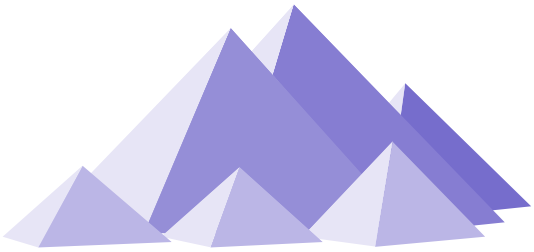

Budding Architects
Exploring the designs of pyramids and prisms

 

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The context of students as architects is used to examine the similarities and differences between prisms and pyramids. Leavy and Hourigan use the Van Hiele Model as a tool to support teachers to develop expectations for differentiating geometry in the classroom using practical examples.

Introduction

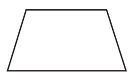
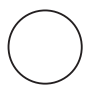
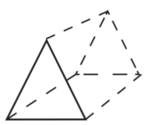
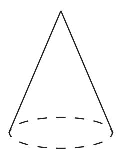
> I am still searching for ways to help my students identify and differentiate pyramids and prisms. While they readily recognise pyramids, they often confuse the terms and the structures. How can I help them identify solids as being pyramids or prisms and recognise the characteristics of these three-dimensional solids?
>
> (Ella, Grade 6 Teacher)

Is this situation familiar to you? When students explore geometric structures, a number of forms must be recognised and differentiated from others. Many international curriculum documents recommend a focus on three-dimensional (3D) shapes and their nets as students approach the middle primary grades. The Australian Curriculum and Assessment Reporting Authority [ACARA] recommends that by Year 5, students should be able to connect 3D objects with their nets and move to constructing simple prisms and pyramids by Year 6 (ACARA, 2014). Similarly, in the United States, by sixth grade not only should students be able to "Represent three-dimensional figures using nets made up of rectangles and triangles" (CCSS.Math.Content.6.G.A.4), but they should also be able to "apply these techniques in the context of solving real-world and mathematical problems" (CCSS.Math.Content.6.G.A.4) (CCSSM, 2010). In this article, we describe how we coordinated these common curriculum goals through the use of a real-world context to motivate the representation and exploration of prisms and pyramids with Year 6 students.

Defining pyramids and prisms

Pyramids and prisms have several properties in common: they are 3D shapes and polyhedrons.

Table 1. Important definitions.

Shape	Definition	Example	Non-example
Polygon	Closed 2D shapes with straight sides.		
Polyhedron	3D shapes consisting of the union of polygonal faces resulting in an enclosed region without any holes.		

A polyhedron is a 3D solid with flat faces. Each of the faces are polygons (Table 1) i.e., hexagons, squares, triangles. The fact that each face is a polygon has implications for what can be classed as a polyhedron (Table 1). 3D shapes with curved faces, such as cylinders and cones, are not polyhedrons due to their circular bases.

Initial observations of pyramids and prisms

The teacher stated: "This morning we will be working as architects and examining the design of two different types of buildings. Architects have good observation skills and focus on the features of buildings to better help them in their designs. I want you to look at these two buildings and identify similarities and differences between them." Following this introduction, the teacher displayed images of two different buildings (Figures 1, 2). Figures 1 and 2 depict a city skyscraper and an Egyptian pyramid respectively. The skyscraper is a square-based prism and the pyramid is also square-based. A birds-eye view of the pyramid (Figure 3) was also displayed in an effort to focus students' attention to the shape of each structure's base.

Figure 1. The skyscraper (square-based prism).

Figure 2. The square-based pyramid.

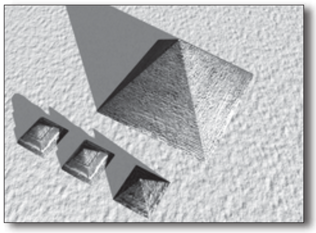


Figure 3. Birds-eye view of the pyramid.

Students then worked in pairs to discuss their observations in relation to the similarities and differences between the buildings. They shared their ideas during the teacher-led whole class discussion:

Teacher: What similarities did you notice between the buildings?

Sylvia: From the birds-eye view they both look the same. They look square.

Peter: Yes. The bases of both of them are square.

Teacher: What differences did you notice between the buildings?

Francis: The sides of the pyramid aren't straight, they seem diagonal. But the sides of the skyscraper are straight – they are vertical.

Cian: The skyscraper has a flat roof but the pyramid doesn't. The roof of the pyramid has all the sides joining up at one point.

Niamh: The Skyscraper is a cuboid and the pyramid is a pyramid.

Teacher: Is there another name for the shape of the skyscraper? It is a cuboid, but can you call it something else?

Analysis of student discourse, such as that in the dialogue above, is not a trivial endeavour for teachers. Instruction in geometry, and the analysis of student responses, may be informed from many areas such as the work of Piaget, Vygotsky or the *SOLO* taxonomy. In this research we used the Van Hiele Model (Crowley, 1987; Van de Walle, 2007) to inform instruction. This model consists of five levels, which describe how students reason about shapes and other geometric ideas. This theory asserts that the levels are a product of experience and instruction. In outlining the theory, the Van Hiele's present a treatment of both the subject matter (i.e., geometry) and the role of the teacher (i.e., specific instruction approaches and techniques). Relationships between each of the levels are described and suggestions are made for ways to support and precipitate the advancement of students' reasoning. The framework emphasises the role of the teacher in providing suitable geometrical experiences. Indeed "Van Hiele specifically states that the inability of many teachers to match instruction with their learners' levels of geometrical understanding is a contributing factor to their failure to promote meaningful understandings in this topic" (Feza and Webb, 2005). Hence, these factors motivated the selection of the Van Hiele framework as a guiding framework to inform the instructional decisions made in this research.

The final question posed by the teacher was met with silence from the students. This activity and the associated dialogue provide a number of insights into the geometric reasoning of these students.

One observation is that students readily recognised the shapes as pyramids and cuboids. They analysed the shapes based on properties as indicated by their references to faces and the orientation of faces. In the dialogue cited, student comments are indicative of Level 1 behaviours in the Van Hiele Model of Geometric Reasoning (Table 2). However, the lack of awareness of a cuboid as being a member of the class of shapes called prisms indicates that students are not yet functioning at Level 2 of the Van Hiele Model of Geometric Reasoning. Students at Level 2 are expected to engage in informal deductive reasoning involving classifying shapes and making generalisations about shapes in hierarchies. Hence a student functioning at Level 2 would see the class of prisms as containing a variety of 3D shapes such as cubes, cuboids, and so on.

Classifying pyramids and prisms

Following student observations of the buildings, we wanted to shift the focus of instruction to the properties you might use to classify solids as belonging to prisms or pyramids. Students notice many properties when exploring solids and their attention is usually drawn to: the number of faces, the shapes of those faces and the presence or absence of one or more bases. A description of the properties of pyramids and prisms is presented in Table 3; this list is not exhaustive and focuses on properties that are salient and relevant in school mathematics.

Table 2. The Van Hiele Model of Geometric Reasoning for 3D shapes (concentrating on the first three levels within which primary and middle school students are usually classified).

Level	Name of Level	Expected behaviour in relation to 3D solids	When presented with a cube:
0	Visualisation	Solids are judged and identified visually and holistically, with little or no explicit consideration of components or properties.	I know this is a cube because it looks like one.
1	Analysis	The learner identifies components of solids and informally describes solids using isolated mathematical properties, although properties are not logically related. Explanations are based on observation.	I know this is a cube because it has six faces and each face is a square.
2	Informal Deduction	Learners are able to logically classify solids and understand the logic of definitions. Statements are based on informal mathematical justifications.	This cube is a prism because it has two bases and all the sides are flat. If you slice it in a few places the cross-section section is always a square that is the same size.

Table 3. Common characteristics of pyramids and prisms .

	Pyramids	Prisms
Bases	One base and a point not on the same plane as the base. The point is called the apex.	Two opposite faces that are identical polygons. These faces are referred to as the bases.
Shape of the base	Polygon	Polygon
Lateral faces	The apex is connected with line segments to each vertex of the base resulting in lateral faces. Lateral faces are triangles.	Vertices of the bases are joined to form lateral faces. Lateral faces are parallelograms.
Categories	If lateral faces are isosceles triangles then the shape is a right pyramid. Otherwise, the shape is an oblique pyramid.	If lateral faces are rectangles then the shape is a right prism. Otherwise, the shape is an oblique prism.
Cross-section	Same polygon as the base, but the dimensions get smaller the closer you get to the apex.	Same polygon and dimensions as the base.

The cross-sections of pyramids and prisms

Students were informed that the cuboid was part of a group of shapes known as ‘prisms’ (Oberdorf & Taylor-Cox, 1999; Van De Walle, 2007). The focus of the next section of the lesson was exclusively upon allowing students to discover what we considered to be one of the differentiating characteristics between pyramids and prisms: the cross-section. The goal was to support students in using the outcomes of an examination of the cross-sections of 3D shapes to inform subsequent classification of shapes as pyramids or prisms.

Instruction in 3D geometry usually presents students with planar two-dimensional (2D) representations of solids rather than with actual models (Battistia, 1999). We know that students have great difficulties conceptualising 3D shapes that are presented on 2D surfaces (Koester, 2003; Parzysz, 1988). Hence, the design of the instructional sequence we describe prioritises the use of 3D models of pyramids and prisms (Cockcroft & Marshall, 1999) as opposed to pictures, drawings or other 2D representations of these solids.

The teacher presented a cake that is a square-based prism (otherwise known as a cuboid) and asked students which building best resembles the cake. The cake we used is commonly known as a Battenberg and students readily identified it as the skyscraper. The teacher cut a slice off the cake (using a horizontal cut as seen on figure 4) and revealed that the cross-section is square. The teacher continued to cut slices to demonstrate that the cross-section is always the same —in this case, a square with the same dimensions. The teacher used this demonstration to conclude that a prism has cross-sections of the same dimension all along its length.

The teacher then presented a model of a pyramid (figure 5). This clay model was designed and constructed so that the teacher could make horizontal cuts across the pyramid, thus facilitating the display of the cross-sections. Several cuts were made and students easily noted that in pyramids while the cross-sections are the same shape (in this case, squares), the dimensions of the cross-section changes depending on where the cut is made.

Figure 4. Battenberg cake.

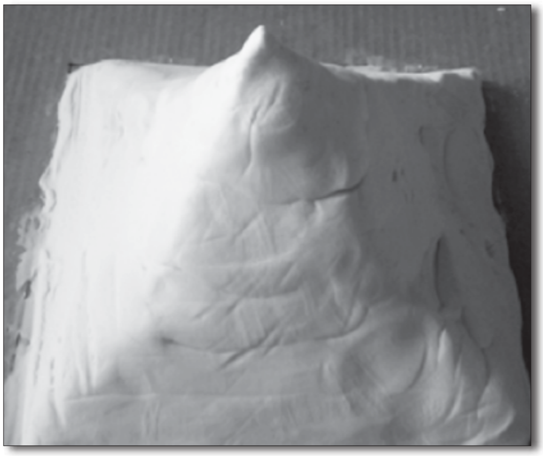


Figure 5. Clay Model of Pyramid.

After this process, the focus moved to the possible implications of these distinguishing features for architects:

Teacher: Why is this information about cross-sections important if you are an architect?

Orla: It would be easier to build a skyscraper cause all the floors are the same shape.

Connor: Yes, and if it was the pyramid, you'd have to make each floor smaller as you go from the ground to the top.

Teacher: What do you notice about the bases of the pyramid and prism?

Kate: The prism has a floor on top and a floor on the bottom and they are the same. But the Pyramid has only the bottom floor.

Sam: An architect could design a viewing area on the top of a building if it is a prism.

Sorcha: Or you could put a helicopter pad on the top of a prism. But it would be impossible to land a helicopter on the top of a pyramid!

Examining faces: exploring models of pyramids and prisms

The teacher posed the following task: "I want you to imagine that you are architects, some of you specialise in the design of pyramids and others in the design of prisms. Your task is to sort a collection of 3D shapes into pyramids and prisms. Use your knowledge of the cross-sections to help you classify the shapes. Then, I want you to closely observe the (lateral) faces of the pyramids and prisms and see if you can find similarities or differences".

Students worked in pairs or groups of three (Figure 6) to sort prisms (cuboid, triangular prism, hexagonal prism) and pyramids (tetrahedron, square-based pyramid, and hexagonal-based pyramid). Students experienced few difficulties imagining the cross-sections of the shapes and readily used this criterion for their classification. Students were reminded to first sit their shape on the base and use this orientation to serve as a guide for where the horizontal cut would be made. This task provided insights into students' ability to identify prisms and distinguish between pyramids and prisms. It also provided the opportunity to explore another distinguishing feature of prisms and pyramids relating to the shape and orientation of the faces.

Figure 6. Sorting 3D shapes.

The following is a discussion that took place with one group. The group had classified their shapes into pyramids and prisms and was exploring the faces of each shape in the respective groups.

Teacher: What do you observe about the faces of the prisms and pyramids?

Seamus: They all have a different number of faces. But, on the pyramids all the sides are triangles and they are slanted, like diagonally (orienting his hand to mimic the angle of the side).

Teacher: Do you all agree with Seamus or have anything to add?

Fiona: On the pyramid the triangle sides slant and meet together at the top of the pyramid. In a point.

Teacher: How are the faces of the prism different from the pyramid?

Sarah: In the prism, the sides aren't triangles.

Fiona: And they are not slanty, they are straight. Vertical.

Creating nets for pyramids and prisms

Net construction is a complex visualisation task that requires students to make translations between 3D objects and 2D nets by carefully studying and moving between the component parts of the object in both two- and three-dimensional space. Nets have been found to support primary students in observing characteristics of 3D shapes (Mann 2004) and aid pre-service primary teachers in making conjectures about the area, perimeter and fold lines of cube nets (Jeon 2009).

While most students were familiar with nets, the teacher reminded them that the net of a 3D shape is what the shape would look like if it were opened out flat. The teacher stated: "If you are an architect, a net of the building is extremely valuable as it provides a 'map' of the shape of the exterior walls and of the ground and top floors." To illustrate the relationship between a 3D shape and its net, the NCTM Illuminations tool (NCTM, nd) (http://illuminations.nctm.org/Activity.aspx?id=3521) and a model of a cube were used. The Illuminations tool supports the display of a 3D object, its rotation, and the subsequent unfolding to form a net. The teacher selected a cube on the Illuminations tool, unfolded the cube (virtually) to form the net and then reassembled the net to form a cube (Figure 7, see image on white board). The focus then turned to a cube made with Polydrons (these are construction materials that are used to build 3D structures). The teacher carefully opened the cube to show the net (figure 7, see material in teachers hand). The teacher then repeated the process making a different net the second time and concluded by stating that there are 11 possible nets for a cube and similarly other 3D shapes have several different nets.

A variety of 3D shapes (tetrahedron, square-based pyramid, and a square-based prism) were distributed. All shapes were made using Polydrons as these structures are easily opened to show nets and then reassembled into the 3D structures. Students were encouraged to first observe the shape and predict the net (Figure 8). Then, they opened the shape and sketched the net (Figure 9). They were then encouraged to check their prediction and discover and sketch as many other nets for the same shape as possible (Figure 10). Students discussed with their partner why they might have drawn different nets. Each time a new net was discovered, students were encouraged to draw it and then use this net and reassemble the 3D shape.

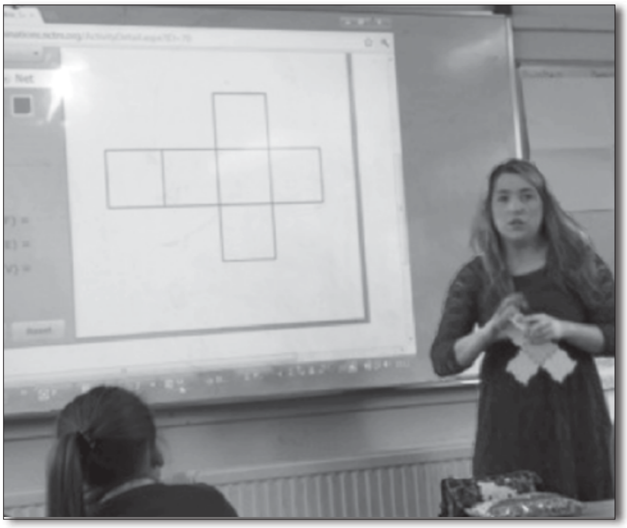


Figure 7. Explaining nets using the NCTM illuminations tool and polydrons.

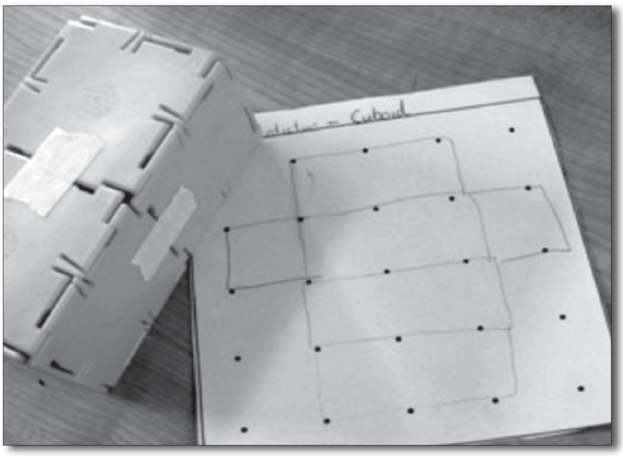


Figure 8. Predicting the net of a square-based prism.

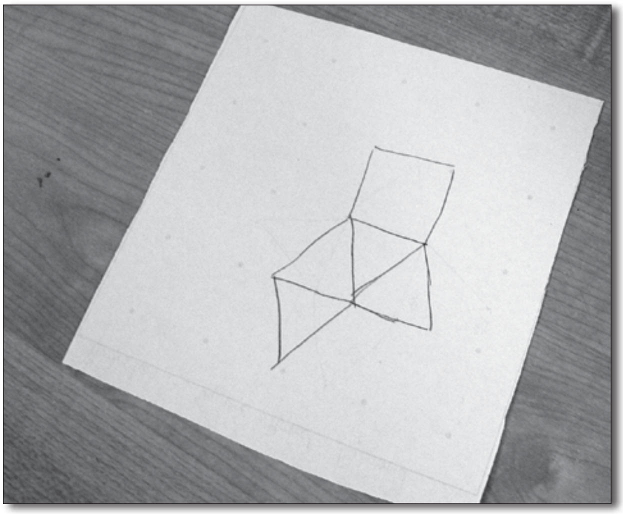


Figure 9. Sketching the net of a tetrahedron based on the open 3D shape.

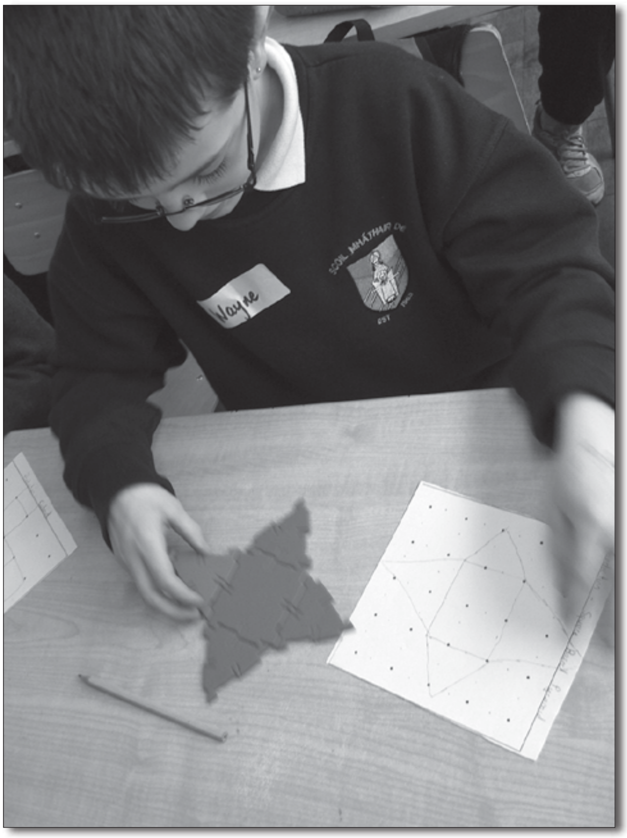


Figure 10. Alternative net for a square-based pyramid.

Conclusion

The study of 3D shapes can often be impoverished with respect to the use of a rich variety of materials, as compared to other geometric concepts such as 2D shapes and symmetry. This paper demonstrates how a context can be used to motivate students to discover and use the properties of 3D shapes. In this case, the context facilitated students to become aware of the properties which are common among all prisms and pyramids, as well as the distinguishing features between the two figures. It also provided a realistic justification for creating and examining the nets of these shapes. The paper describes how students in Year 6 can actively explore the properties of 3D shapes using a mix of traditional and non-traditional materials alongside the purposeful use of technology. It is possible that these activities could be usefully modified for other class levels or the instruction accelerated to support gifted learners. It is our experience that the Van Hiele Model serves as a valuable tool in supporting teachers develop a range of expectations for students of different abilities thus supporting differentiation in the classroom.

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