Symmetric powers of trace forms on symbol algebras

Ronan Flatley

Mary Immaculate College, Limerick, Ireland

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k-fold exterior powers of V

Let K be a field with $\mathrm{char} K \neq 2$.

Let V a vector space over K, $\dim_K V = m < \infty$.

$$V^{\otimes k} := \underbrace{V \otimes_K \cdots \otimes_K V}_{k\text{times}} \text{ for } k > 0,$$

$$V^{\otimes 0} := K,$$

$$V^{\otimes k} := 0 \text{ for } k < 0.$$

For $k \geq 0$ we define the *k-fold exterior power of V*, denoted $\Lambda^k V$, to be the quotient space of $V^{\otimes k}$ by the subspace generated by all $v_1 \otimes \cdots \otimes v_k$ with two of the vectors equal, the projection being

$$p : V^{\otimes k} \to \Lambda^k V, \; p(v_1 \otimes \cdots \otimes v_k) = v_1 \wedge \cdots \wedge v_k.$$

k-fold exterior powers of V

If $\{v_1, \ldots, v_m\}$ is a basis for V, then a basis for $\Lambda^k V$ is given by the set of k-fold wedge products $\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < \cdots < i_k \leq m\}$ and there are $\binom{m}{k}$ such expressions.

$\Lambda^k V$ has dimension $\binom{m}{k}$.

k-fold exterior powers of φ

Definition (Bourbaki)

Let $\varphi : V \times V \to K$ be a bilinear form and let k be a positive integer not greater than m. We define the k-fold exterior power of φ,

$$\Lambda^k \varphi : \Lambda^k V \times \Lambda^k V \to K$$

by

$$\Lambda^k \varphi(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det \left(\varphi(x_i, y_j)\right)_{1 \leq i,j \leq k} .$$

We define $\Lambda^0 \varphi := \langle 1 \rangle$, the identity form of dimension 1. For $k > m$, we define $\Lambda^k \varphi$ to be the zero form, since $\Lambda^k V = 0$ for all $k > m$.

Remark

$\Lambda^k \varphi$ is a bilinear form and is symmetric if φ symmetric.

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Remark

$\Lambda^k \varphi$ is a bilinear form and is symmetric if φ symmetric.

Useful facts

Let φ be a symmetric bilinear form over K with $\varphi \simeq \langle a_1, \ldots, a_m \rangle$. Then

$$\Lambda^k \varphi \simeq \perp_{1 \leq i_1 < \cdots < i_k \leq m} \langle a_{i_1} \cdots a_{i_k} \rangle.$$

In particular,

$$\Lambda^k (m \times \langle 1 \rangle) \simeq \binom{m}{k} \times \langle 1 \rangle \text{ and } \Lambda^k (m \times \langle -1 \rangle) = \binom{m}{k} \times \langle (-1)^k \rangle.$$

Let φ and ψ be symmetric bilinear forms over K. Then

$$\Lambda^k (\varphi \perp \psi) \simeq \perp_{i+j=k} \Lambda^i \varphi \otimes \Lambda^j \psi \ .$$

Examples of exterior powers of forms

Suppose $\varphi = \langle a, b, c, d \rangle$. Then

$$\Lambda^2 \varphi = \langle ab, ac, ad, bc, bd, cd \rangle$$
$$\Lambda^3 \varphi = \langle abc, abd, acd, bcd \rangle \ .$$

So if $\varphi = 2 \times \mathbb{H} = 2 \times \langle 1, -1 \rangle$ then

$$\Lambda^2 \varphi = \langle -1, 1, -1, -1, 1, -1 \rangle \simeq \langle -1, -1 \rangle \perp 2 \times \mathbb{H}$$
$$\Lambda^3 \varphi = \langle -1, 1, -1, 1 \rangle = 2 \times \mathbb{H} \ .$$

k-fold symmetric powers of V

We define the k-fold symmetric power of V, denoted $S^k(V)$, to be the quotient space of $V^{\otimes k}$ by the subspace generated by

$$v_1 \otimes \cdots \otimes v_k \; - \; v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

for all $v_i \in V$ and all permutations $\sigma : \{1, \cdots, m\} \to \{1, \cdots, m\}$.

If $\{v_1, \ldots, v_m\}$ is a basis for V, then a basis for $S^k V$ is

$$\{v_{i_1}^{k_{i_1}} \cdots v_{i_\ell}^{k_{i_\ell}} : 1 \leq i_1 < \cdots < i_\ell \leq m, k_{i_1} + \cdots + k_{i_\ell} = k\}$$

$S^k(V)$ has dimension $\binom{m+k-1}{k}$.

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k-fold symmetric powers of φ

Definition

Let V be an m-dimensional vector space over K. Let $\varphi = \langle a_1, \cdots, a_m \rangle$ be the diagonalisation of a symmetric bilinear form on V and let k be a positive integer. We define the k-fold symmetric power of φ,

$$S^k\varphi : S^kV \times S^kV \to K$$

by

$$S^k\varphi = \perp_{\substack{1 \leq i_1 < \cdots < i_\ell \leq m \\ k_{i_1} + \cdots + k_{i_\ell} = k}} \langle a_{i_1}^{k_{i_1}} \cdots a_{i_\ell}^{k_{i_\ell}} \rangle .$$

We define $S^0\varphi := \langle 1 \rangle$. Clearly, $S^1\varphi = \varphi$.

Useful facts

Theorem 1 (McGarraghy, 2002)

Let φ be an m-dimensional symmetric bilinear form over K. Then

$$S^k\varphi \simeq \perp_{i=0}^{\lceil k/2 \rceil} \binom{m+i-1}{i} \times \Lambda^{k-2i}\varphi.$$

$$S^k(\varphi \perp \psi) \simeq \perp_{i+j=k} S^i\varphi \otimes S^j\psi$$

Remark

$\Lambda^k\varphi$ is always a subform of $S^k\varphi$.

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Examples of symmetric powers of forms

Suppose $\varphi =< a, b, c, d\rangle$. Then

$$S^2\varphi = \langle a^2, b^2, c^2, d^2\rangle \perp \langle ab, ac, ad, bc, bd, cd\rangle$$
$$= \langle a^2, b^2, c^2, d^2\rangle \perp \Lambda^2\varphi.$$

So if $\varphi = 2 \times \mathbb{H} = 2 \times \langle 1, -1\rangle \simeq \langle 1, 1, -1, -1\rangle$ then

$$S^3\varphi \simeq 10 \times \mathbb{H} = 8 \times \mathbb{H} \perp \Lambda^3\varphi.$$

Hyperbolic forms: results

Proposition 1

Let $\phi \simeq h \times \mathbb{H}$ where $h \in \mathbb{N}$ and k odd with $1 \leq k \leq 2h-1$. Then

$$\Lambda^k \phi \simeq \frac{1}{2}\binom{2h}{k} \times \mathbb{H}.$$

Proposition 2

Let $\phi \simeq h \times \mathbb{H}$ where $h \in \mathbb{N}$, $k = 2\ell$ and $0 \leq \ell \leq h$. Then

$$\Lambda^k \phi = \Lambda^{2\ell} \phi \simeq \binom{h}{\ell} \times \langle (-1)^\ell \rangle \perp \frac{1}{2}\left(\binom{2h}{2\ell} - \binom{h}{\ell}\right) \times \mathbb{H}.$$

Remark

McGarraghy (2001) proved Proposition 2 for ordered fields.

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Hyperbolic forms: results

Let $\phi \simeq h \times \mathbb{H}$.

When K is an ordered field then
$\Lambda^k \phi$ hyperbolic $\iff$ k odd. (McGarraghy, 2001)

Proposition 2 shows that this is not true for fields in general.

e.g. If K contains $\sqrt{-1}$ and $\phi \simeq 4 \times \mathbb{H}$, we have

$$\Lambda^2\phi \simeq \binom{4}{1} \times \langle -1 \rangle \perp \frac{1}{2}\left(\binom{8}{2} - \binom{4}{1}\right) \times \mathbb{H}$$
$$\simeq 14 \times \mathbb{H}.$$

Hyperbolic forms: results

Proposition 3

Let k be any positive odd integer, $h \in \mathbb{N}$. Then

$$S^k(h \times \mathbb{H}) = \frac{1}{2}\binom{2h+k-1}{k} \times \mathbb{H}.$$

Remark

In the proof we use a Vandermonde-like identity, namely,

$$\sum_{i=0}^{k} \binom{h+i-1}{i}\binom{h+k-i-1}{k-i} = \binom{2h+k-1}{k}$$

for h and k arbitrary natural numbers (Gould, 1956).

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Hyperbolic forms: results

Proposition 4

Let k be any non-negative even integer, $k = 2\ell, h \in \mathbb{N}$. Then

$$S^k(h \times \mathbb{H}) \simeq \binom{h+\ell-1}{\ell} \times \langle 1 \rangle \perp \frac{1}{2}\left(\binom{2h+2\ell-1}{2\ell} - \binom{h+\ell-1}{\ell}\right) \times \mathbb{H}.$$

Remark

The identity

$$\sum_{i=0}^{\ell} \binom{2h+i-1}{i}\binom{h}{\ell-i}(-1)^{\ell-i} = \binom{h+\ell-1}{\ell}.$$

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Hyperbolic forms: summary

$\varphi \simeq h \times \mathbb{H}$

k	$\Lambda^k \varphi$	$S^k \varphi$
odd	Hyp	Hyp
even	$\binom{h}{\frac{k}{2}} \times \langle (-1)^{\frac{k}{2}} \rangle \perp \mathrm{Hyp}$	$\binom{h + \frac{k}{2} - 1}{\frac{k}{2}} \times \langle 1 \rangle \perp \mathrm{Hyp}$

Symbol algebras

Let A be a central simple algebra of degree n over K.

We write $T_A \colon A \to K$ for the quadratic trace form

$$T_A(z) = \mathrm{Trd}_A(z^2) \quad \text{for } z \in A,$$

where Trd_A is the reduced trace of A.

Symbol algebras

Suppose K contains a primitive n-th root of unity ω, $n \in \mathbb{N}$ arbitrary. Let $a, b \in K^\times$ and let S be the symbol algebra over K generated by elements x and y with

$$x^n = a, \quad y^n = b \quad \text{and} \quad yx = \omega xy.$$

We denote S as $(a, b; n, K, \omega)$.

Proposition 5

We have

$$\begin{array}{lll} \text{(i)} & T_S \simeq \langle n \rangle \perp \mathrm{Hyp} & n \text{ odd} \\ \text{(ii)} & T_S \simeq \langle n \rangle \langle 1, a, b, (-1)^{n/2} ab \rangle \perp \mathrm{Hyp} & n \text{ even.} \end{array}$$

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Exterior powers of T_S

$$T_S \simeq \langle n \rangle \perp \mathrm{Hyp}, \text{ for } n \text{ odd}$$

Proposition 6

Let n be odd and k an integer such that $0 \leq k < n^2$. Then

$$\Lambda^k T_S \simeq \begin{cases} \binom{\frac{n^2-1}{2}}{\frac{k-1}{2}} \times \langle (-1)^{\frac{k-1}{2}} \rangle \perp \mathrm{Hyp}, & \text{if } k \text{ is odd;} \\ \binom{\frac{n^2-1}{2}}{\frac{k}{2}} \times \langle (-1)^{\frac{k}{2}} \rangle \perp \mathrm{Hyp}, & \text{if } k \text{ is even.} \end{cases}$$

Exterior powers of T_S

Let n be even. We write $T_S \simeq q_S \perp \mathrm{Hyp}$ where

$$q_S \simeq \langle n \rangle \langle 1, a, b, (-1)^{\frac{n}{2}} ab \rangle \ .$$

Proposition 7

Let n be even. Then, for $0 \leq k < n^2$,

(a) If k is odd,

$$\Lambda^k T_S \simeq \binom{\frac{n^2-2}{2}}{\frac{k-1}{2}} \times \langle (-1)^{\frac{n(k-1)}{4}} \rangle q_S \perp \mathrm{Hyp}.$$

(b) If k is even,

$$\ldots$$

Exterior powers of T_S

Proposition 7

(b) If k is even,

$$\Lambda^k T_S \simeq \begin{cases} \binom{\frac{n^2}{2}}{\frac{k}{2}} \times \langle 1 \rangle \perp \mathrm{Hyp}, & \text{if } n \equiv 0 \pmod 4; \\ \left(1 - \frac{2k}{n^2}\right) \binom{\frac{n^2}{2}}{\frac{k}{2}} \times \langle (-1)^{\frac{k}{2}} \rangle \perp \mathrm{Hyp}, & \\ \qquad \text{if } k \leq \frac{n^2}{2} \text{ and } n \equiv 2 \pmod 4; & \\ \left(\frac{2k}{n^2} - 1\right) \binom{\frac{n^2}{2}}{\frac{k}{2}} \times \langle (-1)^{\frac{k+2}{2}} \rangle \perp \mathrm{Hyp}, & \\ \qquad \text{if } k > \frac{n^2}{2} \text{ and } n \equiv 2 \pmod 4. & \end{cases}$$

Exterior powers of T_A, A csa$_K$ of degree 4

Let K be a field containing a primitive 4$^{\text{th}}$ root of unity.

Let A be a central simple algebra of degree 4 over K.

Proposition 8 (Rost, Serre, Tignol, 2006)

In $W(K)$, for $j = 1, \ldots, 15$, we have:

$$\Lambda^j T_A = \begin{cases} 0 & \text{for } j \text{ even}, \\ T_A & \text{for } j \text{ odd}. \end{cases}$$

Corollaire 2, "La forme trace d'une algèbre simple centrale de degré 4", C. R. Acad. Sci. Paris, Ser. I 342 (2006) 83-87.

Exterior powers of T_S

Corollary 9 (to Prop. 7)

Suppose $n = 2^r p_1^{s_1} \dots p_\ell^{s_\ell}$ where $r \geq 2$ and the p_i are odd primes, $s_i \geq 0$. If $k = 2^u p_1^{v_1} \dots p_\ell^{v_\ell}$, $1 \leq u \leq 2r - 1$ and $0 \leq v_i \leq 2s_i$, then $\Lambda^k T_S$ is hyperbolic.

Corollary 10 (to Prop. 7)

Suppose $n = 2p_1^{s_1} \dots p_\ell^{s_\ell}$ where the p_i are odd primes, $s_i \geq 0$ If $k = 2p_1^{v_1} \dots p_\ell^{v_\ell}$, $0 \leq v_i \leq 2s_i$, then $\Lambda^k T_S$ is hyperbolic.

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Symmetric powers of T_S

Proposition 11

Let n be odd and $k \geq 0$. Then

$$S^k T_S = \begin{cases} \binom{\frac{n^2+k-2}{2}}{\frac{k-1}{2}} \times \langle 1 \rangle \perp \mathrm{Hyp}, & \text{if } k \text{ is odd;} \\ \binom{\frac{n^2+k-1}{2}}{\frac{k}{2}} \times \langle 1 \rangle \perp \mathrm{Hyp}, & \text{if } k \text{ is even.} \end{cases}$$

We use the identity

$$\sum_{i=0}^{\ell} \binom{2h+i}{i} \binom{h}{\ell - i} (-1)^{\ell - i} = \binom{h+\ell}{h}.$$

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Symmetric powers of T_S

Remark

For odd degree n, power k, the k^{th} symmetric power of T_S has dimension

$$\binom{n^2+k-1}{k}.$$

When k is even, the corresponding number of copies of $\langle 1 \rangle$ is

$$\binom{\frac{n^2+k-1}{2}}{\frac{k}{2}}.$$

Symmetric powers of T_S

Proposition 12

Let n be even, k odd.

We write $T_S \simeq q_S \perp \mathrm{Hyp}$ where $q_S \simeq \langle n \rangle \langle 1, a, b, (-1)^{\frac{n}{2}} ab \rangle$.

$$S^k = \binom{\frac{n^2+k-1}{2}}{\frac{k-1}{2}} \times q_S \perp \mathrm{Hyp} \; .$$

(TO BE CHECKED)