

ALGEBRAIC DIMENSION OF TWISTOR SPACES WHOSE FUNDAMENTAL SYSTEM IS A PENCIL

NOBUHIRO HONDA AND BERND KREUSSLER

ABSTRACT. We show that the algebraic dimension of a twistor space over $n\mathbb{C}\mathbb{P}^2$ cannot be two if $n > 4$ and the fundamental system (i.e. the linear system associated to the half-anti-canonical bundle, which is available on any twistor space) is a pencil. This means that if the algebraic dimension of a twistor space on $n\mathbb{C}\mathbb{P}^2$, $n > 4$, is two, then the fundamental system either is empty or consists of a single member. The existence problem for a twistor space on $n\mathbb{C}\mathbb{P}^2$ with algebraic dimension two is open for $n > 4$.

1. INTRODUCTION

If Z is a compact complex manifold, the algebraic dimension of Z , usually denoted by $a(Z)$, is defined to be the complex dimension of a projective algebraic variety, the rational function field of which is isomorphic to that of Z . We always have $a(Z) \leq \dim_{\mathbb{C}} Z$, and in case of equality, Z is called Moishezon. Any projective algebraic manifold is Moishezon. In the other extreme case, $a(Z) = 0$, Z has no non-constant rational function.

Not so many compact differential manifolds admit complex structures whose algebraic dimension ranges from zero to half of its real dimension. Complex tori of dimension $d \geq 2$ are examples of such manifolds. In complex dimension two, only complex tori and K3 surfaces are examples for which all three possible values of the algebraic dimension actually occur. In dimension three, *twistor spaces* associated to self-dual metrics on 4-manifolds [1] are good candidates of such manifolds.

By a result of Campana [3], if Z is a Moishezon twistor space, then the base 4-manifold is homeomorphic to $n\mathbb{C}\mathbb{P}^2$, the connected sum of $n \geq 0$ copies of complex projective planes; by convention $0\mathbb{C}\mathbb{P}^2 = S^4$. If $n < 4$, any twistor space on $n\mathbb{C}\mathbb{P}^2$ is Moishezon as long as the corresponding self-dual metric has positive scalar curvature [15, 22]. No example seems to be known of a self-dual metric of non-positive scalar curvature on these manifolds. Thus if $n < 4$, all known twistor spaces are Moishezon.

The situation is very different for $n \geq 4$. First, if $n = 4$, for any $a \in \{1, 2, 3\}$, there actually exists a twistor space Z over $4\mathbb{C}\mathbb{P}^2$ which satisfies $a(Z) = a$, see [22, 6, 12, 13]. Also, it is known that $a(Z) \neq 0$ as long as the self-dual metric on $4\mathbb{C}\mathbb{P}^2$ is of positive scalar curvature [22]. Moreover, no example seems to be known of a self-dual metric on $4\mathbb{C}\mathbb{P}^2$ of non-positive scalar curvature. Thus our understanding of possible values of the algebraic dimension is quite satisfactory in case $n = 4$, too.

The main focus of this article is on the case $n > 4$. For any $n > 4$ and $a \in \{0, 1, 3\}$ it is known that there exists a twistor space Z on $n\mathbb{C}\mathbb{P}^2$ which satisfies $a(Z) = a$, [7, 22, 18]. Moreover, in [8, Main Theorem], it is stated that there exists a twistor space Z on $n\mathbb{C}\mathbb{P}^2$ with $a(Z) = 2$ for any $n > 4$. These spaces satisfy $\dim |K^{-1/2}| = 1$, where $K^{-1/2}$ is the

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natural square root of the anti-canonical line bundle on Z , which is available on any twistor space [10]. This is in contrast to the following theorem, which is main result of this article.

Theorem 1.1. *If $n > 4$ and Z is a twistor space on $n\mathbb{CP}^2$ such that $\dim |K^{-1/2}| = 1$, then $a(Z) \neq 2$.*

At the end of this article we give a detailed explanation of the contradiction between this result and the result of [8].

By a result of [17], if a twistor space Z over $n\mathbb{CP}^2$, $n > 4$, satisfies $\dim |K^{-1/2}| > 1$, then Z is Moishezon. Therefore from Theorem 1.1, we obtain

Proposition 1.2. *If a twistor space Z over $n\mathbb{CP}^2$, $n > 4$, satisfies $a(Z) = 2$, then the system $|K^{-1/2}|$ consists of a single member, or is empty.*

To the best of the authors' knowledge, existence of this kind of twistor spaces is not known. Thus the existence of a twistor space on $n\mathbb{CP}^2$ with algebraic dimension two seems to be not known.

Here is an outline of our proof of Theorem 1.1. Let Z be a twistor space over $n\mathbb{CP}^2$ which satisfies $\dim |K^{-1/2}| = 1$ and assume that the base locus of the pencil $|K^{-1/2}|$ constitutes a cycle of smooth rational curves. This assumption is always satisfied if $n > 4$ and $a(Z) = 2$ (Proposition 3.7). By blowing up this base curve and then taking a simultaneous small resolution of all ordinary double points that appear by the blow-up, we get the diagram

$$(1.1) \quad \begin{array}{ccc} Z_1 & \xrightarrow{\mu_1} & Z \\ f_1 \downarrow & & \\ \mathbb{CP}^1 & & \end{array}$$

Here, μ_1 is the composition of the blow-up and the small resolution, and f_1 is a surjective morphism induced by the pencil $|\mu_1^* K^{-1/2} - E|$, where E is the exceptional divisor of the birational morphism μ_1 . Smooth fibres of f_1 are naturally identified with members of the pencil $|K^{-1/2}|$, and they are rational surfaces.

From this fibration, if S denotes a *generic* member of the pencil, we have the equality $a(Z) = 1 + \kappa^{-1}(S)$, where $\kappa^{-1}(S)$ is the anti-Kodaira dimension [17, Corollary 4.3]. Hence if $a(Z) = 2$, generic members of the pencil satisfy $\kappa^{-1}(S) = 1$. Moreover, for any smooth member S of $|K^{-1/2}|$, we have $a(Z) \leq 1 + \kappa^{-1}(S)$. Hence if $a(Z) = 2$, we have $\kappa^{-1}(S) \geq 1$ for any smooth member S of the pencil. In Proposition 3.4 we show that if some member of the pencil satisfies $\kappa^{-1}(S) = 2$, then $a(Z) = 3$. Hence if $a(Z) = 2$, any smooth member S of the pencil must satisfy $\kappa^{-1}(S) = 1$. We shall show by contradiction that this situation can never happen. This implies that even if *some* member S of the pencil satisfies $\kappa^{-1}(S) = 1$, *generic* members of the pencil necessarily satisfy $\kappa^{-1} = 0$, which then implies that $a(Z) = 1$. The main tool of our analysis is the Zariski decomposition for a divisor on a surface [25].

Finally, a remark about notation. If $|L|$ is a complete linear system on some compact complex manifold and if we write

$$|L| = |L'| + D$$

for some effective divisor D , then D is a fixed component of $|L|$. But this does not imply that the system $|L'|$ is without fixed component.

We would like to express our sincere gratitude to Professor Fujiki for his generous and helpful comments about the anti-Kodaira dimension of general members of the pencil $|K^{-1/2}|$ on the twistor space. These are reflected not only in the explanation in Section

4 but throughout this article. We would also like to thank the referee for forcing us to improve the clarity of this article.

2. ZARISKI DECOMPOSITION OF AN ANTI-CANONICAL DIVISOR ON A RATIONAL SURFACE

In this section we first recall basic properties of the Zariski decomposition of a divisor on a projective surface, and then investigate the Zariski decomposition of an anti-canonical divisor on a non-singular rational surface of anti-Kodaira dimension zero or one.

Let S be a non-singular projective surface, and C an effective divisor on S . Then the *Zariski decomposition* of C is the decomposition

$$C = P + N,$$

where P and N are effective \mathbb{Q} -divisors (i.e. divisors with positive rational coefficients) or the zero divisor, which satisfy

- (i) P is nef, that is $PD \geq 0$ for any curve D on S ,
- (ii) if $N \neq 0$, N is negative definite in the following sense: if $N = \sum \alpha_i E_i$ with distinct, irreducible E_i and $\alpha_i \neq 0$, the total intersection matrix $(E_i E_j)$ is negative definite,
- (iii) if $N \neq 0$, $PE_i = 0$ for all i .

Any effective divisor C admits a unique Zariski decomposition, see [25, 24]. P is called the *nef part* of C . Note that if $C = P + N$ is the Zariski decomposition of C , then for any integer $m > 0$, $mC = mP + mN$ is the Zariski decomposition of mC . One of the most important properties of the Zariski decomposition is that it disposes of sub-divisors of mC which do not contribute to the dimension $h^0(\mathcal{O}_S(mC))$ in the following sense:

Proposition 2.1. ([24, Lemma 2.4]) *Let S, C and $C = P + N$ be as above, and $m > 0$ an integer. Then we have*

$$|mC| = |[mP]| + [mN],$$

where for an effective \mathbb{Q} -divisor D , $\lfloor D \rfloor$ and $\lceil D \rceil$ denote the (integral) round down and the round up of D respectively. In particular, the divisor $\lceil mN \rceil$ is a fixed component of $|mC|$.

Next let S be a non-singular rational surface. For an integer $m > 0$, the linear system $|mK^{-1}|$ on S is called the m -th anti-canonical system. If this is non-empty, the associated rational map $\phi_m : S \rightarrow \mathbb{C}\mathbb{P}^r$, $r = h^0(mK^{-1}) - 1$, is called the m -th anti-canonical map. The anti-Kodaira dimension, $\kappa^{-1}(S)$, of a rational surface S is defined as

$$\kappa^{-1}(S) := \max_{m>0} \dim \phi_m(S) \in \{2, 1, 0, -\infty\}.$$

Here, $\kappa^{-1}(S) = -\infty$ occurs when $|mK^{-1}| = \emptyset$ for all $m > 0$, but we will not encounter this case in the following. Typical examples of rational surfaces with $\kappa^{-1} = 2$ are Del Pezzo surfaces, while simple examples with $\kappa^{-1} = 1$ are obtained from $\mathbb{C}\mathbb{P}^2$ by blowing up the 9 intersection points of two anti-canonical (i.e. cubic) curves.

The anti-Kodaira dimension of rational surfaces was investigated in more detail by F. Sakai [24]. His main tool was the Zariski decomposition of the anti-canonical divisor. Even if $\kappa^{-1}(S) \geq 0$, the anti-canonical divisor might not be effective. However, Sakai [24, Lemma 3.1] has shown that the divisor K_S^{-1} is pseudo effective iff $\kappa^{-1}(S) \geq 0$. A divisor C on a surface S is called pseudo-effective if $CH \geq 0$ for any ample divisor H . Fujita [9] has shown that any pseudo effective divisor C has a unique Zariski decomposition $C = P + N$ with the same properties as above except that P is no longer required to be effective. Therefore,

if $\kappa^{-1}(S) \geq 0$ and $K_S^{-1} = P + N$ is the Zariski decomposition of an anti-canonical divisor, one can define *the degree* of a surface S to be the self-intersection number

$$(2.1) \quad d(S) := P^2,$$

which is clearly a non-negative rational number. Rational surfaces with $\kappa^{-1} = 2$ may be characterised in terms of the degree as follows.

Proposition 2.2. (*Sakai [23, Proposition 1]*) *Let S be a non-singular rational surface with $\kappa^{-1}(S) \geq 0$, and let $d(S)$ be the degree of S as above. Then $\kappa^{-1}(S) = 2$ if and only if $d(S) > 0$.*

In particular we have $d(S) = 0$ if (and only if) $\kappa^{-1}(S) \in \{0, 1\}$. In the rest of this section we focus on rational surfaces with $d(S) = 0$ which admit a special type of anti-canonical divisor. For this purpose, by a *cycle of rational curves*, we mean either a rational curve with one node, or a connected, reduced, normal crossing divisor

$$(2.2) \quad C = C_1 + \cdots + C_k$$

on S , with all C_i being non-singular rational curves, such that the dual graph of C is a circle. For simplicity, in the expression (2.2), we allow k to be one, in which case $C = C_1$ means a rational curve with one node. We always assume that if $k \geq 2$ in (2.2) the components C_i and C_{i+1} intersect and $C_{k+1} = C_1$. By an *anti-canonical cycle* we mean a cycle of rational curves which belongs to the anti-canonical class.

The following well-known lemma will be used frequently.

Lemma 2.3. ([2, p. 28, Lemma]) *Let S be a smooth surface, C_1, \dots, C_k irreducible curves on S for which $\sum C_i$ is a connected curve, $p_i > 0$ rational numbers so that $P = \sum_{i=1}^k p_i C_i$ satisfies $PC_j = 0$ for $j = 1, \dots, k$.*

Then, the intersection matrix $(C_i C_j)_{1 \leq i, j \leq k}$ is negative semi-definite with one-dimensional kernel generated by P . More precisely, this means that, for $D = \sum_{i=1}^k r_i C_i$ with $r_i \in \mathbb{Q}$ we always have $D^2 \leq 0$ and $D^2 = 0$ occurs iff $D = rP$ for some $r \in \mathbb{Q}$.

For the rest of this section, let S be a non-singular rational surface and suppose that S has an anti-canonical cycle C as in (2.2) and let $C = P + N$ be its Zariski decomposition. Let $m_0 > 0$ be the smallest positive integer for which $m_0 P$ is integral, and write

$$(2.3) \quad m_0 P = l_1 C_1 + \cdots + l_k C_k, \quad (l_i \in \mathbb{Z}_{\geq 0}).$$

Lemma 2.4. (i) *If $P = 0$ then $\kappa^{-1}(S) = 0$.*

(ii) *$P^2 = 0$ if and only if $PC_i = 0$ for $1 \leq i \leq k$.*

Proof. (i) If $P = 0$, the nef part of the Zariski decomposition for mC is 0 for any $m > 0$. By Proposition 2.1 this means that $h^0(mK^{-1}) = 1$ for all $m > 0$, hence $\kappa^{-1}(S) = 0$.

(ii) From (2.3) it is clear that $PC_i = 0$ for $1 \leq i \leq k$ implies $P^2 = 0$.

For the converse, if an index i satisfies $l_i < m_0$ in (2.3), then N includes C_i , and so $PC_i = 0$ from the property (iii) of the Zariski decomposition. Therefore we have

$$(2.4) \quad (m_0 P)^2 = m_0 P \left(\sum_{1 \leq i \leq k} l_i C_i \right) = m_0 \left(\sum_{1 \leq i \leq k} l_i PC_i \right) = m_0^2 \sum_{l_i = m_0} PC_i.$$

Further, as P is nef, we have $PC_i \geq 0$ for any i . Because we assume $P^2 = 0$, (2.4) implies that $PC_i = 0$ even when $l_i = m_0$. \square

By $\text{Pic}^0(C)$ we denote the group of line bundles on the anti-canonical cycle C that are trivial on each component C_i . Thus, part (ii) of Lemma 2.4 says that $P^2 = 0$ is equivalent to $m_0P|_C \in \text{Pic}^0(C)$.

Lemma 2.5. *Let $\pi : S \rightarrow \bar{S}$ be the blow-down of a (-1) -curve on S , and let $\bar{C} := \pi(C)$ be the image of the anti-canonical cycle C , and $x \in \bar{S}$ the image of the (-1) -curve. Define $m = \text{mult}_x \bar{C}$. Then,*

- (i) \bar{C} is an anti-canonical cycle on \bar{S} and $m \in \{1, 2\}$;
- (ii) if $\bar{C} = \bar{N} + \bar{P}$ is the Zariski decomposition of \bar{C} and $m = 2$, then $P = 0$ iff $\bar{P} = 0$.

Proof. (i) Let $E \subset S$ be the exceptional curve of the blow-up π . As $EC = EK^{-1} = 1$, either E is a component of C or it intersects C transversally at one point. Let C' be the strict transform of $\bar{C} = \pi(C)$, then $C' = C$ if E is not a component of C , or $C' = C - E$ if E is a component of C . In both cases, \bar{C} clearly is a cycle of rational curves. Because $m = C'E$ and $EC = 1$, we obtain $m = 1$ if E is not a component of C , and $m = 2$ if E is a component of C . In particular, $C = C' + (m-1)E$ in both cases. Because $\pi^*\bar{C} = C' + mE$ and $K^{-1} = \pi^*K_{\bar{S}}^{-1} - E$, we now obtain $\bar{C} \in |K_{\bar{S}}^{-1}|$.

(ii) As $m = 2$, the exceptional curve E is a component of C . We fix notation so that $C_k = E$ and $\bar{C} = \bar{C}_1 + \dots + \bar{C}_{k-1}$, where $\bar{C}_i = \pi(C_i)$. Consider the intersection matrix $\bar{M} = (\bar{C}_i \bar{C}_j)_{1 \leq i, j \leq k-1}$. The intersection matrix $M = (C_i C_j)_{1 \leq i, j \leq k}$ is obtained from \bar{M} by adding an extra row and a column with $C_1 C_k = C_{k-1} C_k = 1$ and $C_k^2 = -1$ being their only non-zero entries. In addition, four entries in \bar{M} have to be changed to get M , namely $C_1^2 = \bar{C}_1^2 - 1$, $C_{k-1}^2 = \bar{C}_{k-1}^2 - 1$ and $C_1 C_{k-1} = 0$, whereas $\bar{C}_1 \bar{C}_{k-1} = 1$.

By adding the k -th column of M to columns 1 and $k-1$, the values of the entries of the part of M that corresponds to \bar{M} are restored to the original values they had in \bar{M} . It now follows easily from Sylvester's criterion that the matrix M is negative definite if and only if \bar{M} is so. Because $\bar{P} = 0$ is equivalent to \bar{M} being negative definite and $P = 0$ if and only if M is negative definite, the claim now follows. \square

We will frequently need the following more detailed properties of the Zariski decomposition $C = P + N$ of the anti-canonical cycle C on the surface S .

Proposition 2.6. *Suppose that $P \neq 0$ and $P^2 = 0$.*

- (i) We have $l_i > 0$ for all i .
- (ii) If $\pi : \tilde{S} \rightarrow S$ is the blow-up of a point $x \in C$ and $m = \text{mult}_x C$, then

$$\kappa^{-1}(\tilde{S}) = \kappa^{-1}(S) \quad \text{if } m = 2$$

$$\kappa^{-1}(\tilde{S}) = 0 \quad \text{if } m = 1.$$

- (iii) We have $l_i = 1$ for some index i .
- (iv) If $K^2 < 0$, then $m_0 > 1$, $k \geq 2$, and we have $l_i \neq l_j$ for some indices i and j .

Proof. For (i), if $l_i = 0$ for some i , we have $m_0 P C_i = l_{i-1} + l_{i+1}$ since C is a cycle. But as $P^2 = 0$, Lemma 2.4 (ii) shows that $P C_i = 0$ for all i , hence $l_{i-1} = l_{i+1} = 0$. Repeating this argument, we obtain $l_i = 0$ for any i , but this contradicts $P \neq 0$.

To prove (ii) we recall from [17, Lemma 5.1] that $\kappa^{-1}(\tilde{S}) \leq \kappa^{-1}(S) = 1$ with equality if $m \geq 2$. Because C is a cycle of rational curves, the points on C have multiplicity 1 or 2. We assume now $m = 1$ and show $\kappa^{-1}(\tilde{S}) = 0$. From (i), Lemma 2.4 (ii) and Lemma 2.3, for any divisor $D = \sum_{i=1}^k n_i C_i$, we have $D^2 \leq 0$ with equality only if $D = rP$ for some $r \in \mathbb{Q}$.

Let \tilde{C}_i be the strict transform of C_i and $\tilde{C} = \sum_{i=1}^k \tilde{C}_i$. Because we assumed $m = 1$, \tilde{C} is an anti-canonical cycle on \tilde{S} . We now show that the intersection matrix of the components of \tilde{C} is negative definite. To fix notation, we assume $x \in C_1$, so that $\tilde{C}_1^2 = C_1^2 - 1$. Consider $\tilde{D} = \sum_{i=1}^k n_i \tilde{C}_i$ and let $D = \sum_{i=1}^k n_i C_i$, then $\tilde{D}^2 = D^2 - n_1^2 \leq D^2 \leq 0$. If $\tilde{D}^2 = 0$ we then have $D^2 = 0$ and $n_1 = 0$, hence $D = rP = \sum_{i=1}^k r l_i C_i$ for some $r \in \mathbb{Q}$ and so $r l_1 = n_1 = 0$. As $l_1 \neq 0$ this implies $r = 0$, hence $D = 0$, which gives $\tilde{D} = 0$. This shows that the intersection matrix $(\tilde{C}_i \tilde{C}_j)_{1 \leq i, j \leq k}$ is negative definite. This implies that $\tilde{P} = 0$ in the Zariski decomposition $\tilde{C} = \tilde{P} + \tilde{N}$ of the anti-canonical divisor \tilde{C} , hence $\kappa^{-1}(\tilde{S}) = 0$ by Lemma 2.4 (i).

To prove (iii), we recall from [24, Theorem 3.4] that there is a birational morphism $\varphi : S \rightarrow S_0$ with S_0 being non-singular, such that $\kappa^{-1}(S_0) = \kappa^{-1}(S)$ and any anti-canonical divisor $C_0 \in |K_{S_0}^{-1}|$ is nef, that is $N_0 = 0$ in the Zariski decomposition $C_0 = P_0 + N_0$ of C_0 . Because $P^2 = 0$ and $\kappa^{-1}(S_0) = \kappa^{-1}(S)$, Proposition 2.2 implies that $K_{S_0}^2 = P_0^2 = 0$ as well.

The morphism φ is a composition of blow-ups. By Lemma 2.5 (i) the image of C in each of these partial blow-ups is an anti-canonical cycle containing the blown-up point. Moreover, as we have seen in the proof of (ii) above, if the blown-up point had multiplicity one on the anti-canonical cycle, the nef part of the Zariski decomposition would vanish after the blow-up. Because of Lemma 2.5 (ii), this would lead to $P = 0$ on S , in contradiction to our assumption. Therefore, it follows that at each step a double point of the anti-canonical cycle is blown-up and the nef part of the Zariski decomposition does not vanish.

Because the anti-Kodaira dimension does not increase under blow-up, [17, Lemma 5.1], and $\kappa^{-1}(S_0) = \kappa^{-1}(S)$, all partial blow-ups in this process have $\kappa^{-1} = \kappa^{-1}(S)$, in particular the self-intersection number of the nef part of the Zariski decomposition is equal to zero.

From the above we infer that $C_0 := \varphi(C)$ is an anti-canonical cycle on S_0 which coincides with the nef part, P_0 , of its Zariski decomposition. Thus, $m_0 = 1$ and all $l_i = 1$ on S_0 , and so the assertion (iii) holds for the pair (S_0, C_0) . To finish the proof of (iii) by induction, it remains to show that, if (S, C) is a pair consisting of a rational surface and an anti-canonical cycle on it that satisfies the assumptions of the current proposition and has $l_i = 1$ for one index i , and if $\pi : \tilde{S} \rightarrow S$ is the blow-up at a double point of C , then the nef part of the new anti-canonical cycle \tilde{C} on \tilde{S} also satisfies (iii).

To show this, we continue to use the notation introduced above, like (2.2), for S, C and P . To fix notation, we suppose that the point $C_k \cap C_1$ is blown up by π . Let \tilde{C}_{k+1} be the exceptional divisor of π and write \tilde{C}_i for the strict transform of the component C_i ($1 \leq i \leq k$). We recall that $l_i > 0$ ($1 \leq i \leq k$), define $l_{k+1} = l_1 + l_k$ and consider the divisor \tilde{D} on \tilde{S}

$$(2.5) \quad \tilde{D} = \sum_{i=1}^{k+1} l_i \tilde{C}_i.$$

It is easy to see that $\tilde{D} \tilde{C}_i = 0$ for $i = 1, \dots, k+1$, and therefore \tilde{D} is nef and $\tilde{D}^2 = 0$. Hence, again by Lemma 2.3, the intersection matrix $(\tilde{C}_i \tilde{C}_j)_{1 \leq i, j \leq k+1}$ is negative semi-definite, and if \tilde{P} is the nef part of $\tilde{C} = \sum \tilde{C}_i$ and \tilde{m}_0 is the smallest positive integer for which $\tilde{m}_0 \tilde{P}$ is integral, we have $\tilde{m}_0 \tilde{P} = r \tilde{D}$ for some $r \in \mathbb{Q}_{>0}$. Since $l_i = 1$ for some index i ($i \leq k$) by the inductive assumption, it follows that $r \in \mathbb{Z}_{>0}$.

If $r \neq 1$, all the coefficients of $\tilde{m}_0 \tilde{P}$ would be divisible by r . If all \tilde{C}_i appeared in the negative part, \tilde{N} , of the Zariski decomposition of \tilde{C} , condition (ii) in the definition of the

Zariski decomposition would imply $\tilde{P}^2 < 0$. Thus, $\tilde{P}^2 = 0$ implies that at least one of the \tilde{C}_i is missing in \tilde{N} . Hence, at least one of the coefficients of $\tilde{m}_0\tilde{P}$ is equal to \tilde{m}_0 . But then r would divide \tilde{m}_0 and all coefficients of $\tilde{m}_0\tilde{P}$, which contradicts the choice of \tilde{m}_0 . This shows that $r = 1$, and we obtain $\tilde{m}_0\tilde{P} = \tilde{D}$. Thus from the expression (2.5) we now see that the cycle \tilde{C} satisfies the property (iii).

The final item (iv) follows now easily, because $K^2 < 0$ implies that the morphism φ in the proof of (iii) is not an isomorphism and so there was at least one blow-up carried out. The expression (2.5) for m_0P shows then that at least one of the l_i is greater than 1. Also, note that m_0 is equal to the largest coefficient in m_0P . \square

Lemma 2.7. *Suppose $P \neq 0$, $P^2 = 0$ and that there exists an integer $\nu > 0$ for which $h^0(\nu m_0P) > 1$, then $m_0P|_C \in \text{Pic}^0(C)$ has finite order. Moreover, if r is the smallest positive integer for which $h^0(rm_0P) > 1$, then $rm_0P|_C \simeq \mathcal{O}_C$.*

Proof. Let r be the smallest positive integer for which $h^0(rm_0P) > 1$ and let $s \in H^0(rm_0P)$ be a non-zero element that satisfies $(s) \neq rm_0P$. Put $D := (s) \in |rm_0P|$ and write

$$D = D' + \sum_{1 \leq i \leq k} a_i C_i,$$

where D' (which may be 0 at this moment) does not include C_i for any i . As $D \neq rm_0P = \sum r l_i C_i$, we have $a_i \neq r l_i$ for at least one i . Since $D \in |rm_0P|$, we have a linear equivalence

$$D' + \sum_{1 \leq i \leq k} a_i C_i \sim \sum_{1 \leq i \leq k} r l_i C_i.$$

Collecting all indices that satisfy $a_i > r l_i$ (if any) on the left-hand side, we obtain

$$(2.6) \quad D' + \sum_{a_i > r l_i} (a_i - r l_i) C_i \sim \sum_{a_j \leq r l_j} (r l_j - a_j) C_j.$$

If RHS is the zero-divisor, then so has to be LHS, which contradicts $a_i \neq r l_i$ for some i . So both sides of (2.6) are effective divisors. Because LHS and RHS do not have a common irreducible component, we obtain that the self-intersection number of RHS is non-negative. Since $P^2 = 0$, by Lemma 2.4 (ii) we have $PC_i = 0$ for any i , $1 \leq i \leq k$. Because of Proposition 2.6 (i) we can apply Lemma 2.3 to obtain that the total intersection matrix $(C_i C_j)_{1 \leq i, j \leq k}$ is negative semi-definite. Hence the self-intersection number of RHS has to be zero. By Lemma 2.3, this can happen only when

$$\sum_{a_j \leq r l_j} (r l_j - a_j) C_j = r' m_0 P,$$

for some $r' \in \mathbb{Q}_{>0}$. Because $l_i > 0$ for all i by Proposition 2.6 (i), we can conclude that $r l_i - a_i > 0$ for all i . Therefore from (2.6) we obtain a linear equivalence

$$(2.7) \quad D' \sim r' m_0 P, \quad r' \in \mathbb{Q}_{>0}.$$

On the other hand, we have $l_i = 1$ for some index i by Proposition 2.6 (iii). Because D' is integral, (2.7) implies now that $r' \in \mathbb{Z}$. Hence by the minimality of r , and as $D' \leq D$, we have $r' = r$.

Because we have chosen D' not to have any C_i as a component and because $D' C_i = r' m_0 P C_i = 0$, the divisor D' does not intersect any C_i . As $D' \sim r m_0 P$, this means that $rm_0P|_C \simeq \mathcal{O}_C$. Hence $m_0P|_C$ is of finite order in $\text{Pic}^0(C)$. \square

We will use this result to show the following property regarding pluri-anti-canonical systems on S , which will be used in the next section.

Proposition 2.8. *If $P \neq 0$, $P^2 = 0$ and $m_0P|_C \in \text{Pic}^0(C)$ is of finite order τ , then*

- (i) $|\tau m_0K^{-1}| = |\tau m_0P| + \tau m_0N$, $\text{Bs } |\tau m_0P| = \emptyset$, $\dim |\tau m_0P| = 1$, and the associated morphism $S \rightarrow \mathbb{C}\mathbb{P}^1$ is an elliptic fibration;
- (ii) $\kappa^{-1}(S) = 1$;
- (iii) for any $\nu > 0$, the system $|\nu \tau m_0P|$ is composed with the pencil $|\tau m_0P|$, i.e. each element of $|\nu \tau m_0P|$ is a sum of elements of $|\tau m_0P|$;
- (iv) for any integer $\nu > 0$, we have

$$|\nu \tau m_0P - C| = |(\nu - 1)\tau m_0P| + (\tau m_0P - C).$$

Proof. Because C is an anti-canonical divisor with Zariski decomposition $C = P + N$, we obtain $|\tau m_0K^{-1}| = |\tau m_0P| + \tau m_0N$. By Lemma 2.7 we have

$$(2.8) \quad h^0(\nu m_0P) = 1, \quad \text{if } 0 < \nu < \tau.$$

Note that $\tau m_0P - C$ is an effective divisor by Proposition 2.6 (i). Using Serre-duality this implies $H^2(\tau m_0P - C) = 0$. Using $P^2 = 0$, the definition of the Zariski decomposition implies that

$$K^2 = C^2 = P^2 + 2PN + N^2 = N^2 \leq 0$$

with equality iff $N = 0$. If $(\tau m_0P - C)^2 = C^2 = K^2 < 0$, the non-empty system $|\tau m_0P - C|$ has a (non-zero) fixed component. Indeed, if L is a line bundle on a smooth surface so that $|L|$ is not empty and $L^2 < 0$, then $|L|$ must have a fixed component, as can be seen as follows. Let $Y = \sum d_i Y_i \in |L|$ with $d_i > 0$ and prime divisors Y_i . Then there exists k such that $LY_k < 0$, because otherwise we would have $L^2 = L \sum d_i Y_i = \sum d_i LY_i \geq 0$. If now Y' is any element of $|L|$, then $Y'Y_k = LY_k < 0$, hence Y_k is a component of Y' , i.e. a fixed component of $|L|$.

If $K^2 < 0$ we let D be the (maximal) fixed component of $|\tau m_0P - C|$. By Lemma 2.4 (ii), Proposition 2.6 (i) and Lemma 2.3, the intersection matrix formed by the components of the cycle C is negative semi-definite, hence $(\tau m_0P - C - D)^2 \leq 0$. But if this was negative, $|\tau m_0P - C - D|$ would still have a fixed component, in contradiction to the choice of D . Hence $(\tau m_0P - C - D)^2 = 0$. If $K^2 = 0$, we simply take $D = 0$ to obtain $(\tau m_0P - C - D)^2 = 0$. Therefore, by Lemma 2.3 again, we have

$$(2.9) \quad \tau m_0P - C - D = sm_0P, \quad s \in \mathbb{Q}_{>0}.$$

Moreover, s is an integer since m_0P has a component of multiplicity one, by Proposition 2.6 (iii), and the left-hand side is an integral divisor. From (2.9) we have $s < \tau$, and so $h^0(sm_0P) = 1$ by (2.8). Therefore,

$$h^0(\tau m_0P - C) = h^0(\tau m_0P - C - D) = h^0(sm_0P) = 1.$$

By Riemann-Roch we readily obtain $\chi(\tau m_0P - C) = 1$. Because $H^2(\tau m_0P - C) = 0$ and $h^0(\tau m_0P - C) = 1$, we must have $H^1(\tau m_0P - C) = 0$. Since $h^0(\tau m_0P|_C) = h^0(\mathcal{O}_C) = 1$, the standard exact sequence

$$0 \longrightarrow \tau m_0P - C \longrightarrow \tau m_0P \longrightarrow \tau m_0P|_C \longrightarrow 0$$

implies now that $h^0(\tau m_0P) = 2$. In addition, this sequence provides us with a surjection $H^0(\tau m_0P) \rightarrow H^0(\tau m_0P|_C) \simeq H^0(\mathcal{O}_C)$, from which we obtain $\text{Bs } |\tau m_0P| = \emptyset$.

The general fibre of the morphism $\phi : S \rightarrow \mathbb{CP}^1$ associated to the pencil $|\tau m_0 P|$ is non-singular, because S is smooth. The smooth (and hence all) fibres of ϕ are connected. To see this, let F be a smooth fibre of ϕ , then $h^0(F) = h^0(\tau m_0 P) = 2$ and $\mathcal{O}_S(F) \cong \phi^* \mathcal{O}(1)$, hence $\mathcal{O}_F(F) \cong \phi^* \mathcal{O}(1)|_F \cong \mathcal{O}_F$. Because $h^1(\mathcal{O}_S) = 0$, the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow F \longrightarrow \mathcal{O}_F(F) \longrightarrow 0$$

implies $2 = h^0(F) = h^0(\mathcal{O}_S) + h^0(\mathcal{O}_F) = 1 + h^0(\mathcal{O}_F)$. Therefore, $h^0(\mathcal{O}_F) = 1$, i.e. F is connected. Then, since $P^2 = PK^{-1} = 0$, the genus formula implies that the general fibre of ϕ is an elliptic curve. This completes the proof of (i).

To prove (ii) we just have to observe that (i) immediately implies $\kappa^{-1}(S) \geq 1$. The assumption $P^2 = 0$, on the other hand, gives $\kappa^{-1}(S) < 2$, and so $\kappa^{-1}(S) = 1$.

To prove assertion (iii), first observe that $|\nu \tau m_0 P| = |\phi^* \mathcal{O}(\nu)|$. As ϕ has connected fibres, we have $\phi_* \mathcal{O}_S \cong \mathcal{O}_{\mathbb{P}^1}$ and the projection formula implies $\phi_* \phi^* \mathcal{O}(\nu) \cong \mathcal{O}(\nu)$. Hence, $H^0(\nu \tau m_0 P) \cong H^0(\phi^* \mathcal{O}(\nu)) \cong H^0(\mathcal{O}(\nu))$, which shows that each element of $|\nu \tau m_0 P|$ is a sum of fibres, i.e. a sum of elements of $|\tau m_0 P|$.

Finally, for (iv), let $\phi : S \rightarrow \mathbb{CP}^1$ be as above and let $t_1 \in \mathbb{CP}^1$ be the point for which $\tau m_0 P = \phi^{-1}(t_1)$. Let $D \in |\nu \tau m_0 P - C|$ be any member. Then since $D + C \in |\nu \tau m_0 P|$ and each member of this linear system is a sum of members of $|\tau m_0 P|$, i.e. fibres of ϕ , and because $C \subset \phi^{-1}(t_1)$ there exist points t_2, t_3, \dots, t_ν such that

$$D + C = \sum_{1 \leq i \leq \nu} \phi^{-1}(t_i) \quad \text{and so} \quad D = \sum_{2 \leq i \leq \nu} \phi^{-1}(t_i) + (\phi^{-1}(t_1) - C).$$

This implies assertion (iv), because $h^0(\tau m_0 P - C) = 1$, as we have shown above. \square

Corollary 2.9. *Suppose that $P \neq 0$ and $P^2 = 0$. Then*

$$\kappa^{-1}(S) = 1 \quad \iff \quad m_0 P|_C \text{ has finite order in } \text{Pic}^0(C).$$

Proof. If $\kappa^{-1}(S) = 1$ there exists an integer $\nu > 0$ for which $h^0(\nu m_0 P) = h^0(\nu m_0 K^{-1}) > 1$, hence $m_0 P|_C$ has finite order by Lemma 2.7. The converse is Proposition 2.8 (ii). \square

3. TWISTOR SPACES WHOSE FUNDAMENTAL SYSTEM IS A PENCIL

Let Z be a twistor space on $n\mathbb{CP}^2$, and F be the natural square root of the anti-canonical bundle over Z , which is known to exist on any twistor space [10]. Following Poon [22], we call F the *fundamental line bundle*, and the associated linear system $|F|$ the *fundamental system*. Basic properties of the fundamental line bundle are

$$FL = 2, \text{ and } \sigma^* F \simeq \overline{F},$$

where L is a fibre of the twistor projection $Z \rightarrow n\mathbb{CP}^2$ which is called a *real twistor line*, and $\sigma : Z \rightarrow Z$ is a natural anti-holomorphic involution called the *real structure*.

By the works of Poon [20, 22], Kreussler and Kurke [15], when $n \leq 3$, for any twistor space Z over $n\mathbb{CP}^2$ whose self-dual metric is of positive scalar curvature, we have $\dim |F| \geq 3$, and the structure of these twistor spaces is well understood through the rational map associated to the fundamental system $|F|$. In particular, all of these twistor spaces are Moishezon.

Let now Z be a twistor space on $n\mathbb{CP}^2$, $n \geq 4$, and assume that $\dim |F| \geq 2$. In this situation $|F|$ needs to contain an irreducible divisor, because otherwise there would exist two pencils, $|D|$ and $|\overline{D}|$, of divisors of degree one that define a surjective rational map to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and for which $D + \overline{D}$ is a fundamental divisor. But then the image of $|F|$ would

be at least two-dimensional and Bertini's Theorem implies that not all elements in $|F|$ could be reducible. This allows us to apply the results of [17].

First, by [17, Theorem 3.6], if $\dim |F| \geq 3$, we always have the equality $\dim |F| = 3$, and Z has to be a so-called LeBrun twistor space [18], the structure of which is also well-understood. In particular, such twistor spaces are Moishezon. Second, by [17, Theorem 3.7], if $\dim |F| = 2$ and $n \geq 5$, then Z has to be one of the twistor spaces investigated in [5], and they are again Moishezon. Third, if $\dim |F| = 2$ and $n = 4$, then Z is either one of the Moishezon twistor spaces studied in [5], or a twistor space that satisfies $a(Z) = 2$. The former happens exactly when $\text{Bs}|F| \neq \emptyset$ ([13, Proposition 2.4]), and if the latter is the case, the morphism $Z \rightarrow \mathbb{C}\mathbb{P}^2$ associated to the net $|F|$ is an elliptic fibration which is an algebraic reduction of Z . Thus the basic structure of Z is also well-understood if $\dim |F| = 2$.

For the rest of this paper we let Z be a twistor space over $n\mathbb{C}\mathbb{P}^2$ ($n \geq 4$) and suppose $\dim |F| = 1$. Then general members of the fundamental system $|F|$ are irreducible, since otherwise we readily have $\dim |F| \geq 3$. This implies that the self-dual metric on $n\mathbb{C}\mathbb{P}^2$ has positive scalar curvature, see [17, Proposition 2.4].

Let $S \in |F|$ be a smooth fundamental divisor and recall that $H^1(\mathcal{O}_Z) = 0$ because Z is simply connected. As we assume $h^0(F) = 2$, the standard exact sequence

$$(3.1) \quad 0 \longrightarrow \mathcal{O}_Z \longrightarrow F \longrightarrow K_S^{-1} \longrightarrow 0,$$

implies $h^0(K_S^{-1}) = 1$. This means that the anti-canonical system $|K_S^{-1}|$ consists of a single member, say C . In particular, we have $\kappa^{-1}(S) \geq 0$. From the surjectivity of the restriction map $H^0(F) \rightarrow H^0(K_S^{-1})$ we have

$$(3.2) \quad \text{Bs}|F| = C.$$

By a theorem of Pedersen and Poon [19], any real irreducible member $S \in |F|$ is non-singular and obtained from $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ by blowing up $2n$ points. On such a surface, the anti-canonical curve C is real (i.e. $\sigma(C) = C$) since S and so K_S^{-1} are real. Moreover we have the following result on the structure of the base curve C .

Proposition 3.1. *Let $Z \rightarrow n\mathbb{C}\mathbb{P}^2$, $n \geq 4$, be a twistor space satisfying $\dim |F| = 1$, and let the curve C be the base locus of the pencil $|F|$, as above. If C is non-singular, it is an elliptic curve. If C is singular, it is a cycle of rational curves which is of the form*

$$(3.3) \quad C = C_1 + \cdots + C_k + \overline{C}_1 + \cdots + \overline{C}_k$$

for some $k \geq 1$, where \overline{C}_i means $\sigma(C_i)$, and in the presentation (3.3) two components intersect iff they are adjacent, or they are C_1 and \overline{C}_k .

Proof. Recall that $C \subset S$ for each smooth real $S \in |F|$. If C is non-singular, the adjunction formula immediately implies that C is an elliptic curve. If C is singular it is a cycle of rational curves, by [16, Proposition 3.6]. If C arises from item (I) or (III) in [16, Proposition 3.6], it consists of conjugate pairs of rational curves, as required in (3.3). If C arose from item (II) in [16, Proposition 3.6] it would have exactly two irreducible components, one of them a real twistor line. Because a real twistor line in S generates a pencil, we would readily have $h^0(K_S^{-1}) = 2$, in contradiction to our observations just after the exact sequence (3.1). Therefore C is a cycle of rational curves as described in (3.3). \square

In the sequel, for simplicity of notation, we often write C_{k+1} for \overline{C}_1 and \overline{C}_{k+1} for C_1 for components of the cycle (3.3). This cycle will be significant throughout our proof of the main result.

Another important property of twistor spaces satisfying $\dim |F| = 1$, which we next explain, concerns reducible members of the pencil $|F|$. The cycle (3.3) can be split into connected halves in exactly k ways. For example, if $k = 3$, the possibilities are:

$$\begin{aligned} (C_1 + C_2 + C_3) + (\overline{C}_1 + \overline{C}_2 + \overline{C}_3), \\ (C_2 + C_3 + \overline{C}_1) + (\overline{C}_2 + \overline{C}_3 + C_1), \\ (C_3 + \overline{C}_1 + \overline{C}_2) + (\overline{C}_3 + C_1 + C_2). \end{aligned}$$

The following proposition, which was proved in [16], implies that if the base curve is singular, these subdivisions are nicely realised by reducible members of the pencil $|F|$.

Proposition 3.2. ([16, Proposition 3.7]) *If the base curve of the pencil $|F|$ is a cycle of rational curves as in (3.3), then $|F|$ has exactly k reducible members. Moreover each of them is real and of the form $S_i^+ + S_i^-$ ($1 \leq i \leq k$), where S_i^+ and S_i^- are non-singular irreducible divisors satisfying $\overline{S}_i^+ = S_i^-$. Furthermore, the divisor $S_i^+ + S_i^-$ splits the cycle C into halves in the following manner:*

- if L_i denotes the real twistor line joining the two points $C_i \cap C_{i+1}$ and $\overline{C}_i \cap \overline{C}_{i+1}$, then $S_i^+ \cap S_i^- = L_i$,
- the intersections $S_i^+ \cap C$ and $S_i^- \cap C$ are connected.

Note that all the reducible fundamental divisors $S_i^+ + S_i^-$ are singular along the twistor line $L_i = S_i^+ \cap S_i^-$. Therefore, all smooth fundamental divisors are automatically irreducible. We will also need the following property of the Zariski decomposition of the cycle C .

Proposition 3.3. *In the situation of Proposition 3.1, let S be a smooth member of the pencil $|F|$. Then, the degree of S (see (2.1)) and the Zariski decomposition of the cycle $C \subset S$ are independent of the choice of S .*

Proof. Let $S \in |F|$ be any smooth member, which is irreducible as we have seen above, and let $C \subset S$ the base locus of the pencil $|F|$. For the self-intersection numbers in S of components of the cycle C we have

$$C_i^2 = -2 + K_S^{-1}C_i = -2 + FC_i,$$

hence these self-intersection numbers in S are independent of the choice of the smooth member S . Obviously the intersection numbers between different components of C are independent of the choice of S as well.

Let $C = P + N$ be the Zariski decomposition of the cycle C regarded as a curve in S . Then P is nef as a divisor in S . In particular $PC_i \geq 0$ for any index i . As the intersection number PC_i is determined by the coefficients of P and the intersection numbers $C_j C_k$, it is independent of the choice of S and it follows that P is nef also in any other smooth member of $|F|$. Similarly, N is negative definite not only in S but also in any other smooth member of $|F|$. By the same reason, we have $PC_i = 0$ for each C_i which is included in N , not only in S but also in any other smooth member of $|F|$. Thus all the properties (i), (ii) and (ii) that characterise the Zariski decomposition are satisfied for $P + N$ in all smooth members of the pencil $|F|$. Therefore, the Zariski decomposition of the cycle C is independent of the choice of a smooth member $S \in |F|$. This implies that the self-intersection number $P^2 = d(S)$ is also independent of the choice of S . Hence we obtain the proposition. \square

Note that, even though the Zariski decomposition $C = P + N$ is independent of S , it is possible that the divisor $m_0 P|_C$ in $\text{Pic}(C)$ does depend on S . The reason is that the

restriction $m_0P|_C$ actually is $\mathcal{O}_S(m_0P) \otimes \mathcal{O}_C$ and like $\mathcal{O}_S(C) \otimes \mathcal{O}_C$, the normal bundle of C in S , this may depend on S . The independence of the Zariski decomposition just means that the rational coefficients of P and N do not depend on S .

Since the works of Poon [21] and Campana [4], it has long been well realised that the algebraic dimension $a(Z)$ of a twistor space Z on $n\mathbb{C}\mathbb{P}^2$ is closely related to the anti-Kodaira dimension of a smooth member of the system $|F|$. Actually, if S is a smooth member of $|F|$, the standard exact sequence

$$(3.4) \quad 0 \longrightarrow (m-1)F \longrightarrow mF \longrightarrow mK_S^{-1} \longrightarrow 0, \quad m > 0$$

and the equality $a(Z) = \kappa(Z, F)$ imply the inequality

$$(3.5) \quad a(Z) \leq 1 + \kappa^{-1}(S).$$

Does equality hold in (3.5) when Z satisfies $\dim |F| = 1$? For instance, if at least one smooth member S of the pencil $|F|$ satisfies $\kappa^{-1}(S) = 0$, we clearly have the equality $a(Z) = 1$. The following proposition shows that the same is true if some $S \in |F|$ satisfies $\kappa^{-1}(S) = 2$.

Proposition 3.4. *If at least one smooth member S of the pencil $|F|$ satisfies $\kappa^{-1}(S) = 2$, then equality holds in (3.5), and the twistor space Z is Moishezon.*

Proof. Let $S \in |F|$ be smooth with $\kappa^{-1}(S) = 2$. If the base curve C of the pencil $|F|$ is non-singular, we readily obtain $\kappa^{-1}(S) \leq 1$ from $C^2 = K_S^2 = F^3 = 8 - 2n \leq 0$. Hence C is singular, and by Proposition 3.1, C is a cycle of rational curves on S as in (3.3). Let $C = P + N$ be the Zariski decomposition of C on S . For the degree of S we then have $d(S) > 0$ by Proposition 2.2. Since the degree is independent of the choice of a smooth member S of $|F|$ by Proposition 3.3, we obtain that $d(S') > 0$ for any smooth member $S' \in |F|$. By Proposition 2.2, this implies $\kappa^{-1}(S') = 2$ for any smooth $S' \in |F|$.

Once this is obtained, the equality $a(Z) = 3$ follows from a general estimate of the algebraic dimension from below in the case of a fibre space. We refer to [8, Proposition 4.1] and [17, Corollary 4.3] for details. \square

In order to determine the algebraic dimension of Z if $\dim |F| = 1$, we are left with the case where the pencil $|F|$ contains a smooth member S satisfying $\kappa^{-1}(S) = 1$. This is well understood for the case $n = 4$, which we next explain in order to clarify the difference to the case $n > 4$.

Let Z be a twistor space on $4\mathbb{C}\mathbb{P}^2$ satisfying $\dim |F| = 1$ and let C be the base locus of $|F|$. Suppose that Z has a smooth fundamental divisor of anti-Kodaira dimension one. Using (3.5), this implies $a(Z) \leq 2$. As we assume $n = 4$, we can apply [16, Theorem 6.2] to conclude that F is nef. Therefore, $K_S^{-1} \simeq F|_S$ is nef for each smooth $S \in |F|$, i.e. $N = 0$ in the Zariski decomposition of C on S . As $K_S^2 = 0$, Corollary 2.9 shows now that $\kappa^{-1}(S) = 1$ if and only if the line bundle $K_S^{-1}|_C$ is of finite order in $\text{Pic}^0(C)$. Because

$$(3.6) \quad K_S^{-1}|_C \simeq (F|_S)|_C \simeq F|_C,$$

this order does not depend on S . Hence the anti-Kodaira dimension of smooth members of $|F|$ is constant. From this, exactly like at the end of the proof of Proposition 3.4, we obtain the following result.

Proposition 3.5. *Let Z be a twistor space over $4\mathbb{C}\mathbb{P}^2$ satisfying $\dim |F| = 1$. Suppose that there exists a smooth member S of the pencil $|F|$ satisfying $\kappa^{-1}(S) = 1$. Then $a(Z) = 2$.*

We should mention that a much more concrete result was obtained in [6, Theorem 3.4] without using the general estimate on the algebraic dimension of a fibred space. Namely, if τ denotes the order of the line bundle $K_S^{-1}|_C$ in $\text{Pic}^0(C)$, then $\dim |\tau F| = \tau + 1$, $\text{Bs } |\tau F| = \emptyset$, and the associated morphism $Z \rightarrow \mathbb{C}\mathbb{P}^{\tau+1}$ provides an algebraic reduction of Z which is an elliptic fibration. Strictly speaking, the paper [6] assumes that the base curve C is smooth, but the proof equally works even if it is a cycle of rational curves because $K_S^{-1}|_C$ is trivial on each component of C in the situation considered here. Although we are assuming $\dim |F| = 1$ here, the case $\tau = 1$ actually happens, and then $\dim |F| = 2$, see [11, 12].

Now we are ready to state the main result of this article. It says that, in contrast to the case $n = 4$, when $n > 4$ and $\kappa^{-1}(S) = 1$ for some $S \in |F|$, equality in (3.5) never holds.

Theorem 3.6. *Let $n > 4$ and Z be a twistor space on $n\mathbb{C}\mathbb{P}^2$ satisfying $\dim |F| = 1$. Suppose there exists a smooth member S of the pencil $|F|$ such that $\kappa^{-1}(S) = 1$, then $a(Z) = 1$.*

If such a member $S \in |F|$ exists, inequality (3.5) implies $a(Z) \leq 2$. On the other hand we have $a(Z) \geq 1$ from the presence of the pencil. Thus for the proof of Theorem 3.6 it is enough to show $a(Z) \neq 2$. We emphasize here that in Theorem 3.6 we are *not* assuming that the member S is generic in the pencil $|F|$. Indeed, by [17, Corollary 4.3], under the assumption of Theorem 3.6, if S is generic in the pencil $|F|$, equality in (3.5) holds, which means

$$(3.7) \quad a(Z) = 1 + \kappa^{-1}(S) = 2.$$

Hence Theorem 3.6 implies that *the member $S \in |F|$ in the theorem cannot be generic in the pencil $|F|$* . In other words, even if a twistor space Z on $n\mathbb{C}\mathbb{P}^2$, $n > 4$, with $\dim |F| = 1$ possesses a smooth member $S \in |F|$ which satisfies $\kappa^{-1}(S) = 1$, a *generic* member S' of the pencil has to satisfy $\kappa^{-1}(S') = 0$.

We note that, by investigating small deformations of twistor spaces of Joyce metrics [14], the existence of twistor spaces that fulfil the properties of Theorem 3.6 was shown in [8]. Our result shows that they have algebraic dimension 1.

The proof of Theorem 3.6 will be completed at the end of this section. In preparation for this proof, we first note that under the assumptions of Theorem 3.6 the base curve C of $|F|$ cannot be smooth and moreover $k \geq 2$ holds in (3.3).

Proposition 3.7. *Let Z and S be as in Theorem 3.6, and C be the unique anti-canonical curve on S . Then C is a cycle of rational curves as in (3.3) with $k \geq 2$.*

Proof. By Proposition 3.1, the base curve C is either a smooth elliptic curve or a cycle of rational curves as in (3.3). If C is smooth, from $K^2 = 8 - 2n < 0$ we easily obtain $h^0(mK_S^{-1}) = 1$ for any $m > 0$, but then $\kappa^{-1}(S) = 0$. Therefore C cannot be smooth.

If C is as in (3.3) with $k = 1$, then $C = C_1 + \overline{C}_1$, and $C_1\overline{C}_1 = 2$ as C is a cycle. Together with $K^2 = 8 - 2n$, this means $C_1^2 = \overline{C}_1^2 = 2 - n$. Hence the intersection matrix for the cycle C becomes

$$\begin{pmatrix} 2 - n & 2 \\ 2 & 2 - n \end{pmatrix}.$$

If $n > 4$, this is negative definite. Hence $P = 0$ in the Zariski decomposition $C = P + N$ of C , which implies $\kappa^{-1}(S) = 0$, see Lemma 2.4 (i). Therefore $k \geq 2$. \square

Let Z, S and C be as in Theorem 3.6 and Proposition 3.7 and let $\mu : \hat{Z} \rightarrow Z$ be the blow-up with centre C . The space \hat{Z} has singularities, but we discuss this later. Let E_i and

\overline{E}_i ($1 \leq i \leq k$) be the exceptional divisors over the curves C_i and \overline{C}_i , respectively. Write

$$E := (E_1 + \cdots + E_k) + (\overline{E}_1 + \cdots + \overline{E}_k)$$

for the sum of all exceptional divisors. Any two distinct smooth members S' and S'' of the pencil $|F|$ intersect transversally along the cycle C in the sense that $S''|_{S'} = C$ as a divisor on S' . Therefore, the blow-up $\mu : \hat{Z} \rightarrow Z$ composed with the rational map $Z \rightarrow \mathbb{CP}^1$ associated to the pencil $|F|$ is a morphism

$$\hat{f} : \hat{Z} \rightarrow \mathbb{CP}^1,$$

the fibres of which are the strict transforms of the members of the pencil $|F|$. This is nothing but an elimination of the indeterminacy locus of the rational map $Z \rightarrow \mathbb{CP}^1$. We will use the same letters to denote divisors in Z and their strict transforms in \hat{Z} .

Recall from Proposition 3.2 that the pencil $|F|$ has exactly k reducible members $S_i^+ + S_i^-$, and that $k \geq 2$ by Proposition 3.7. Let $\lambda_i \in \mathbb{CP}^1$, $1 \leq i \leq k$, be the point over which the reducible fibre $S_i^+ \cup S_i^-$ is lying. Evidently we have the basic relation

$$(3.8) \quad \hat{f}^* \mathcal{O}(1) \simeq \mu^* F - E.$$

For any index i , the exceptional divisor E_i comes with two fibrations $\hat{f}|_{E_i} : E_i \rightarrow \mathbb{CP}^1$ and $\mu|_{E_i} : E_i \rightarrow C_i \simeq \mathbb{CP}^1$. Both are clearly \mathbb{CP}^1 -bundles, and fibres of different fibrations intersect transversally at one point. Hence we obtain an isomorphism $E_i \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$ for any i . The same is true for the real conjugate divisor \overline{E}_i from reality.

Since the centre C of the blow-up μ is singular, being a cycle of rational curves, the space \hat{Z} has ordinary double points over the singularities of the cycle C . Concretely, these are exactly the points where the four divisors

$$(3.9) \quad S_i^+, S_i^-, E_i \text{ and } E_{i+1} \quad (1 \leq i \leq k)$$

meet and the real conjugate points of these, see Figure 1 where these points for $i = 1$ are indicated by black circles. Thus \hat{Z} has precisely $2k$ ordinary double points in total. Since each ordinary double point admits two small resolutions, the number of simultaneous small resolutions of all the double points, which are compatible with the real structure, is 2^k in total. Let $Z_1 \rightarrow \hat{Z}$ be any one of these small resolutions. We denote by $\mu_1 : Z_1 \rightarrow Z$ the composition $Z_1 \rightarrow \hat{Z} \rightarrow Z$ of the small resolution and the blow-up. We continue to use the same letters E_i and \overline{E}_i to denote the strict transforms of the exceptional divisors in Z_1 . Also, E denotes the total sum $(E_1 + \cdots + E_k) + (\overline{E}_1 + \cdots + \overline{E}_k)$ in Z_1 . We write $f_1 : Z_1 \rightarrow \mathbb{CP}^1$ for the composition $Z_1 \rightarrow \hat{Z} \rightarrow \mathbb{CP}^1$. From (3.8) we obtain the relation

$$(3.10) \quad f_1^* \mathcal{O}(1) \simeq \mu_1^* F - E,$$

and f_1 has reducible fibre $S_i^+ \cup S_i^-$ over the point λ_i , $1 \leq i \leq k$.

If we regard the cycle C as a curve on a real smooth member of the pencil $|F|$, it has a Zariski decomposition with P and N both real

$$(3.11) \quad C = P + N.$$

By Proposition 3.3, this Zariski decomposition is independent of the choice of the smooth member. As in Section 2, let m_0 be the smallest integer for which $m_0 P$ is integral, and write

$$m_0 P = l_1 C_1 + \cdots + l_k C_k + l_1 \overline{C}_1 + \cdots + l_k \overline{C}_k.$$

For later use, we use these coefficients l_i and define a \mathbb{Q} -divisor \mathbf{P} on Z_1 by the equation

$$(3.12) \quad m_0 \mathbf{P} = l_1 E_1 + \cdots + l_k E_k + l_1 \overline{E}_1 + \cdots + l_k \overline{E}_k.$$

Similarly we define a \mathbb{Q} -divisor \mathbf{N} on Z_1 by the equation

$$(3.13) \quad \mathbf{P} + \mathbf{N} = E.$$

Thus \mathbf{P} and \mathbf{N} may be regarded as enlargements of the nef part and the negative part, respectively, of the Zariski decomposition (3.11) to divisors in Z_1 . The key feature of these divisors is that if we restrict (3.13) to a smooth fibre of the projection $\mu_1 : Z_1 \rightarrow \mathbb{C}\mathbb{P}^1$, we obtain the Zariski decomposition (3.11).

Because we assume $\kappa^{-1}(S) = 1$ for a given $S \in |F|$, we obtain from Proposition 2.2 and Lemma 2.4 (i) that $P^2 = 0$ and $P \neq 0$ in S . Because of Proposition 3.3 the same is true for all smooth fundamental divisors. In particular, we can apply Proposition 2.6 to get $l_i > 0$ for all i and $l_i = 1$ for at least one i . Moreover, by Proposition 2.6 (iv), as $n > 4$, the k integers l_i cannot all be equal to each other. Continuing to write $l_{k+1} := l_1$, it follows that there is an index i for which $l_i > l_{i+1}$ holds. After a cyclic permutation of the indices we may assume that $l_1 > l_2$.

For arbitrary integers r, ρ we define a line bundle $\mathcal{M}(r, \rho)$ on Z_1 by

$$(3.14) \quad \mathcal{M}(r, \rho) := f_1^* \mathcal{O}(r) + \rho m_0 \mathbf{P}.$$

Proposition 3.8. *Let S' be any smooth fibre of the fibration $f_1 : Z_1 \rightarrow \mathbb{C}\mathbb{P}^1$, and identify the pair $(S', S' \cap E)$ in Z_1 with (S', C) in Z by the map $\mu_1 : Z_1 \rightarrow Z$. Then for any $r \in \mathbb{Z}$ and $\rho \in \mathbb{Z}$, we have*

$$(3.15) \quad \mathcal{M}(r, \rho)|_{S' \cap E} \in \text{Pic}^0(C).$$

Proof. Recall that $\text{Pic}^0(C)$ denotes the group of those line bundles on C that are of degree zero on each component of C . By the identifications $E_i \cap S' \simeq C_i$ and $\overline{E}_i \cap S' \simeq \overline{C}_i$ induced by the birational morphism μ_1 , we obtain that the restriction of $\mathcal{M}(r, \rho)$ to S' is isomorphic to $\rho m_0 P$. Here we have disposed of $f_1^* \mathcal{O}(r)$ since S' is a fibre of f_1 . Because $P^2 = 0$ on S' , Lemma 2.4 (ii) implies that $m_0 P|_{S' \cap E} \in \text{Pic}^0(C)$, hence so is $\rho m_0 P|_{S' \cap E}$. \square

We now take a closer look at the ordinary double point $S_1^+ \cap S_1^- \cap E_1 \cap E_2$ in \hat{Z} . Without loss of generality, we may suppose that S_1^+ and S_1^- are distinguished by the property that the intersection $S_1^+ \cap E_1$ is (not a curve but) a point, see Figure 1.

The small resolution $Z_1 \rightarrow \hat{Z}$ blows up one of the pairs $\{S_1^-, E_2\}$ or $\{S_1^+, E_1\}$. Because there is no essential difference between these two, we may suppose that the pair $\{S_1^-, E_2\}$ is blown up. We denote the exceptional curve over the point $S_1^+ \cap E_1$ by Δ_1 .

Since μ blows up the cycle C , we cannot speak about strict transforms of the components C_i of C into Z_1 . But, if we specify a fibre of $f_1 : Z_1 \rightarrow \mathbb{C}\mathbb{P}^1$, we can. In particular, we denote the strict transforms of C_i and \overline{C}_i in the fibre $f_1^{-1}(\lambda_1)$ by

$$(3.16) \quad C_{1,i} \quad \text{and} \quad \overline{C}_{1,i}, \quad 1 \leq i \leq k.$$

These are identified with C_i and \overline{C}_i , respectively, by the birational morphism $\mu_1 : Z_1 \rightarrow Z$. The first index, 1, indicates that the curves are included in the fibre $f_1^{-1}(\lambda_1) = S_1^+ \cup S_1^-$, and the second index reflects that they correspond to C_i and \overline{C}_i respectively. Unlike in the original space Z , the union of all the curves (3.16) is not a cycle because the exceptional curves Δ_1 and $\overline{\Delta}_1$ are inserted between $C_{1,1}$ and $C_{1,2}$, and between $\overline{C}_{1,1}$ and $\overline{C}_{1,2}$,

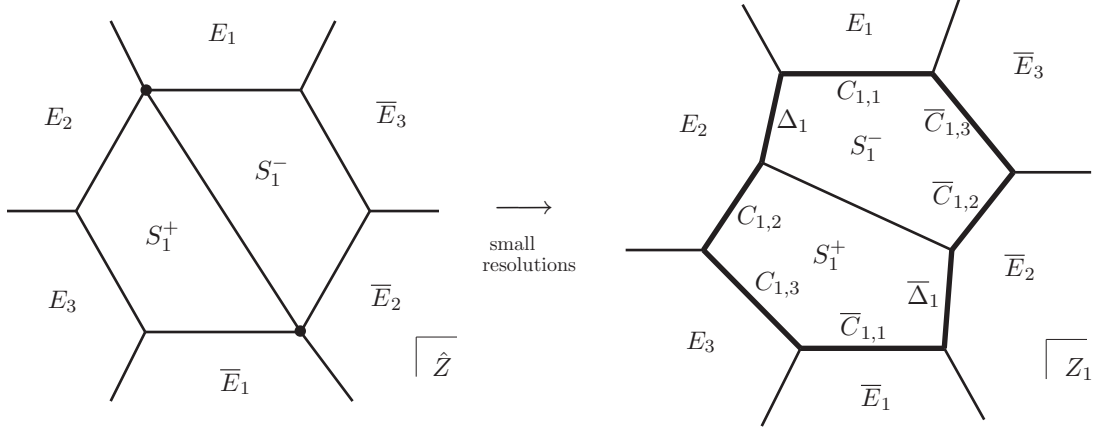


FIGURE 1. The small resolution of the ordinary double points on the fibre $\hat{f}^{-1}(\lambda_1)$ depicted for $k = 3$. The bold lines in the right figure mark the cycle \mathcal{C} that is defined below.

respectively, see Figure 1. Hence the union of all the curves (3.16) consists of two connected chains

$$(3.17) \quad C_{1,2} \cup C_{1,3} \cup \cdots \cup C_{1,k} \cup \bar{C}_{1,1} \quad \text{and} \quad \bar{C}_{1,2} \cup \bar{C}_{1,3} \cup \cdots \cup \bar{C}_{1,k} \cup C_{1,1},$$

and by adding Δ_1 and $\bar{\Delta}_1$ we get a cycle of rational curves, which we denote by \mathcal{C} . We clearly have

$$\mathcal{C} = f_1^{-1}(\lambda_1) \cap E$$

in the sense that \mathcal{C} is the restriction of the divisor E to the fibre $f_1^{-1}(\lambda_1)$ or, equivalently, \mathcal{C} is an element of the linear system $|(f_1|_E)^*\mathcal{O}(1)|$ on the surface E .

The intersection numbers of the line bundle $\mathcal{M}(r, \rho)$ with the components of the cycle \mathcal{C} will play an essential role in the subsequent proofs.

Lemma 3.9. *We have*

$$(3.18) \quad \mathcal{M}(r, \rho) \cdot C_{1,i} = \begin{cases} 0 & i \neq 2 \\ -\rho(l_1 - l_2) & i = 2. \end{cases}$$

and

$$(3.19) \quad \mathcal{M}(r, \rho) \cdot \Delta_1 = \rho(l_1 - l_2).$$

Proof. Let S' be any smooth fibre of $f_1 : Z_1 \rightarrow \mathbb{CP}^1$. If $i \neq 2$, the curve $C_{1,i}$ is homologous in E_i to $C_i = S' \cap E_i$. Therefore, using Proposition 3.8, we have

$$(3.20) \quad \mathcal{M}(r, \rho) \cdot C_{1,i} = 0 \quad \text{for all } i \neq 2.$$

The case $i = 2$ requires more attention. Dropping components that are disjoint from $C_{1,2}$, see Figure 1, we first obtain

$$(3.21) \quad \mathcal{M}(r, \rho) \cdot C_{1,2} = \rho(l_2 E_2 + l_3 E_3) C_{1,2}.$$

Further, as $C_{1,2} \cap E_3$ is one point and the intersection is transverse, we have $E_3 C_{1,2} = 1$. Next, $E_2 C_{1,2} = (C_{1,2})_{S_1^+}^2$ since $E_2 \cap S_1^+ = C_{1,2}$ and the intersection is transverse. Moreover, since the pairs $(S_1^+, C_{1,2})$ in Z_1 and (S_1^+, C_1) in the original twistor space Z are isomorphic,

the self-intersection number $C_{1,2}^2$ in $S_1^+ \subset Z_1$ is equal to C_2^2 calculated in $S_1^+ \subset Z$. We let $a_2 := -C_2^2$, calculated in $S \subset Z$. By the adjunction formula and because $S_1^+ + S_1^- = F$ on Z , we have

$$\begin{aligned} (C_2)_{S_1^+}^2 &= -2 + K_{S_1^+}^{-1}C_2 = -2 + (K_Z^{-1} - S_1^+)C_2 = -2 + (F + S_1^-)C_2 \\ &= -2 + FC_2 + S_1^-C_2 = -2 + K_S^{-1}C_2 + 1 = (C_2)_S^2 + 1 = -a_2 + 1. \end{aligned}$$

By Lemma 2.4 (ii) we have $m_0PC_2 = 0$ on the original surface $S \subset Z$ of which we assumed $\kappa^{-1}(S) = 1$. Thus, we have the relation

$$(3.22) \quad l_1 - a_2l_2 + l_3 = 0,$$

with which we obtain

$$(l_2E_2 + l_3E_3)C_{1,2} = l_2(-a_2 + 1) + l_3 = l_2 - l_1.$$

Therefore, from (3.21), we get

$$(3.23) \quad \mathcal{M}(r, \rho) \cdot C_{1,2} = -\rho(l_1 - l_2).$$

For the remaining intersection number (3.19) we notice that the curve $\Delta_1 \cup C_{1,2}$, regarded as a curve in the surface E_2 , is homologous to the fibre $S \cap E_2 \simeq C_2$ of the projection $f_1|_{E_2} : E_2 \rightarrow \mathbb{CP}^1$. Because $\mathcal{M}(r, \rho)|_S = m_0P$ and $\mathcal{M}(r, \rho) \cdot (S \cdot E_2) = (\mathcal{M}(r, \rho) \cdot S) \cdot E_2 = (m_0PC_2)_S$ is zero, we obtain

$$\mathcal{M}(r, \rho) \cdot (\Delta_1 + C_{1,2}) = 0,$$

and (3.19) follows from (3.23). \square

Let $\rho > 0$ be an integer and recall that we have chosen notation so that $l_1 > l_2$. From (3.18) for $i = 2$ we then obtain $\mathcal{M}(r, \rho) \cdot C_{1,2} < 0$. Using (3.18) for $i \neq 2$ and reality it follows now that the two chains

$$(3.24) \quad C_{1,2} \cup C_{1,3} \cup \cdots \cup C_{1,k} \cup \bar{C}_{1,1} \quad \text{and} \quad \bar{C}_{1,2} \cup \bar{C}_{1,3} \cup \cdots \cup \bar{C}_{1,k} \cup C_{1,1}$$

are contained in the base locus of the linear system $|\mathcal{M}(r, \rho)|$. This will be a stepping stone for the following stronger statement, which will play a key role in the proof of the main theorem.

Proposition 3.10. *For any $r \in \mathbb{Z}$ and $\rho > 0$, we have*

$$H^0(E, \mathcal{M}(r, \rho)|_E) = 0.$$

In particular, if $\rho > 0$ the divisor E is a fixed component of the linear system $|\mathcal{M}(r, \rho)|$ on Z_1 .

Proof. Recall that the cycle of rational curves \mathcal{C} belongs to the linear system $|(f_1|_E)^*\mathcal{O}(1)|$ on E and that $\mathcal{M}(r, \rho) = \mathcal{M}(r-1, \rho) \otimes f_1^*\mathcal{O}(1)$. Therefore, multiplication by a section of $f_1^*\mathcal{O}(1)|_E$ with zero locus \mathcal{C} provides an exact sequence

$$0 \longrightarrow \mathcal{M}(r-1, \rho)|_E \longrightarrow \mathcal{M}(r, \rho)|_E \longrightarrow \mathcal{M}(r, \rho)|_{\mathcal{C}} \longrightarrow 0.$$

To first show that $\phi = 0$ in the induced exact sequence

$$(3.25) \quad 0 \longrightarrow H^0(\mathcal{M}(r-1, \rho)|_E) \longrightarrow H^0(\mathcal{M}(r, \rho)|_E) \xrightarrow{\phi} H^0(\mathcal{M}(r, \rho)|_{\mathcal{C}}),$$

we let

$$s \in H^0(E, \mathcal{M}(r, \rho)|_E)$$

be a non-zero section and note that $\phi(s) = s|_{\mathcal{C}}$.

Suppose that s identically vanishes on a component of E . Then since, for each i the degree of the line bundle $\mathcal{M}(r, \rho)$ on a general fibre of $f_1|_{E_i} : E_i \rightarrow \mathbb{CP}^1$ is zero by Proposition 3.8, we obtain that s vanishes identically on E . Hence s cannot vanish identically on any component of E . Therefore we can consider the effective Cartier divisor (s) on E , and the restriction $(s|_{E_i})$ is a curve on E_i for any i .

From the intersection numbers (3.18) we have seen above that the divisor (s) includes the two connected chains (3.24) for $\rho > 0$ and all $r \in \mathbb{Z}$. Hence the curve $(s|_{E_2})$ passes through the point $C_{1,1} \cap \Delta_1$, see Figure 1. This point is not on $C_{1,2}$, because $C_{1,1}$ and $C_{1,2}$ are disjoint curves in $f_1^{-1}(\lambda_1) \subset Z_1$. If the divisor $(s|_{E_2})$ had a component other than Δ_1 , containing this point, then we would have $s|_{E_2} = 0$ by Proposition 3.8, since the fibre of $f_1|_{E_2}$ through this point is $\Delta_1 + C_{1,2}$ and so $(s|_{E_2})$ would have to intersect any fibre of the projection $f_1 : E_2 \rightarrow \mathbb{CP}^1$. Therefore such a component does not exist. Hence the divisor $(s|_{E_2})$, and so (s) , includes Δ_1 . The same argument shows that $(s) \supset \overline{\Delta}_1$. We have seen before that (s) includes all the other components $C_{1,i}$ and $\overline{C}_{1,i}$ of $\mathcal{C} = f_1^{-1}(\lambda_1) \cap E$ and so we obtain $\mathcal{C} \subset (s)$, which implies $\phi(s) = s|_{\mathcal{C}} = 0$. This shows that $\phi = 0$.

From the exact sequence (3.25) we obtain now for all $\rho > 0$ and all $r \in \mathbb{Z}$ that

$$(3.26) \quad H^0(E, \mathcal{M}(r-1, \rho)|_E) \cong H^0(E, \mathcal{M}(r, \rho)|_E).$$

On the other hand, using (3.14) and the projection formula for $f_1 : E \rightarrow \mathbb{CP}^1$, we obtain

$$\begin{aligned} f_{1*}\mathcal{M}(r, \rho)|_E &\cong f_{1*}(f_1^*\mathcal{O}_{\mathbb{CP}^1}(r) \otimes \mathcal{O}_E(\rho m_0\mathbf{P})) \\ &\cong \mathcal{O}_{\mathbb{CP}^1}(r) \otimes f_{1*}\mathcal{O}_E(\rho m_0\mathbf{P}). \end{aligned}$$

For any coherent sheaf \mathcal{F} on \mathbb{CP}^1 and for sufficiently large j , $H^0(\mathcal{O}_{\mathbb{CP}^1}(-j) \otimes \mathcal{F}) = 0$. Taking j so that this holds for $\mathcal{F} = f_{1*}\mathcal{O}_E(\rho m_0\mathbf{P})$, we obtain from (3.26) that

$$\begin{aligned} H^0(E, \mathcal{M}(r, \rho)|_E) &\cong H^0(E, \mathcal{M}(-j, \rho)|_E) \\ &\cong H^0(\mathbb{CP}^1, f_{1*}\mathcal{M}(-j, \rho)|_E) \\ &\cong H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-j) \otimes \mathcal{F}) = 0 \end{aligned}$$

for all $r \in \mathbb{Z}$ and all $\rho > 0$. □

By Propositions 3.8 the restriction of $m_0\mathbf{P}$ to $f^{-1}(\lambda) \cap E \simeq C$ gives an element $\mathcal{P}_\lambda \in \text{Pic}^0(C)$ for each value of λ for which the fibre $f_1^{-1}(\lambda)$ is smooth.

Proposition 3.11. *Assume that \mathcal{P}_λ has the same finite order τ for all smooth fibres $f_1^{-1}(\lambda)$. Then, for any $r \in \mathbb{Z}$ and $\nu > 0$, we have*

$$|\mathcal{M}(r, \nu\tau)| = |f_1^*\mathcal{O}(r)| + \nu\tau m_0\mathbf{P}.$$

Proof. Recall that

$$\mathcal{M}(r, \nu\tau) = f_1^*\mathcal{O}(r) + \nu\tau m_0\mathbf{P}.$$

By Proposition 3.10 the divisor E is a fixed component of $|\mathcal{M}(r, \nu\tau)|$ and so

$$(3.27) \quad |\mathcal{M}(r, \nu\tau)| = |f_1^*\mathcal{O}(r) + \nu\tau m_0\mathbf{P} - E| + E.$$

Let S' be a general fibre of $f_1 : Z_1 \rightarrow \mathbb{CP}^1$, and identify the pair $(S', S' \cap E)$ with the pair (S', C) included in the original space Z by the birational morphism $\mu_1 : Z_1 \rightarrow Z$. Because we assumed \mathcal{P}_λ to have finite order, we can use Proposition 2.8 (iv) to obtain

$$(3.28) \quad |\nu\tau m_0\mathbf{P} - C| = |(\nu-1)\tau m_0\mathbf{P}| + (\tau m_0\mathbf{P} - C)$$

on S' . As this is valid for any smooth fibre of f_1 , we obtain that the divisor $\tau m_0 \mathbf{P} - E$ on Z_1 is a fixed component of the system $|f_1^* \mathcal{O}(r) + \nu \tau m_0 \mathbf{P} - E|$. Hence, from (3.27) and using definition (3.14), we obtain

$$\begin{aligned} |\mathcal{M}(r, \nu \tau)| &= |f_1^* \mathcal{O}(r) + (\nu - 1) \tau m_0 \mathbf{P}| + \tau m_0 \mathbf{P} \\ &= |\mathcal{M}(r, (\nu - 1) \tau)| + \tau m_0 \mathbf{P}. \end{aligned}$$

Using induction on $\nu > 0$ we finally obtain

$$|\mathcal{M}(r, \nu \tau)| = |\mathcal{M}(r, 0)| + \nu \tau m_0 \mathbf{P} = |f_1^* \mathcal{O}(r)| + \nu \tau m_0 \mathbf{P},$$

as required. \square

We are now ready to prove our main result.

Proof of Theorem 3.6. We first show that if the order of the restriction \mathcal{P}_λ of $m_0 \mathbf{P}$ to $C \simeq f^{-1}(\lambda) \cap E$ is not the same for all smooth fibres $f_1^{-1}(\lambda)$ then we have $a(Z) = 1$. To see this, we note that the set of points of finite order in $\text{Pic}^0(C)$, which is isomorphic to \mathbb{C}^* , has the property that any continuous path that connects two distinct points of finite order necessarily contains points of infinite order. Moreover we have a continuous function $\lambda \mapsto \mathcal{P}_\lambda \in \text{Pic}^0(C)$ which is defined on the open set of $\mathbb{C}\mathbb{P}^1$ over which f_1 has smooth fibres. Therefore, if the order of \mathcal{P}_λ is not constant, the image of the continuous function mentioned above contains an element of infinite order. If $S' \in |F|$ is the member corresponding to such an element of infinite order, Corollary 2.9 shows that $\kappa^{-1}(S') \neq 1$. As $d(S') = d(S) = 0$, Proposition 2.2 implies $\kappa^{-1}(S') \neq 2$, and we conclude $\kappa^{-1}(S') = 0$. As we have seen from (3.5), this implies $a(Z) = 1$.

Thus it remains to consider the situation in which the order of \mathcal{P}_λ is the same for all smooth fibres $f_1^{-1}(\lambda)$. We will show by contradiction that this situation never happens.

So suppose that the restriction \mathcal{P}_λ has the same finite order τ for all λ for which $f_1^{-1}(\lambda)$ is smooth. Then, by Corollary 2.9, we have $\kappa^{-1}(S) = 1$ for each smooth $S \in |F|$. Hence $\kappa^{-1} = 1$ for generic members of the pencil $|F|$ and, using [17, Corollary 4.3], we obtain that $a(Z) = 2$.

On the other hand, with the aid of Proposition 3.11, we are able to show $a(Z) = 1$ as follows. For each integer $\nu > 0$, (3.11) gives a Zariski decomposition in S

$$\nu \tau m_0 C = \nu \tau m_0 P + \nu \tau m_0 N,$$

and since the Zariski decomposition of the cycle C is independent of the smooth member S by Proposition 3.3, this is the Zariski decomposition of the divisor $\nu \tau m_0 C$ on any smooth member of the pencil $|F|$. Therefore, by Proposition 2.1, for any non-singular fundamental divisor $S' \in |F|$, the divisor $\nu \tau m_0 N$ is a fixed component of the system $|\nu \tau m_0 K^{-1}|$ on S' . Hence the pull-back $\mu_1^*(\nu \tau m_0 F)$ to Z_1 has the divisor $\nu \tau m_0 \mathbf{N}$ as a fixed component; the divisor \mathbf{N} was defined in (3.13). We define a line bundle \mathcal{L} on Z_1 by

$$(3.29) \quad \mathcal{L} := \mu_1^*(\tau m_0 F) - \tau m_0 \mathbf{N} \simeq f_1^* \mathcal{O}(\tau m_0) + \tau m_0 \mathbf{P} = \mathcal{M}(\tau m_0, \tau),$$

where we have used the relation (3.10) to get the isomorphism. The line bundle \mathcal{L} is equipped with a natural real structure coming from that on F on Z . The morphism $\mu_1 : Z_1 \rightarrow Z$ provides a bijection between the members of the two linear systems $|\nu \tau m_0 F|$ on Z and $|\mathcal{L}^{\otimes \nu}|$ on Z_1 . Moreover, the composition of μ_1 with the rational map defined by $|\nu \tau m_0 F|$ is the rational map which is given by $|\mathcal{L}^{\otimes \nu}|$. Because

$$|\mathcal{L}^{\otimes \nu}| = |\mathcal{M}(\nu \tau m_0, \nu \tau)| = |f_1^* \mathcal{O}(\nu \tau m_0)| + \nu \tau m_0 \mathbf{P},$$

by Proposition 3.11, we see that the rational map associated to $|\mathcal{L}^{\otimes \nu}|$ has one-dimensional image, equal to the image of a Veronese embedding of \mathbb{CP}^1 . Hence, for all $\nu > 0$, the rational map defined by $|\nu \tau m_0 F|$ has one-dimensional image as well, and this implies that $a(Z) = \kappa(Z, F) = 1$. This contradicts our previous conclusion $a(Z) = 2$. Hence it is impossible that \mathcal{P}_λ has finite order not depending on λ . \square

It might be interesting to observe that in the argument that leads to the contradiction in the second part of the above proof, the constancy of the order of \mathcal{P}_λ is only needed to apply Proposition 3.11.

4. COMMENT ON A RESULT IN THE PAPER [8]

In [8], the existence of a twistor space on $n\mathbb{CP}^2$ with algebraic dimension two is claimed for any $n > 4$. The proof of this assertion consists essentially of the following two parts.

1. For a non-singular surface S which admits an anti-canonical cycle, it is shown in [8, Lemma 3.2] that the finiteness of the order of a certain line bundle L_a on a cycle of rational curves implies $\kappa^{-1}(S) = 1$. With the aid of formula (2.5) it can be shown that the pull-back of the line bundle L_a to S is isomorphic to the line bundle $\mathcal{O}(m_0 P)|_C$ studied in Section 2.
2. If a twistor space Z on $n\mathbb{CP}^2$, $n > 4$, admits a divisor in $|F|$ which is biholomorphic to the surface S in the above part 1 satisfying $\kappa^{-1}(S) = 1$, then we always have $\dim |F| = 1$. It is claimed in the proof of [8, Lemma 4.3] that the isomorphism class of the line bundle L_a from part 1 is the same for generic members of the pencil $|F|$.

It follows that a generic fundamental divisor S in a twistor space Z as in part 2 above satisfies $\kappa^{-1}(S) = 1$. Using [8, Proposition 4.1] it is then concluded that $a(Z) = 2$. This conclusion is in conflict with our Theorem 1.1.

To resolve this contradiction, we first recall some notation of [8] and then focus on a key step in the proof that the order of the line bundle L_a does not depend on S .

Each non-singular fundamental divisor S_t ($t \in \mathbb{CP}^1$) is given as an iteration of blow-ups

$$(4.1) \quad S_t \xrightarrow{w_t} T' \xrightarrow{u} T \xrightarrow{\alpha} \mathbb{CP}^1 \times \mathbb{CP}^1.$$

Here, α is the blow-up of 4 points, all of which are nodes of an anti-canonical cycle on $\mathbb{CP}^1 \times \mathbb{CP}^1$ consisting of 4 components. The morphism u is a composition of blow-ups of which each centre is a node of the anti-canonical cycle. Both T' and T are toric and their isomorphism classes are independent of $t \in \mathbb{CP}^1$. On the other hand, w_t blows up 4 points, where each centre belongs to the strict transforms of the 4 exceptional curves of α which are disjoint to each other. Thus the variation of the complex structure on S_t is entirely encoded in the variation of the 4 points blown up under w_t . It is important to keep in mind that the sequence (4.1) is constructed for each non-singular S_t individually, and there is no natural way to identify the surface T' (and T and $\mathbb{CP}^1 \times \mathbb{CP}^1$ also) for different choices of $t \in \mathbb{CP}^1$.

Let C be the anti-canonical cycle on S_t , which is the base curve of the pencil $|F|$, and $B' = w_t(C)$. The argument that the order of the line bundle L_a does not depend on the surface S in the fundamental pencil is built in an essential way on the assertion that the pull-back $(w_t|_C)^*(K_{T'}^{-1}|_{B'})$ does not depend on t . This independence is based on the claim that the morphism $w_t|_C : C \rightarrow B'$ does not depend on t .

We now derive a contradiction from the assumption that $w_t|_C : C \rightarrow B'$ is the same for two nearby values of t for which S_t is a real smooth member of the fundamental pencil. This confirms that there is a gap in the proof of [8, Lemma 4.3] on line 10, page 1105.

Let S and S' be two real irreducible fundamental divisors on a twistor space Z over $n\mathbb{CP}^2$ such that both are obtained by a sequence of blow-ups from $\mathbb{CP}^1 \times \mathbb{CP}^1$ as in (4.1). The linear system of twistor lines on each of these smooth surfaces gives, see [19], a morphism with target \mathbb{CP}^1 and we can choose the morphisms in (4.1) so that their composition with the first projection $p_1 : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is the morphism given by the pencil of twistor lines. We now identify the surfaces T' , T and $\mathbb{CP}^1 \times \mathbb{CP}^1$ obtained by blowing down some curves on S with those obtained by blowing down some curves on S' . As mentioned above, there is no natural way to do so.

From Proposition 3.2 we can deduce that the two components of the cycle C that are met by real twistor lines in a real fundamental divisor S_t do not change as long as t varies within one connected component of the set of real irreducible fundamental divisors, which is the complement of a finite set of points on a circle. We assume that S and S' belong to such a connected component and denote the morphisms w_t for these two surfaces by $w : S \rightarrow T'$ and $w' : S' \rightarrow T'$, respectively. We let π be the composition

$$\pi : S \xrightarrow{w} T' \xrightarrow{u} T \xrightarrow{\alpha} \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{p_1} \mathbb{CP}^1$$

and define $\pi' : S' \rightarrow \mathbb{CP}^1$ similarly. Both restrictions $w|_C$ and $w'|_C$ have the same image $B' \subset T'$. Because $p_1 \alpha u : T' \rightarrow \mathbb{CP}^1$ is the same for both surfaces S and S' , the assumption that $w|_C = w'|_C$ implies that the two compositions

$$\pi|_C : C \xrightarrow{w|_C} B' \subset T' \xrightarrow{u} T \xrightarrow{\alpha} \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{p_1} \mathbb{CP}^1$$

and

$$\pi'|_C : C \xrightarrow{w'|_C} B' \subset T' \xrightarrow{u} T \xrightarrow{\alpha} \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{p_1} \mathbb{CP}^1$$

coincide. Let now $L \subset S$ be a real twistor fibre, then $L \cap C = \{p, \bar{p}\}$ is a set of two conjugate points. Consider the fibre $L' \subset S'$ of π' over the point $\pi(p) = \pi(\bar{p})$. As π' has only finitely many reducible fibres and the real twistor lines in S and S' meet the same components of C , we can choose L so that L' is irreducible. Because, by assumption, $\pi|_C = \pi'|_C$, we have $L' \cap C = \{p, \bar{p}\}$ as well. However, as there do not exist two real twistor fibres in Z through the point p and $S \cap S'$ does not contain a real twistor fibre, L' cannot be a real member of the pencil $|L'|$ in S' . Hence, L' and its conjugate are two different members of the same pencil on S' and they intersect at least at the two points p and \bar{p} . But this is absurd as the irreducible curve L' has self-intersection number 0 on S' . This contradiction shows that $w_t|_C : C \rightarrow B'$ cannot be the same for any two values of t for which S_t is real.

A detailed analysis of the argument in [8, p. 1105] reveals that the core problem is whether for $s \neq t$ the automorphism $(w_s|_C) \circ (w_t|_C)^{-1} : B' \rightarrow B'$ extends to an automorphism $T' \rightarrow T'$, or more precisely, to an isomorphism between neighbourhoods of B' . This seems to have been overlooked in [8]. A priori there is no reason for this to hold, and our Theorem 1.1 shows that for generic $s \in \mathbb{CP}^1$, such an extension cannot exist and even if some member in the pencil $|F|$ satisfies $\kappa^{-1} = 1$, the general member of the pencil needs to satisfy $\kappa^{-1} = 0$.

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Nobuhiro Honda

Mathematical Institute, Tohoku University, and

Department of Mathematics, Tokyo Institute of Technology, Tokyo, JAPAN (current affiliation)

honda@math.titech.ac.jp

Bernd Kreußler

Department of Mathematics and Computer Studies, Mary Immaculate College, Limerick, IRELAND

bernd.kreussler@mic.ul.ie