Using a simulation to explore the Law of Large Numbers

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Introduction

Let's imagine we flip a coin. There are two probable outcomes – a head or a tail. As long as our coin is a fair coin then both outcomes are equally likely. However, we know from experience that just because both outcomes are equally probable, it does not mean that when we flip a coin 10 times we will get heads 50% of the time and tails 50% of the time. Figure 1 below shows the outcome of flipping a virtual coin 10 times – the outcome is far from what we would expect from the generating theoretical probabilities. As you can see we got a head 7 times and a tail 3 times.

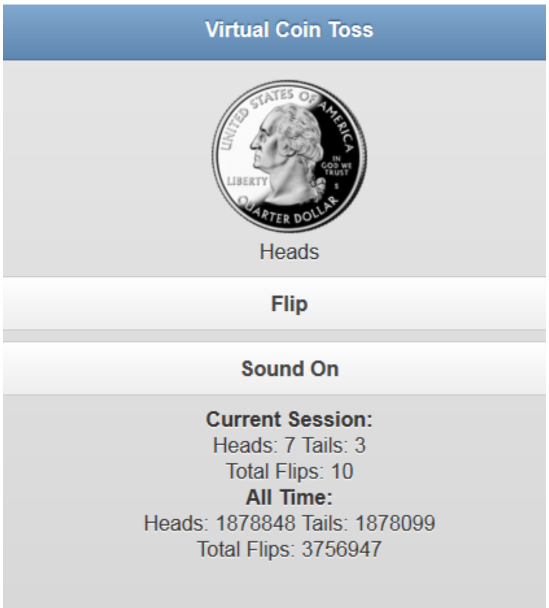


Figure 1: Virtual coin flip 10 times
http://www.virtualcointoss.com/

However, if we flip the coin a very large number of times (say 10000 times), the possible outcome is a lot more predictable than if we performed the trials a small number of times (say 10 times like we did above). If you examine image 1 again, you can see that the coin has been flipped in total 3,756,948 times and heads appeared 1,878,848 times and tails 1,878,099 times. We can determine that the small sample of 10 flips generated heads 70% of the time, in contrast to the large sample of 3,756,948 flips where heads appeared 50.009% of the time. In statistical terms, as the sample size increased, the probability of getting a head or tail (i.e. a random variable) came closer to the expected probability for the whole population i.e. the theoretical and experimental probability move closer together. In other words, as a sample increases in size the sample mean will get close to the expected value of the population as a whole (i.e. the population mean). **This is known as the Law of Large Numbers (LLN).**

The Law of Large Number in School Mathematics

The Law of Large Numbers is a focus of the Irish second level school probability and statistics curricula. The tenets of the law are a focus of study in the Senior cycle handbooks (http://www.projectmaths.ie/teachers/strand1-senior.asp in Strand 1 of Project Mathematics). The suggested activities aim to support students in determining "*the relative frequency for each outcome by experiment and note how it approaches the theoretical probability as the number of trials increases*".

However, there is potential to engage children, as young as 5th and 6th class of primary school, in investigations of the Law of Large Numbers. These children possess the fundamental skills required to explore the effect of increasing sample size. Such skills involve the ability to "*estimate the likelihood of occurrence of events,* to *construct and use frequency charts and tables* to *record the outcomes of experiments carried out a large number of times*" (PSMC, p. 111).

This study is part of Lesson Study research carried out by the authors of this guide (Leavy 2010; Leavy, Hourigan & McMahon 2013).

The sequence of activities presented would be suitable for junior cycle or transition year mathematics also.

Sequence of instruction

Warm up: reviewing probability concepts

We designed a simple activity to provide practice in recording probabilities, to review the language of uncertainty and engage students in discussions relating to the outcomes of random events. This introduction may not be necessary with older students.

Students were placed in small groups. A bag of six counters was displayed: 3 yellow and 3 red. The teacher mixed up the counters in the bag and said he was going to select one counter. He then posed a series of questions:

- What are the possible outcomes? What colour might the counter be?
- What are the chances of choosing a yellow counter? Why?
- Do you think this is a fair game? Is there an equal chance of getting a yellow or a red?
- Let's agree that this is a fair game and the chances of choosing a yellow square are 50:50 or ½ of the time. If I play the game in exactly the same way 6 times, how many times do you think I will choose a yellow square?

Students discussed the questions in their groups and recorded their answers. They then played the game 6 times. Each time they were instructed to shake the bag, reach in without looking, choose a counter and record its colour on their recording sheet, and then put the counter back into the bag (Figure 2).

Figure 2 Group completing warm-up game

When they had played the game 6 times, students added up their totals (number of red and yellow counters). The following discussion occurred with one group:

Teacher	If I was to do this six times, how many yellows would you expect to get?
Mark	Three yellows. Because there are 3 yellow and 6 counters altogether.
Cian	Well you might get 3 yellows. But you don't have to. You could get 4 reds and 2 yellows.

Children engaged in the activity. Six red and zero yellow counters were removed.

Teacher	Our prediction was 3 yellows and we got 6 reds. Do you have any idea why that happened?
Rebecca	We were just unlucky.
Teacher	If we were to play it again would we get the same result?
Rebecca	No. We mightn't be unlucky that time.
Teacher	Is this a fair game?
Alan	Well there are 3 reds and 3 yellows. So you could get a red or a yellow but we aren't guaranteed to lose or to win.

As is evident from the transcript, these primary children can estimate the likelihood of the occurrence of an event and also realise that they cannot be certain of the outcome of a random event.

The Spinner Activity

We used the electronic spinner available at http://illuminations.nctm.org/ActivityDetail.aspx?ID=79. This activity allows students to gain experience in recording the outcomes of a random event. The use of the online virtual spinner provides the opportunity to increase substantially the number of trials and record the outcomes in a very short amount of time. Other technology was used to present the results graphically.

The spinner (a circular region) can be divided into as many regions as you wish. We divided the circle into 12 regions. Our rule was that: **The spinner has to land on a wedge in the upper half to win the game.** The game was first demonstrated on the interactive white board. The nature of the game and probability of winning were discussed (50% or ½). Then the game was played on laptops (Figure 3 and 4).

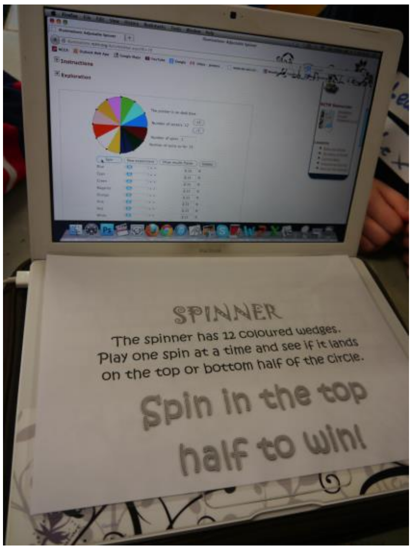
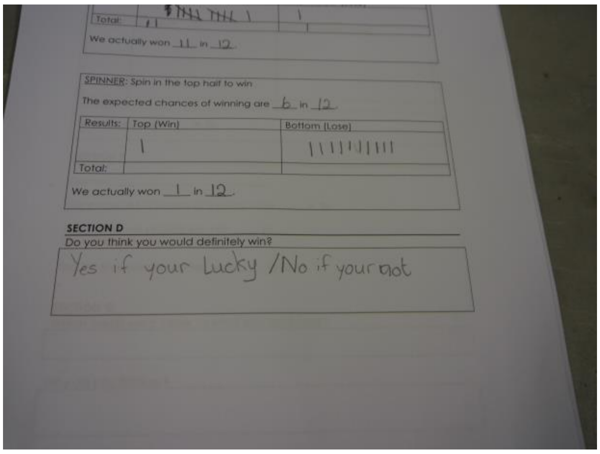


Figure 3: The virtual spinner

Students first calculated and recorded the chance of winning (6 in 12) (Figure 5). They then spun the spinner 12 times and recorded how many times the spinner landed in the top half of the circle (Figures 4 and 5).
A whole class discussion then focused on the outcomes from the spinner activity. Students became aware that different groups had different outcomes. The teacher suggested that they investigate the effect of increasing the number of spins.


Figure 4: Recording the spinner outcomes


We actually won 11 in 12.
SPINNER: Spin in the top half to win
The expected chances of winning are 6 in 12.

Results:	Top (Win)	Bottom (Lose)
	|	|||||||||||
Total:		

We actually won 1 in 12.
SECTION D
Do you think you would definitely win?
Yes if your lucky / No if your not

Figure 5: Recording the spinner outcomes

What happens when we increase the number of trials?
The teacher informed the children that they would try spinning the spinner 36 times. Prior to the activity, everyone predicted the outcome. Once finished, students compared their results from the 12 spins against the 36 spins. We asked:

- What differences did you notice between 12 and 36 spins?
- Which results were closer to the chances that we expected: the 1st time or the 2nd time? Why do you think that is? Discuss this in your groups and come up with a reason why.

Most groups found that the chance of winning the game was closer to their predictions when they spun 36 times. Some referred to the role of 'luck' which might be considered as akin to the notion of randomness. Older students are likely to come to the conclusion that the greater number of tries brings the actual outcomes (experimental probability) closer to the theoretical probability.

Does increasing the number of trials really matter?
One group's results from the 12 and 36 spins were displayed (Figure 6). The results prompted further discussion. As can be seen, while the outcome for 12 spins was not as expected (8 vs. 4), the outcomes of 36 spins was closer to the expected probability of ½.

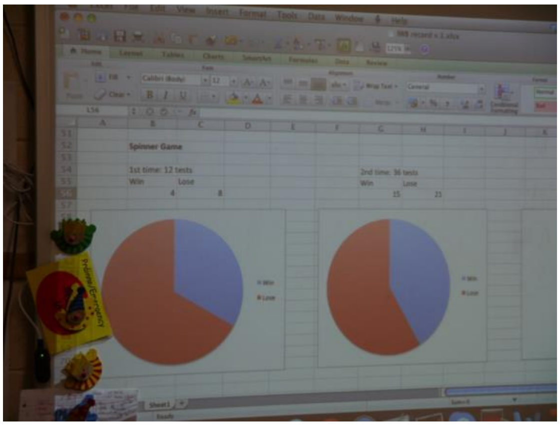


Figure 6: Comparing the outcomes of 12 and 36 spins

The teacher told students that they were going to try make the experimental probability move as close as possible to the theoretical probability by increasing the number of trials again.

The class were then told that they were going to try a really large test and spin the interactive spinner 1000 times. The students were encouraged to predict how many times they expected to win. Students will likely predict that the outcome will be close to 500 times; and, again, younger students may refer to the role that luck would play. The teacher projected the Illuminations online spinner on the whiteboard, completed 1000 spins and represented the new results alongside the original investigations (Figure 7).

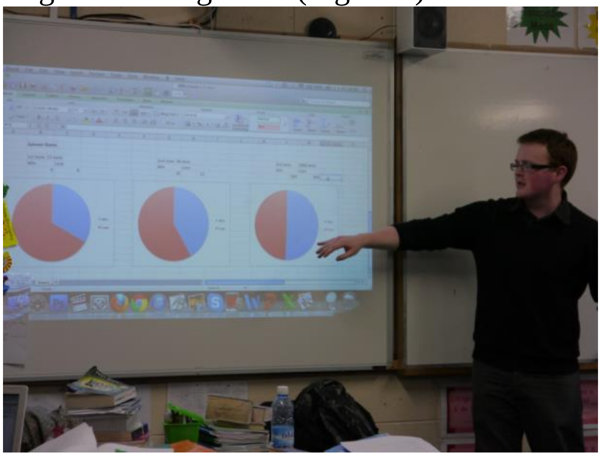


Figure 7: Comparing the outcomes from 12, 36 and 1000 spins.

We used the following questions to guide the classroom discussion:

- When we spun 1000 times, how many times did we win?
- In your groups, compare our large test with our small test and see which one was closer to the expected value of 50:50.
- The outcome of which test was closer to our expected chances of winning?

Students were able to confidently and competently utilize the language of probability in these final discussions, could readily identify the difference between theoretical (expected) and experimental (actual) results, and explain the effect of larger trials on the relationship between theoretical and experimental probability.

Reflections

Our research found that although this concept is not on the primary curriculum, senior primary students were ready to consider the discrepancy between theoretical and experimental probability and the impact of large trials on this relationship. This readiness was to a large extent due to the fact that these children had ample opportunities to use the language of chance to predict, describe and compare outcomes of practical experiments. In addition, the use of visual images supported developing their understanding.

References

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