathfrak{g} \otimes \mathfrak{g}$ the closure of X_0 in the Zariski topology. This means that X is the smallest subset of the vector space $\mathfrak{g} \otimes \mathfrak{g}$ which contains X_0 and which is the zero set of finitely many polynomials. As r is meromorphic and not necessarily rational, the dimension of X could be larger than one. Let now P, Q be rational functions as in Proposition 4.4. Because $P(x, y) \in X_0$ for $x, y \in X_0$, the restriction of P is a rational map $X \times X \longrightarrow X$. Because $r(u+v) = P\big(r(u), r(v)\big)$ for all $u, v \in \mathbb{C}$ and addition on $\mathbb{C}$ is associative and commutative, we obtain $P(x, y) = P(y, x)$ and $P(x, P(y, z)) = P(P(x, y), z)$ for all $x, y, z \in X_0$. As the set where two rational functions coincide is Zariski-closed, these two identities hold for all $x, y, z \in X$ where they are defined.

The only thing left to verify that (X, P) is a birational group law is the birationality of the maps Φ and Ψ from Definition 4.2. We use the rational map Q from Proposition 4.4 to define an inverse of Φ on $X_0 \times X_0$ by $(x, z) \mapsto (x, Q(z, x))$. Similarly, an inverse of Ψ on $X_0 \times X_0$ is given by $(z, y) \mapsto (Q(z, y), y)$. Using the same argument as before, these maps are rational inverses of Φ and Ψ respectively on $X \times X$, hence these two maps are birational.

By Weil's Theorem 4.3, there exists an algebraic group G and a birational map $\psi : X \to G$ under which the group law of G corresponds to P. Denote by $f : \mathbb{C} \to G$ the meromorphic map $\psi \circ r$ and by $\widetilde{R} : G \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ the rational map ψ^{-1} followed by the embedding of X into $\mathfrak{g} \otimes \mathfrak{g}$.

As (X, P) was commutative, G is a commutative algebraic group. By Chevalley's Theorem, the group G is an extension of an abelian variety by a finite product of additive and multiplicative groups, see for example [5, §9.2, Thms. 1, 2] and [8, Ch. IV, §3, no. 6]. This implies that the variety G is isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b \times \mathbb{A}$, where $\mathbb{A}$ is an abelian variety. Therefore, the universal covering of G is a vector space $\mathbb{C}^n$, with $n = a + b + \dim(\mathbb{A})$. If $\pi : \mathbb{C}^n \longrightarrow G$ denotes a covering map which is a homomorphism of groups, we obtain the following commutative diagram

$$\begin{tikzcd}
& \mathbb{C}^n \arrow[d, "\pi"] \arrow[dr, "R"] & \\
\mathbb{C} \arrow[ur, "\bar{f}"] \arrow[r, "f"] \arrow[dr, "r"'] & G \arrow[r, "\widetilde{R}"] & \mathfrak{g} \otimes \mathfrak{g} \\
& X \arrow[u, "\psi"'] \arrow[ur] &
\end{tikzcd}$$

in which $\bar{f}$ is the unique lift of f. By definition, f is a homomorphism of groups, and therefore $\bar{f}$ as well. This implies that there exists $\mathbf{e} \in \mathbb{C}^n$ such that $\bar{f}(u) = u\mathbf{e}$. The commutativity of the diagram implies now $r(u) = R(u\mathbf{e})$. Because addition on an open dense subset of $\mathbb{C}^n$

corresponds to P under $\psi^{-1} \circ \pi$, we have $R(\mathbf{e} + \mathbf{e}') = P(R(\mathbf{e}), R(\mathbf{e}'))$ for all $\mathbf{e}, \mathbf{e}' \in \mathbb{C}^n$. From the proof of Proposition 4.4 we see that this implies that R satisfies (3.1) with $u, v \in \mathbb{C}^n$ and so we can apply Lemma 4.7. By $V \subset \mathbb{C}^n$ we denote a hyperplane which satisfies the conditions of Lemma 4.7.

We choose a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{C}^n$ such that π maps $W_{\mathrm{rat}} = \langle \mathbf{e}_1, \ldots, \mathbf{e}_a \rangle$ isomorphically onto $\mathbb{C}^a$ and $W_{\mathrm{trig}} = \langle \mathbf{e}_{a+1}, \ldots, \mathbf{e}_{a+b} \rangle$ onto $(\mathbb{C}^*)^b$ in such a way that $\pi\left(\sum_{k=a+1}^{a+b} x_k \mathbf{e}_k\right) = \big(\exp(x_{a+1}), \ldots, \exp(x_{a+b})\big) \in (\mathbb{C}^*)^b$. Let $W = W_{\mathrm{rat}} \oplus W_{\mathrm{trig}}$ and $\widetilde{\Gamma} = \pi^{-1}(0)$ be the full period lattice of the quotient map. Then $\widetilde{\Gamma}_k = \widetilde{\Gamma} \cap \langle \mathbf{e}_k \rangle$ is equal to zero if $k \le a$ and equal to the free abelian group $2\pi i \mathbb{Z} \mathbf{e}_k$ of rank 1 if $a < k \le a + b$. This is the full lattice of periods of the function $r_k(u) = R(u\mathbf{e}_k)$. Moreover, if $k \le a + b$, $G_k = \langle \mathbf{e}_k \rangle / \widetilde{\Gamma}_k \subset G$ is one of the additive or multiplicative subgroups, in particular it is a closed subvariety of G.

Let $\gamma_1, \ldots, \gamma_{2m} \in \widetilde{\Gamma}$, $m = n - a - b = \dim(\mathbb{A})$, be such that $\mathbb{A}$ is the quotient of $\mathbb{C}^n/W$ under the lattice of maximal rank generated by the images $\overline{\gamma}_1, \ldots, \overline{\gamma}_{2m}$ of $\gamma_1, \ldots, \gamma_{2m}$ in $\mathbb{C}^n/W$. Then $\widetilde{\Gamma} = (2\pi i \mathbb{Z})^b \oplus \langle \gamma_1, \ldots, \gamma_{2m} \rangle$. Let $\lambda_j \in \mathbb{C}$ be determined by $\gamma_j - \lambda_j \mathbf{e} \in V$. As the set of poles of R is invariant under V (see Lemma 4.7), we have $\lambda_j \in \Gamma$.

If $W \subset V$, then the images of $\gamma_1, \ldots, \gamma_{2m}$ under the surjection $\mathbb{C}^n \to \mathbb{C}^n/W \to \mathbb{C}^n/V$ generate a lattice of maximal rank. The one-dimensional subspace $\langle \mathbf{e} \rangle$ maps isomorphically onto $\mathbb{C}^n/V$ and the image of $\lambda_j \mathbf{e}$ coincides with the image of γ_j. Hence, the assumption $W \subset V$ implies $\mathrm{rk}(\Gamma) = 2$.

If $\mathrm{rk}(\Gamma) < 2$ we, therefore, find $\mathbf{e}_k \notin V$ with $1 \le k \le a+b$ and there exists a non-zero constant λ such that $\mathbf{e} - \lambda \mathbf{e}_k \in V$, hence $r_k(\lambda u)$ and $r(u)$ are equivalent by Lemma 4.7. This implies that both functions have the same set of poles, which is a discrete subgroup by Proposition 4.6. As the rational function $\widetilde{r}_k = \widetilde{R}_{|G_k}$ has only finitely many poles, the poles and the periods of $r_k(\lambda u)$ have to be lattices of the same rank, that is $\mathrm{rk}\,\widetilde{\Gamma}_k = \mathrm{rk}\,\Gamma$. Hence, if $\mathrm{rk}\,\Gamma = 1$, $\mathbf{e}_k \in W_{\mathrm{trig}}$ and $r_k(\lambda u)$ is a rational function of $\exp(\lambda u)$ and in the case $\mathrm{rk}\,\Gamma = 0$ we have $\mathbf{e}_k \in W_{\mathrm{rat}}$ and $r_k(\lambda u)$ is a rational function.

If $\mathrm{rk}\,\Gamma = 2$ we cannot argue in the same way, because it is not clear that we can find a one-dimensional subspace, not contained in W, which contains two independent elements of the lattice $\langle \gamma_1, \ldots, \gamma_{2m} \rangle$. Instead we use the homomorphism $\Gamma \longrightarrow \mathsf{Aut}(\mathfrak{g})$ from Proposition 4.6 sending γ to A_γ and prove directly that r is an elliptic function.

If the image $\Gamma' \subset \mathsf{Aut}(\mathfrak{g})$ of this homomorphism was infinite, the smallest algebraic subgroup $\overline{\Gamma}'$ of $\mathsf{Aut}(\mathfrak{g})$ which contains Γ' would be commutative and of positive dimension. Therefore, its Lie algebra, which is a commutative subalgebra of $\mathfrak{g}$, contains a non-zero element a. For this element we have $A_\gamma(a) = a$ for all $\gamma \in \Gamma$. Let $f : \mathbb{C} \longrightarrow \mathfrak{g}$ be the meromorphic map given by $f(u) = T_u^*(a)$, where $T_u = \varphi_1(r(u))$. We have $f(u + \gamma) = T_{u+\gamma}^*(a) = T_u^* A_\gamma^*(a) = T_u^*(a) = f(u)$, because $T_{u+\gamma} = A_\gamma \circ T_u$ and $A_\gamma^* = A_\gamma^{-1} = A_{-\gamma}$ (Proposition 4.6). This means that f is Γ-periodic. But f has a simple pole at each point of Γ and no other poles, which is impossible for a doubly periodic function. This contradiction shows that Γ' is a finite group.

Let γ_1, γ_2 be generators of Γ. Then $A_{\gamma_1}, A_{\gamma_2}$ are both of finite order $n_i = \mathrm{ord}\,(A_{\gamma_i})$. From $T_{u+\gamma} = A_\gamma \circ T_u$ we obtain that $T_{u+n_1\gamma_1} = T_u = T_{u+n_2\gamma_2}$ and this means that T_u and so also r is an elliptic (doubly periodic with periods $n_1\gamma_1, n_2\gamma_2$) function. □

References

[1] Artin M 1970 Théorème de Weil sur la construction d'un groupe à partir d'une loi rationelle *Schémas en Groupes* SGA 3 Exp. XVIII (Berlin: Springer) pp 632–53

[2] Belavin A and Drinfeld V 1982 Solutions of the classical Yang-Baxter equation for simple Lie algebras *Funktsional. Anal. i ego Prilozhen.* **16** 1

[3] Belavin A and Drinfeld V 1983 The classical Yang-Baxter equation for simple Lie algebras *Funktsional. Anal. i ego Prilozhen.* **17** 69

[4] Belavin A and Drinfeld V 1998 *Triangle Equations and Simple Lie Algebras* (Amsterdam: Harwood Academic Publishers)
[5] Bosch S, Lütkebohmert W and Raynaud M 1990 *Néron Models* (Berlin: Springer)
[6] Burban I and Kreussler B 2007 Vector bundles on degenerations of elliptic curves and Yang–Baxter equations *Preprint* math.AG/0708.1685, to appear in *Memoirs of the AMS*
[7] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[8] Demazure M and Gabriel P 1970 *Groupes Algébriques I* (Paris, Amsterdam: Masson, North-Holland)
[9] Etingof P and Schiffmann O 2002 *Lectures on Quantum Groups* (Somerville: International Press)
[10] Polishchuk A 2002 Classical Yang-Baxter equation and the A_∞-constraint *Adv. Math.* **168** 56
[11] Weil A 1955 On Algebraic Groups of Transformations *Amer. J. Math.* **77** 355