Mathematics teaching matters: Making complex mathematical ideas accessible to primary level children.

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MATHEMATICS TEACHING MATTERS: MAKING COMPLEX MATHEMATICAL IDEAS ACCESSIBLE TO PRIMARY LEVEL CHILDREN

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Reform of mathematics curricula has led to the expansion of topics such as algebra, data and probability being taught to primary level children. This new subject matter can be challenging for primary teachers to teach as some teachers have not previously engaged with these topics as students themselves and also some of these areas have traditionally been considered secondary school topics. Furthermore, while there is a wealth of concrete manipulatives available to teach topics such as number and shape and space there is less so for these 'newer' areas of mathematics curricula. This paper reports on the combined efforts of teacher educators, teachers, and pre-service teachers to conceptually engage primary children with algebraic concepts, function and variable, through the use of appropriate models. Lesson Study was the primary method used to support a focus on examining teaching and the effectiveness of models through the design and implementation of 'study lessons'. Insights into children's learning of algebraic concepts are presented in addition to an examination of how good teaching using manipulatives can lead to powerful mathematical reasoning by young children. Video evidence of the findings will be presented and discussed.

INTRODUCTION

The field of mathematics education is dynamic. There has been considerable debate internationally regarding what constitutes *important and powerful* mathematics in primary education. For the most part of the last century, computation skills were considered as a critical foundation for mathematical development into the later years. In reaction to this focus on computational skills there has been considerable emphasis placed on problem solving, on the balance between instrumental and relational understanding (Skemp, 1971) and on procedural and conceptual understanding (Hievert & Lefevre, 1986). Furthermore, there has been debate regarding the nature of the mathematics content considered suitable for primary children. Examination of the national curricula of Ireland, Australia, and the United States (AEC, 1990; DES, 1999; NCTM, 1989) highlights the broadening of mathematical goals through the inclusion of Data Handling, Probability, and Algebra as new foci of study in the past two decades. These developments which emphasize relational and conceptual understanding, combined with the broadening of the mathematical content of primary level curricula, have implications for teaching. Traditional modes of instruction, which draw on didactic approaches and value instructional methods such as drill and practice, do not provide sufficient support for the development of the types of mathematical understandings valued in the new millennium.

PROVIDING ACCESS TO POWERFUL MATHEMATICAL IDEAS

Mathematics education research has been referred to as a 'design science' (Wittman, 1998) wherein a central activity is to concentrate on constructing 'artificial objects' (p. 94) and investigating the differential effects of these objects on the development of mathematical understanding. Such research involves the design, testing, and the process of reiteration and modification of objects and the subsequent measurement of the impact of these objects on learning. The literature suggests a number of ways children can be supported in accessing powerful mathematical ideas with an emphasis on both objects and instructional approaches; all with a view to developing good teaching practices in classrooms. While there are many tools (emerging technologies, for example) and approaches used in primary classrooms to develop understanding of complex mathematical concepts, we briefly outline two approaches we drew on in the design of lessons that supported primary children in gaining access to algebraic concepts: models and representations.

The use of *models and manipulatives* in primary mathematics education is well established, particularly in relation to place value ideas and number work. There is evidence, however, that the manipulatives are not automatically helpful in the development of mathematical understanding. Young children need support in making connections from the analogies embodied by the tools/manipulatives to the mathematical ideas. The following quote exemplifies the thinking around the use of models in Realistic Mathematics Education (RME) wherein 'the starting point is in the contextual situation of the problem that has to be solved .. The premise here is that students who work with these models will be encouraged to (re)invent the more formal mathematics' (Gravemeijer, 1999, p. 159). Research examining children's understanding of mathematics has also identified useable conceptual *representations* that are accessible to children. Valuable representations are being generated that support learning in the emerging areas of primary mathematics, in particular with regard to algebraic thinking (Bellisio & Maher, 1998), data exploration (Curcio, 1987; Jones et al., 2000) and probability (Watson & Moritz, 1998).

SUPPORTING TEACHERS IN DEVELOPING CHILDREN'S MATHEMATICAL UNDERSTANDING

Reform of mathematics curricula has broadened the scope of the mathematics content being taught at primary level and provided a spotlight on a variety of mathematical processes deemed critical to the development of mathematical understanding. The introduction of new mathematics content, such as data and probability, means that some of our teachers are teaching mathematics content that they themselves were not exposed to as learners. Teachers' lack of familiarity with new mathematics can pose challenges for the fidelity of the implementation of revised national curricula. Research in the United States found that, following reform of the mathematics curriculum, the actual mathematical experiences of elementary students reflected little of the experiences of the intended curriculum. Discrepancies between desired and implemented curricula, which frequently occur during the implementation of reform curricula, are not a new problem in mathematics education and neither is it limited to the Irish education system. It is not surprising then, that reviews of the implementation of the Primary School Mathematics Curriculum (NCCA, 2005) identified that teachers feel underprepared to teach certain areas of mathematics. Teaching is a complex activity. Teaching mathematics at the primary level is particularly challenging as teachers are not mathematics content experts i.e. the majority of primary school teachers do not have mathematics to degree level. Primary teachers need considerable support navigating the multiple terrains that contribute to the mathematics education of children. This terrain incorporates attention to learning theory, cognitive development, assessing understanding, and the use of models and tools which help children gain access to powerful mathematical ideas.

This paper focuses on one aspect of good mathematics teaching - the design and selection of tools that mediate learning. We examine the mathematics content area of algebra; a domain traditionally considered to ‘belong’ to the domain of secondary level mathematics. This focus was in part motivated by the results of the 2004 National Assessment of Mathematics Achievement which reported that 38% of inspectors were dissatisfied with the availability of resources for teaching Algebra in fourth class. Furthermore, a sizeable 60% of inspectors were dissatisfied with the use of resources for teaching this same strand (Shiel et al., 2006).

METHOD

Participants

This study was carried out with 21 final year pre-service primary teachers during the concluding semester of their teacher education program. Participants had completed their mathematics education courses (three semesters) and all teaching practice requirements (at junior, middle and senior grades) and self selected into mathematics education as a cognate area of study. Four students were male; the remainder were female. Two students were international Erasmus students.

Lesson Study

All pre-service teachers, and three mathematics educators, engaged in *Japanese Lesson Study* (Fernandez & Yoshida, 2004; Lewis, 2002; Lewis & Tsuchida, 1998). Lesson Study is an approach for studying teaching that utilizes detailed analyses of classroom lessons. Lesson study was used in this study to facilitate the examination of both the planning of lessons and the implementation of those lessons in classrooms and thus provided an avenue to design tools and sequences of instruction to support the development of algebraic reasoning with primary children.

The research was conducted over a 12-week semester. The central activity in lesson study was for participants to work collaboratively on the design and implementation of a study lesson. Participants were organized into groups of 5-6 to engage in the phases of Lesson Study. The first phase involved the *research and preparation* of a study lesson involving researching algebraic topics and the construction of a detailed lesson plan. The *implementation* stage involved one pre-service teacher teaching the lesson in a primary classroom while the other group members, and the researchers, observed and evaluated classroom activity and student learning. Group members then *reflected on and improved* the original lesson design through discussing their classroom observations and modifying the lesson design in line with their observations. The *second implementation* stage involved re teaching the lesson with a second class of primary students and *reflecting* upon observations. The second implementation was videotaped. The cycle concluded with in-class presentations of the outcomes of each of the four lesson study groups.

EXPLORING ALGEBRAIC CONCEPTS IN THE PRIMARY CLASSROOMS

This paper reports on the work of two lesson study groups. Both groups were focusing on algebraic concepts more often associated with secondary school mathematics: function and variable. Research has shown that both concepts, however, can be introduced in primary classrooms provided that adequate supports are provided to children when reasoning about these concepts. We provide a description of the teaching of both concepts, along with a focus on the tools used to facilitate the development of algebraic reasoning, as an argument that 'Good Teaching Matters'.

1. Exploring the concept of 'function'

Function and the Primary School Mathematics Curriculum (PSMC)

While a formal treatment of function does not occur until secondary school in the Irish education system, the concept of function can be introduced informally in the primary classroom. Indeed, function can be introduced in tandem with work on patterns and relations. Work examining relationships with numerical patterns and analysis of how patterns grow and change can be used as a precursor to informal work with functions. However, for primary level children to develop a conceptual understanding of function and successfully analyze the change that comes about as a result of a function, they must be provided the opportunity to model problem situations with objects and use representations to draw conclusions. Within the context of functions, we use the 'function machine' to model the problem situation and provide the use of simple tables to represent the change that occurs as a result of the function.

The function machine

Function machines have been used in mathematics education to support children's understanding of functions. We used a physical model of a function machine, as opposed to traditional pictures or diagrams, to support primary level children in identifying important characteristics of functions. For younger learners, it was critical that these characteristics were represented concretely and explicitly identified. As can be seen in the images below, the function machine itself was a relatively unsophisticated object with a clearly identified input (image 2), output (image 3) and buttons which represent the function (image 1).

Image 1: Front of function machine with input drawer

Image 2: Input section of the function machine (front)

Image 3: Output section of the function machine (back)

Introducing the concept of function to children

The concept of function was introduced using the analogy of baking in a 'faulty oven'. The following scenario is presented by the teacher: *'Yesterday something very strange happened to me! Well… I love baking and yesterday I made 2 cakes* (picture/photo displayed on the white board using projector) *and put them into the oven. When I took the cakes out something strange had happened to them, when I opened the oven there seemed to be more than 2 cakes. Look what I found! There were 4.'* The teacher presented a number of such scenarios from the faulty oven and encouraged pupils to identify the rule/pattern which determined the output from the oven. Links were then made between the 'faulty oven' and the idea of a function/rule machine, the teacher stated *'I soon realised that my oven isn't an oven at all…It's a magic machine called the 'Maths Rule Generator'. Do you want to see it?'*

Developing the concept of function

The 'Function machine' was introduced to children and the language of function was developed through guided exploration of the function machine. The teacher then lead the class through a series of activities the objective of which was to help children explore the relationship between input, output and functions/rules. Prepared materials were placed in the input slot by a child, the function button was then pressed, and the teacher removed the 'baked' materials from an output slot at the back of the box. The teacher leads the class in trying to figure out the rule for each of the buttons. In each case the children were encouraged to predict the rule by looking at the relationship between the input and the output. As a new pair of values (input and output) were presented, such predictions were tested (confirmed or rejected). Throughout the lesson, the input, output, and rule (both guess and actual) were represented on a recording sheet. During the demonstration the rules for the various buttons varied. As this was the children's first introduction to functions, we slowly increased the complexity and challenge as the lesson progressed. Initial 'rules' focused on addition and multiplication (e.g. orange (x2 or double), blue (+15), green (x9)). In order to facilitate challenge, the final button (yellow in this case) was a two-step rule or function (x2+3).

Primary children writing functions

Children were given the opportunity to generate and identify rules/functions while working in collaborative groups. Groups were provided with record sheets, these sheets reflected many of the features of the original function machine – they used the same language (input/output) and contained the same perceptual cues to identify rules/functions (coloured buttons similar to that found on the function machine). One child in each group acted as the 'rule generator'. Others in the group providing inputs (between 0 and 10) and the 'rule generator' worked out and shared the output (using a secret rule e.g. +5). The information was recorded on the recording sheet thereby facilitating prediction and checking of the rule.

As previously outlined only one example of a two-step rule or function was taught during the lesson. However when creating functions pupils were encouraged to devise a two-step rule if they wished. The following is a transcript of the class trying to solve a two-step problem created by one of the children following the work on functions. This example highlights that teaching mathematics using effective models and representations helps develop sophisticated mathematical reasoning and understanding among children.

Teacher	Robert has made up this two-step rule that only works for even numbers. Would someone like to give an input?
Cian	2
Robert	Output is 7
	Robert records 2 as the input and 7 as the output on the white board
Calum	10
Robert	Output is 11
	Robert records 10 as the input and 11 as the output on the white board
David	4
Robert	8
	Robert records 4 as the input and 8 as the output on the white board
Teacher	Can anyone guess Roberts' rule?
	Children nod their heads to indicate they cannot find the rule
Teacher	We'll give you a hint. The first step is addition and the second step is division.
	Children nod their heads to indicate they cannot find the rule
Teacher	The first step is plus 12.
	Children raise hands
Laura	Plus 12 divided by 2.
Robert	You are right.

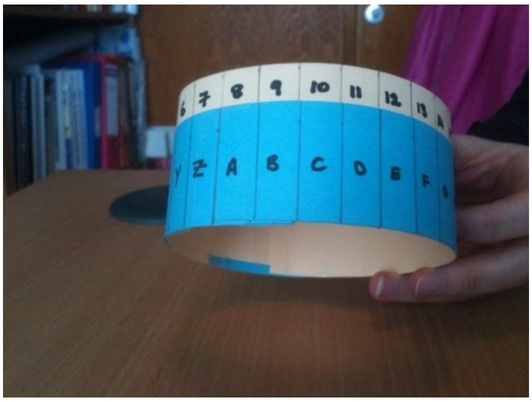
2. Exploring the concept of ‘variable’

Variables are an important concept in algebra. Variables are symbols (e.g. letters/frames) that take the place of numbers or ranges of numbers (Van De Walle, 2004). The National Council of Teachers of Mathematics (2008) outlines three key ideas of variables namely: variables as unknowns e.g. $5 + y = 30$; variables as quantities that vary in joint variation e.g. $A+B =24$; and variables in generalizations of patterns e.g. $a + b = b + a$. For the purposes of this study the first idea of variables, i.e. variables as unknowns, was the focus of study. This aligns with the objectives for this area of Algebra in the Primary School Mathematics Curriculum (1999). To develop primary children’s conceptual understanding of variables it was necessary, like when introducing functions, to present the concept in a concrete and accessible manner. The variable wheel was chosen as a useful model for primary level children. This was chosen as it is a concrete manipulative which introduces the concept of a variable in a fun, interesting and age-appropriate manner, while simultaneously modelling the concept of a variable as an unknown.

The variable wheel

The variable wheel is essentially two strips of paper, one with letters on it and the other with numbers on it. The letter strip is wrapped around the number strip so that the letters align to a number. These strips can be rotated so that any one variable (letter) can have a variety of different values depending on the rotation (images 4 and 5). We commenced activities by having A aligned with 1, B aligned with 2, etc. The variable wheel allowed children to explore the idea of a variable as a letter that can stand for any member of a set of numbers and also enabled children to substitute numbers for variables.

<u>Image 4</u> : Variable Wheel



<u>Image 5</u>: Rotated Variable Wheel

Introducing the concept of a variable to children

The idea of a variable was introduced initially by eliciting from children what they thought the word ‘variable’ meant. To help children the teacher asked questions such as ‘*Does anybody know what it means to vary? For example, the weather varies, or the speed of the car varies on your way to school.’* The first activity carried out with the variable wheel was to have children find the value of the letters in their name. The children found the value of their names, using the variable wheel, e.g. Amy was 39 because A=1, M=13 and Y=25. Answers were compared to their predictions and outcomes discussed.

To introduce the concept of a variable, children were then asked if they thought the results would be the same if they **varied** the values on their variable wheel and worked backwards letting a = 26 and z =1? This was an important step in the lesson as we wanted children to realise that a variable is a letter that can represent **any** number depending on the problem (i.e. a=1 in one problem but a=17 in another problem). Children were allowed to twist the variable wheel and come up with a new value for their name. From this, it was hoped that the children would realise that any letter can represent any number and vice versa and that letters (i.e. variables) don't have to be set values.

In the initial lesson, on completing the above task, pupils demonstrated a misconception that a letter always held the same value e.g. a = 1, e = 5. On reflection it was noted that there was insufficient emphasis placed on the idea of variable and values of letters varying. As a result during the second teaching of the lesson the students placed a greater emphasis on the concept of variables when using the variable wheel. It was found that children grasped the concept easily as a result.

Development of the concept of a variable

The development of the lesson consisted of using relevant story problems that required a variable to solve them. Contexts suitable for primary school students include sport, games, clothes, food, etc. In each problem there was an unknown value to be found. Children were allowed to pick any letter to represent the unknown. A simple table of what was known was drawn up. A number sentence was then written and solved (linkage with equations).

In the first problem the context of sport was chosen. The following problem was presented: '*There were 76 points scored altogether in a rugby match. Ireland scored 40 points. How many points did France score?'* This information was transferred onto a simple table by the teacher and children (see figure 1). Children were told '*As we do not know how many points France scored we can use a letter/variable to represent how many points France scored.'* Children were asked *'What letter might we use as the variable in this problem for France?'* A suggestion was F. This was written in the table (see figure 2). The next step was to write the story problem as a number sentence using the chosen variable i.e. 40 + F = 76. Children solved F = 36. A number of more challenging but similar problems were then presented to provide practice and consolidate the learning related to variable.

Ireland	France	Total
40		76

Figure 1: Table for a variable problem

Ireland	France	Total
40	F	76

Figure 2: Table (with variable) for a variable problem

Primary children writing variable problems

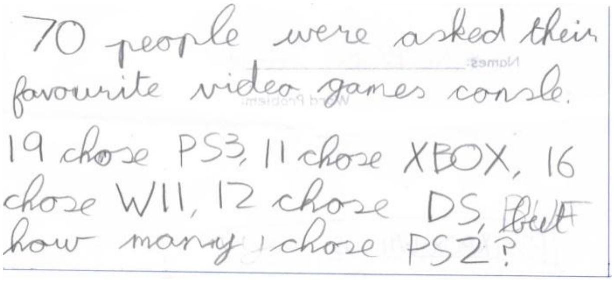
Following whole class work solving story problems using variables the children were provided with an opportunity, in pairs, to create their own story problems that required a variable to solve. To motivate the students we told them that their story problems would be placed in envelopes and given to another class or another group in their class to solve. However, it was important to model to the class how to write story problems and clear directions were given. One way of modelling how to create a story problem was to use information from the class itself and fill in a prepared chart.

There are ______ in a class.
_______ like soccer.
_________ like swimming.
How many children like rugby?

A class survey was conducted and the prepared chart was completed. This made the problem context meaningful for the children. In addition it linked to the data strand of the curriculum.

Figure 3: Model for a story problem

The following is one example of a story problems created by fourth class pupils in a Limerick City primary school.

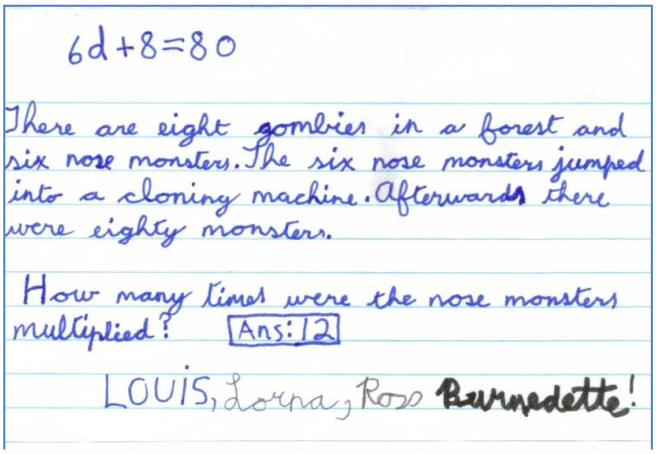

70 people were asked their favourite video games consle. 19 chose PS3, 11 chose XBOX, 16 chose WII, 12 chose DS, how many chose PS2?

70 people were asked their favourite video games console. 19 chose PS3, 11 chose Xbox, 16 chose WII, 12 chose DS. How many chose PS2?

Figure 4: Sample story problem created by group of children

Pushing the boundaries: Extension activity

Once children solved and created story problems involving variables it was decided to challenge the children by presenting a number sentence that had a variable and asking them to write a story problem that represented (the context for) the number sentence. This involved flexible reasoning on the part of students and supported differentiation for more able students. The following are examples of two story problems written by the same group in response to two different number sentences. These examples illustrate the outcomes that good teaching can have on the development of mathematical reasoning.



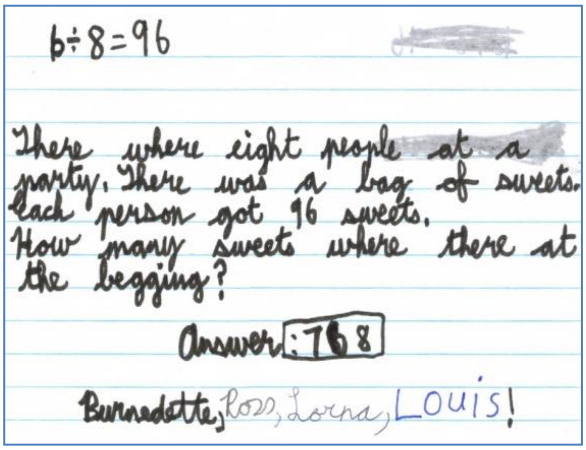
6d+8=80

There are eight zombies in a forest and six nose monsters. The six nose monsters jumped into a cloning machine. Afterwards there were eighty monsters.

How many times were the nose monsters multiplied? Ans: 12

LOUIS, Lorna, Ross Burnadette!

Figure 5: Story problem to represent *6d + 8 = 80*



b÷8=96

There where eight people at a party. There was a bag of sweets. Each person got 96 sweets. How many sweets where there at the begging?

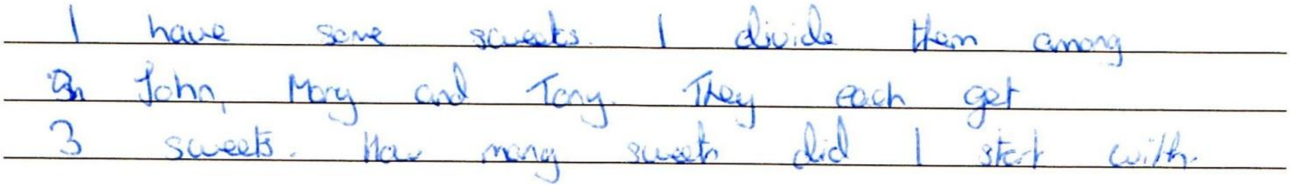
Answer: 768

Burnadette, Ross, Lorna, LOUIS!

Figure 6: Story problem to represent *b÷8 = 96*

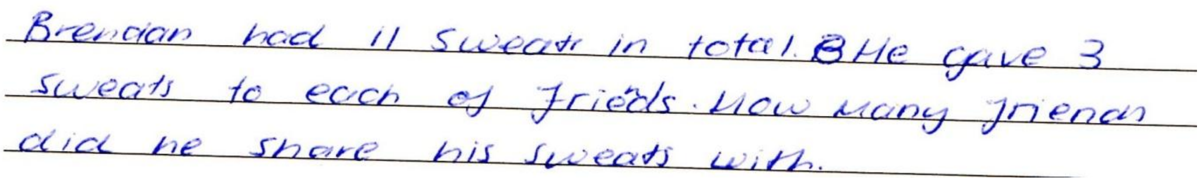
A comparison

The mathematical understandings described in the sections above are not trivial. In particular, the skills required for the group of children to write the story problems (see figure 5 and 6 above), that represented a presented number sentence, are complex. These problems were written following just one hour of instruction on variables and using the variable wheel. As a contrast we present the results of pre-service primary teachers (n=356) when asked to write a number story for the number sentence b ÷ 3 = 11 (similar to problem in figure 6). While a relatively large proportion correctly constructed a word problem (74.4%), a significant proportion of them (25.6%) demonstrated difficulty in constructing a story problem. Examples of incorrect responses from pre-service primary teachers are shown in figures 7 and 8 below.



I have some sweets. I divide them among 3, John, Mary and Tony. They each get 3 sweets. How many sweets did I start with.

Figure 7**:** Sample 1 of pre-service teacher's error



Brendan had 11 sweats in total. He gave 3 sweats to each of friends. How many friends did he share his sweats with.

Figure 8**:** Sample 2 of pre-service teacher's error

Contrast this performance with those of the 4th class children in figures 5 and 6 and we can get some sense of the extent of conceptual understanding of variable held by the 4th class children.

In conclusion

Learning occurred not only for the children involved in the study, but also for the pre-service primary teachers. Pre-service teachers commented on the development of their own mathematical understanding of algebra *'As a result of trial and error that I undertook in the lesson plan my own understanding of algebra and how to write equations improved'* (Variables group: Rosemary). The development of more positive attitudes towards algebra was also mentioned by most participants: *'Algebra is not as abstract as we originally thought, once it is presented in context'* (Variable group presentation).

The development and use of the tools and representations described in this study were critical in supporting relatively young children in reasoning about functions and variables. Our research indicates that some 4th class children exceeded the expectations generally set for algebraic reasoning at the primary level. Yet, these were ordinary Irish children in ordinary Irish classrooms. They were taught by relatively inexperienced teachers (in that they were pre-service teachers). What made the difference to the relatively sophisticated mathematical understandings that were recorded was the attention that we paid, using the Lesson Study framework, to *teaching.* This attention to teaching encompassed concentration on and consideration of pedagogical approaches, mathematical language, the scope and sequence of instruction, and most importantly within the context of algebra – the design of tools and representations that made algebraic concepts accessible to young children. Yes, good teaching matters.

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