POINCARÉ FAMILIES AND AUTOMORPHISMS OF PRINCIPAL BUNDLES ON A CURVE

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Abstract. Let C be a smooth projective curve, and let G be a reductive algebraic group. We give a necessary condition, in terms of automorphism groups of principal G-bundles on C, for the existence of Poincaré families parameterized by Zariski-open parts of their coarse moduli schemes. Applications are given for the moduli spaces of orthogonal and symplectic bundles.

Résumé. **Familles de Poincaré et automorphismes des fibrés principaux sur une courbe.** Soit C une courbe projective lisse, et soit G un groupe algébrique réductif. On donne une condition nécessaire, en termes de groupes d'automorphismes des G-fibrés principaux sur C, pour l'existence des familles de Poincaré paramétrées par des ouverts de Zariski dans leurs schémas de modules grossiers. On donne des applications pour les espaces de modules des fibrés orthogonaux et symplectiques.

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$, and let G be a reductive algebraic group, defined over an algebraically closed field. We consider algebraic principal G-bundles E on C.

Over the complex numbers, the topological type of E is given by an element $d \in \pi_1(G)$. This invariant can also be defined algebraically, in any characteristic; see Definition 2.1. According to [13, 14, 6], there is a coarse moduli scheme $M^{\mathrm{s}}_{G,d}$ of stable principal G-bundles E on C with type $d \in \pi_1(G)$.

Definition 1.1. A *Poincaré family* for an open subscheme $U \subseteq M^{\mathrm{s}}_{G,d}$ is a principal G-bundle $\mathcal{E}$ on $C \times U$ such that for every point $x \in U$, the corresponding isomorphism class of stable G-bundles contains the restriction $\mathcal{E}|_{C\times\{x\}}$.

We address the question whether such Poincaré families exist. Previous results on this question are due to Ramanan [12] in the case $G = \mathrm{GL}_n$, and to Balaji-Biswas-Nagaraj-Newstead [1] in the case $d = 0$.

Section 2 contains our main result, Theorem 2.2. It gives a necessary condition for the existence of a Poincaré family for an arbitrarily small open subscheme $\emptyset \neq U \subseteq M^{\mathrm{s}}_{G,d}$. This condition involves the automorphism groups $\mathrm{Aut}(E)$, which may seem plausible; somewhat surprisingly, it involves all of them, not just those for stable E. As corollaries, we reprove [12, Theorem 2] and generalize [1].

In Section 3, we apply this criterion to orthogonal and to symplectic bundles. It allows us to decide the question at hand in all the cases we consider.

Our main tool in the proofs is the theory of stacks, which helps us to keep track of the automorphism groups $\mathrm{Aut}(E)$ while studying moduli.

2. The main result

Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k. Let G be a connected reductive algebraic group over k. We denote by $Z_G \subseteq G$ the center, and by $T_G \subseteq G$ a maximal torus.

Definition 2.1. The abelian group $\pi_1(G)$ is the quotient of the cocharacter group $\mathrm{Hom}(\mathbb{G}_m, T_G)$ modulo its subgroup generated by the coroots of G.

Let $M^{\mathrm{s}}_{G,d}$ be the coarse moduli scheme of stable principal G-bundles E on C with type $d \in \pi_1(G)$. A stable principal G-bundle E is called *regularly stable* if its group scheme of global automorphisms $\mathrm{Aut}(E)$, which always contains the center Z_G of G, coincides with Z_G. The regularly stable locus

$$M^{\mathrm{rs}}_{G,d} \subseteq M^{\mathrm{s}}_{G,d}$$

is open, according to the slice theorem [11, 2]. We assume that $M^{\mathrm{rs}}_{G,d}$ is non-empty. In characteristic zero, this is proved in [5, Theorem II.6(ii)].

Theorem 2.2. *Suppose that there is a Poincaré family $\mathcal{E}$ for some non-empty open subscheme $U \subseteq M^{\mathrm{s}}_{G,d}$. Then every character $\chi : Z_G \to \mathbb{G}_m$ can be extended to $\mathrm{Aut}(E)$ for every G-bundle E on C with the given type $d \in \pi_1(G)$.*

Proof. Let $\mathcal{M}_{G,d}$ be the moduli stack of principal G-bundles with type d, and let $\mathcal{M}^{\mathrm{rs}}_{G,d} \subseteq \mathcal{M}_{G,d}$ be the open locus of regularly stable bundles. The canonical map

$$\pi : \mathcal{M}^{\mathrm{rs}}_{G,d} \longrightarrow M^{\mathrm{rs}}_{G,d}$$

is a gerbe with band Z_G. Shrinking U if necessary, we may assume $U \subseteq M^{\mathrm{rs}}_{G,d}$, and denote by $\mathcal{U} := \pi^{-1}(U) \subseteq \mathcal{M}^{\mathrm{rs}}_{G,d}$ the corresponding open substack.

The family $\mathcal{E}$ of principal G-bundles on C has a classifying morphism

$$c_{\mathcal{E}} : U \longrightarrow \mathcal{M}_{G,d}.$$

Since $\mathcal{E}$ is a Poincaré family, $c_{\mathcal{E}}$ factors through $\mathcal{U}$, and defines a section for the restricted gerbe $\pi|_{\mathcal{U}} : \mathcal{U} \to U$. Due to [10, Lemme 3.21], the existence of such a section implies that $\mathcal{U}$ is isomorphic to the 'trivial' (neutral) gerbe $U \times BZ_G$ over U. In particular, the character $\chi : Z_G \to \mathbb{G}_m$, which is nothing but a line bundle on the classifying stack BZ_G, defines by pullback a line bundle $\mathcal{L}$ on $\mathcal{U}$.

Let $\mathcal{U}' \subseteq \mathcal{M}_{G,d}$ be a quasi-compact open substack containing $\mathcal{U}$. Since $\mathcal{U}'$ is a smooth Artin stack of finite type over k, we can extend $\mathcal{L}$ to a line bundle $\mathcal{L}'$ on $\mathcal{U}'$. More precisely, [10, Corollaire 15.5] allows us to extend $\mathcal{L}$ to a coherent sheaf on $\mathcal{U}'$, namely to a coherent subsheaf of $j_*(\mathcal{L})$, where $j : \mathcal{U} \hookrightarrow \mathcal{U}'$ is the inclusion; using smoothness, the bidual of this coherent sheaf of rank one on $\mathcal{U}'$ is the required line bundle $\mathcal{L}'$ extending $\mathcal{L}$.

For each point $[E]$ in $\mathcal{U}'(k)$, the group $\mathrm{Aut}(E)$ acts on the fiber of $\mathcal{L}'$ by the definition of line bundles on stacks; hence we obtain a character $\mathrm{Aut}(E) \to \mathbb{G}_m$. If the point $[E]$ lies in $\mathcal{U}(k)$, then this character extends $\chi : Z_G \to \mathbb{G}_m$ by construction; since $\mathcal{U}'$ is connected and $\mathrm{Hom}(Z_G, \mathbb{G}_m)$ is discrete, the same holds for all points $[E]$ in $\mathcal{U}'(k)$. Varying $\mathcal{U}'$, we thus see that χ extends to $\mathrm{Aut}(E)$ for every G-bundle E with type $d \in \pi_1(G)$. □

We first apply this in the case $G = \mathrm{GL}_n$ of vector bundles E on C. Here the type $d \in \pi_1(G) = \mathbb{Z}$ of E is just the degree $\deg(E) \in \mathbb{Z}$.

Corollary 2.3. *If n and d have a common factor $h > 1$, then there is no Poincaré family for any non-empty open subscheme $U \subseteq M^{\mathrm{s}}_{\mathrm{GL}_n,d}$.*

Proof. Choose a vector bundle F of rank n/h and degree d/h on C, and let $E := F^h$ be the direct sum of h copies. Then $\mathrm{Aut}(E)$ contains GL_h, so every character of $\mathrm{Aut}(E)$ restricts to some power of $\det : \mathrm{GL}_h \to \mathbb{G}_m$. Hence the character $\chi = \mathrm{id} : Z_G = \mathbb{G}_m \to \mathbb{G}_m$ doesn't extend to $\mathrm{Aut}(E)$. □

Remark 2.4. Similar arguments apply to vector bundles with fixed determinant, and yield another proof of [12, Theorem 2]. This proof is related to the one given in [4, §5].

The next application is about principal G-bundles of trivial type $d = 0 \in \pi_1(G)$. We obtain the following generalization of [1]:

Corollary 2.5. *There is no Poincaré family for any non-empty open subscheme $U \subseteq M^{\mathrm{s}}_{G,0}$, unless G is a product of a torus and a group with trivial center.*

Proof. We take E to be the trivial G-bundle on C, so $\mathrm{Aut}(E) = G$. If every character $\chi : Z_G \to \mathbb{G}_m$ extends to G, then G has to be such a product. □

Lemma 2.6. *Suppose that G is a product of a torus and a group with trivial center, and let $d \in \pi_1(G)$ be arbitrary. Then there is a Poincaré family for $M^{\mathrm{rs}}_{G,d}$.*

Proof. If the center $Z_G \subseteq G$ is trivial, then $\mathcal{M}^{\mathrm{rs}}_{G,d} \cong M^{\mathrm{rs}}_{G,d}$, so the universal G-bundle on $C \times \mathcal{M}^{\mathrm{rs}}_{G,d}$ yields a Poincaré family for $M^{\mathrm{rs}}_{G,d}$. If $G \cong \mathbb{G}_m$ is a torus of rank one, then $M^{\mathrm{rs}}_{G,d} \cong \mathrm{Pic}^d(C)$ is known to admit a Poincaré family as well. The general case follows by decomposing G into a product of such groups. □

3. Applications to orthogonal and symplectic bundles

We apply the above to the case $G = \mathrm{SO}_n$ of orthogonal bundles. If $n \geq 3$, then $\pi_1(G)$ has two elements. For odd n, the center of SO_n is trivial; this case is covered by Lemma 2.6. For even n, we obtain as a special case of Corollary 2.5:

Corollary 3.1. *If $n \geq 4$ is even, then there is no Poincaré family for any non-empty open subscheme $U \subseteq M^{\mathrm{s}}_{\mathrm{SO}_n,0}$.*

Lemma 3.2. *Assume* $\operatorname{char}(k) \neq 2$. *If* $n \geq 4$ *is even, and* $d \in \pi_1(\mathrm{SO}_n)$ *is nonzero, then there is a Poincaré family for some open subscheme* $\emptyset \neq U \subseteq M^{\mathrm{rs}}_{\mathrm{SO}_n,d}$.

Proof. Let κ be a theta characteristic for C. We denote by $\mathcal{L}_\kappa^{\mathrm{Pfaff}}$ the corresponding Pfaffian line bundle on $\mathcal{M}_{\mathrm{SO}_n,d}$, as constructed in [9, (7.8)]; its fiber over an SO_n-bundle with underlying vector bundle E is $\det \mathrm{H}^0(E \otimes \kappa)$. Assuming that the given type $d \in \pi_1(\mathrm{SO}_n)$ is nontrivial, $\dim \mathrm{H}^0(E \otimes \kappa)$ is always odd.

Let $n \geq 4$ be even; then $Z_{\mathrm{SO}_n} \cong \mu_2$, and its action on the fibers of $\mathcal{L}_\kappa^{\mathrm{Pfaff}}$ is nontrivial. We denote by $\mathcal{E}^{\mathrm{univ}}$ the underlying vector bundle of the universal SO_n-bundle on $C \times \mathcal{M}_{\mathrm{SO}_n,d}$. This μ_2 acts nontrivially on the fibers of $\mathcal{E}^{\mathrm{univ}}$, and hence trivially on the fibers of $\mathcal{E}^{\mathrm{univ}} \otimes \mathrm{pr}_2^*(\mathcal{L}_\kappa^{\mathrm{Pfaff}})$. Its restriction to $C \times \mathcal{M}^{\mathrm{rs}}_{\mathrm{SO}_n,d}$ therefore descends to a vector bundle $\mathcal{E}^{\mathrm{Poinc}}$ on $C \times M^{\mathrm{rs}}_{\mathrm{SO}_n,d}$.

Similarly, $(\mathcal{L}_\kappa^{\mathrm{Pfaff}})^{\otimes 2}$ descends to a line bundle $\mathcal{L}_\kappa^{\det}$ on $M^{\mathrm{rs}}_{\mathrm{SO}_n,d}$. The bilinear form on $\mathcal{E}^{\mathrm{univ}}$ induces a bilinear form $\mathcal{E}^{\mathrm{Poinc}} \otimes \mathcal{E}^{\mathrm{Poinc}} \to \mathrm{pr}_2^*(\mathcal{L}_\kappa^{\det})$, and the given trivialization of $\det(\mathcal{E}^{\mathrm{univ}})$ induces an isomorphism $\det(\mathcal{E}^{\mathrm{Poinc}}) \cong \mathrm{pr}_2^*(\mathcal{L}_\kappa^{\det})^{\otimes n/2}$.

Choosing a trivialization of the line bundle $\mathcal{L}_\kappa^{\det}$ over some open subscheme $\emptyset \neq U \subseteq M^{\mathrm{rs}}_{\mathrm{SO}_n,d}$, these data turn $\mathcal{E}^{\mathrm{Poinc}}$ into an SO_n-bundle over $C \times U$. By construction, this SO_n-bundle is a Poincaré family for U. □

Finally, we apply the previous section to symplectic bundles. Since the symplectic group Sp_{2n} is simply connected, there is only one component $M^{\mathrm{s}}_{\mathrm{Sp}_{2n},0}$; this case is covered by Corollary 2.5.

But we can also consider twisted symplectic bundles. Let $G := \mathrm{Gp}_{2n} \subseteq \mathrm{GL}_{2n}$ be the subgroup generated by the symplectic group $\mathrm{Sp}_{2n} \subseteq \mathrm{SL}_{2n}$ together with the homotheties $\mathbb{G}_m \subseteq \mathrm{GL}_{2n}$. Principal G-bundles correspond to triples (E, b, L) where E is a vector bundle, L is a line bundle, and $b : E \otimes E \to L$ is a symplectic form. The type $d \in \pi_1(G) = \mathbb{Z}$ of such a triple is the degree of L.

Corollary 3.3. *There is no Poincaré family for any non-empty open subscheme* $U \subseteq M^{\mathrm{s}}_{\mathrm{Gp}_{2n},d}$, *unless* n *and* d *are both odd.*

Proof. Suppose that d is even. Choosing a line bundle ξ of degree $d/2$ on C, we get an isomorphism $M^{\mathrm{s}}_{\mathrm{Gp}_{2n},0} \to M^{\mathrm{s}}_{\mathrm{Gp}_{2n},d}$ by sending any triple (E, b, L) to $(E \otimes \xi, b \otimes \mathrm{id}, L \otimes \xi^{\otimes 2})$. Hence this case is covered by Corollary 2.5.

Now suppose that n is even. Let L be a line bundle of degree d on C. We endow the vector bundle $E := \mathcal{O}_C^n \oplus L^n$ with the canonical symplectic form $b : E \otimes E \to L$, such that the subbundles $\mathcal{O}_C^n$ and L^n are both isotropic. If A is an invertible $(n \times n)$-matrix, then $A \oplus (A^{-1})^t \in \mathrm{Aut}(E)$ respects the symplectic form b; thus we obtain an embedding $\mathrm{GL}_n \hookrightarrow \mathrm{Aut}(E, b, L)$.

The center $Z_G \subseteq G$ coincides with the homotheties $\mathbb{G}_m \subseteq G$, so $\mathrm{GL}_n \cap Z_G = \mu_2$ as subgroups of $\mathrm{Aut}(E, b, L)$. The character $\chi = \mathrm{id} : Z_G = \mathbb{G}_m \to \mathbb{G}_m$ is nontrivial on this μ_2, so its restriction $\chi|_{\mu_2}$ doesn't extend to GL_n because n is even. This shows that χ doesn't extend to $\mathrm{Aut}(E, b, L)$. □

Lemma 3.4. *If* n *and* d *are odd, then there is a Poincaré family for* $M^{\mathrm{rs}}_{\mathrm{Gp}_{2n},d}$.

Proof. For each Gp_{2n}-bundle (E, b, L) of type d, the underlying vector bundle E has rank $2n$ and degree $n \cdot d$. As n and d are both odd, the Euler characteristic

$$\chi(E) := \dim \mathrm{H}^0(E) - \dim \mathrm{H}^1(E) = 2n \cdot (1 - g) + n \cdot d$$

is also odd. This number does not depend on E.

Let $(\mathcal{E}^{\mathrm{univ}}, b^{\mathrm{univ}}, \mathcal{L}^{\mathrm{univ}})$ be the universal Gp_{2n}-bundle on $C \times \mathcal{M}_{\mathrm{Gp}_{2n},d}$. The determinant $\mathcal{L}^{\det} := \det(\mathrm{R}\,\mathrm{pr}_{2,*}\,\mathcal{E}^{\mathrm{univ}})$ of the cohomology of $\mathcal{E}^{\mathrm{univ}}$, and the restriction $\mathcal{L}_P^{\mathrm{univ}}$ of $\mathcal{L}^{\mathrm{univ}}$ to a point $P \in C(k)$, are line bundles on $\mathcal{M}_{\mathrm{Gp}_{2n},d}$.

Homotheties $\lambda \in \mathbb{G}_m = Z_G$ act as λ on $\mathcal{E}^{\mathrm{univ}}$, as λ^2 on $\mathcal{L}_P^{\mathrm{univ}}$, and as $\lambda^{\chi(E)}$ on $\mathcal{L}^{\det}$. Using that $\chi(E)$ is odd, Z_G thus acts trivially on the vector bundle

$$\mathcal{E}^{\mathrm{univ}} \otimes \mathrm{pr}_2^*(\mathcal{L}^{\det})^{-1} \otimes \mathrm{pr}_2^*(\mathcal{L}_P^{\mathrm{univ}})^{\otimes(\chi(E)-1)/2} \quad \text{on} \quad C \times \mathcal{M}_{\mathrm{Gp}_{2n},d}.$$

Its restriction to $C \times \mathcal{M}^{\mathrm{rs}}_{\mathrm{Gp}_{2n},d}$ therefore descends to a vector bundle $\mathcal{E}^{\mathrm{Poinc}}$ on $C \times M^{\mathrm{rs}}_{\mathrm{Gp}_{2n},d}$. The universal form b^{univ} induces a symplectic form on $\mathcal{E}^{\mathrm{Poinc}}$, and the resulting Gp_{2n}-bundle is by construction a Poincaré family for $M^{\mathrm{rs}}_{\mathrm{Gp}_{2n},d}$. □

Remark 3.5. Moduli spaces of vector bundles on C with fixed determinant are rational if they admit Poincaré families; cf. [8, 7]. Analogously, [3, Corollary 5.4] shows that moduli spaces of twisted symplectic bundles (E, b, L) on C with fixed line bundle L are rational if $n = \mathrm{rank}(E)/2$ and $d = \deg(L)$ are both odd.

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