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Graph rigidity for unitarily invariant matrix norms ☆

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ABSTRACT

A rigidity theory is developed for bar-joint frameworks in linear matrix spaces endowed with a unitarily invariant matrix norm. Analogues of Maxwell's counting criteria are obtained and minimally rigid matrix frameworks are shown to belong to the matroidal class of (k, l)-sparse graphs for suitable k and l. An edge-colouring technique is developed to characterise infinitesimal rigidity for product norms and then applied to show that the graph of a minimally rigid bar-joint framework in the space of 2×2 symmetric (respectively, hermitian) matrices with the trace norm admits an edge-disjoint packing consisting of a (Euclidean) rigid graph and a spanning tree.

1. Introduction

A *bar-joint framework* is a pair (G, p) consisting of a simple undirected graph $G = (V, E)$ and a mapping of its vertices $p : V \to X$ into a linear space X, with $p(v)$ and $p(w)$ distinct for each edge $vw \in E$. Given such a framework, and a norm on X, one may ask whether it is possible to perturb the elements of $p(V)$ without altering distances between adjacent vertices, and without simply applying an isometry of X to $p(V)$. This generalises to the setting of normed linear spaces a central problem in structural rigidity for Euclidean bar-joint frameworks; a topic with roots in works of Cauchy [5] and Maxwell [20] and a broad spectrum of applications (see for example [26,10]). In 1864, Maxwell observed that the underlying graph G of a rigid framework in Euclidean space necessarily satisfies certain counting conditions. In modern terminology, Maxwell's criterion says that the graph of a generic minimally rigid framework in d-dimensional Euclidean

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space is $(d, d(d+1)/2)$-tight. Much later, in 1970, Laman [14] applied a constructive method to prove that in the Euclidean plane Maxwell's counting conditions are also sufficient, thus providing a complete combinatorial characterisation in dimension 2. In fact, Laman's characterisation had been established many years earlier by Pollaczek-Geiringer in the much overlooked paper [22]. The analogous problem in Euclidean 3-space is much more difficult; simple examples show that Maxwell's criterion is no longer sufficient and there is currently no known combinatorial characterisation of generic rigidity. For a general introduction to graph rigidity in Euclidean space we refer the reader to [2,3,8,25,29].

In recent work, geometric and combinatorial aspects of graph rigidity have been developed in general normed space settings with particular emphasis on applications to polyhedral and ℓ^p norms (see for example [11–13]). Such contexts raise new geometric and combinatorial problems which are interesting in their own right and could suggest new techniques which shed light on the Euclidean context. With this motivation, in this article we extend the development of infinitesimal (first-order) graph rigidity to linear matrix spaces with distance constraints determined by a unitarily invariant norm, obtaining some surprisingly tractable combinatorial conditions and suggesting a 3-dimensional rigidity characterisation for cylindrical normed spaces.

In Section 2 we identify rigid motions for a class of admissible matrix spaces. This class includes the spaces of all $n\times n$ real and complex matrices, the $n\times n$ symmetric matrices and the $n\times n$ hermitian matrices. We then characterise the infinitesimal rigid motions for these spaces (Theorem 13) and, in Section 3, present a rank formula which characterises infinitesimal rigidity for certain matrix frameworks which we call *full* (including those with full affine span). We then provide analogues of the Maxwell counting criteria for Euclidean bar-joint frameworks (Theorem 32) and show that the graphs of minimally rigid matrix frameworks belong to the matroidal class of (k, l)-sparse graphs for suitable values of k and l (Theorem 33). Such graphs satisfy a counting rule which is verifiable by existing polynomial-time pebble game algorithms. Interactions between the algebraic structure of these matrix spaces and the accompanying rigidity theory emerge both in the determination of rigid motions and in the identification of infinitesimal flexes for matrix frameworks.

In Section 4 we consider infinitesimal rigidity in the natural setting of a product norm; providing characterisations of rigid motions and infinitesimal flexes, and developing an edge-colouring technique to completely characterise infinitesimal rigidity in terms of the rigidity of projected monochrome subframeworks (Theorem 47). These results, which may be of independent interest, are applied in Section 5, where we exploit the cylindrical nature of the trace norm on the space of 2×2 symmetric matrices, to show that the graph of a minimally rigid matrix framework is expressible as an edge-disjoint union of a spanning tree and a spanning Laman graph (Theorem 52). We then exhibit a minimally rigid matrix framework for the smallest such graph, the complete graph K_6 with an edge removed, and show that a complete graph K_m admits a placement as a rigid matrix framework if and only if $m \geq 6$. Analogous results are obtained for the space of 2×2 hermitian matrices. In the final section, we discuss sufficient conditions for the existence of a minimally rigid placement in an admissible matrix space and pose some conjectures on connectivity and packing criteria, based on recent work of Cheriyan et al. [6] and Gu [9].

1.1. Preliminaries

We now recall a few standard definitions and fix some notation. If A and B are sets, then A^B denotes the set of all functions from A to B; when A has the structure of a vector space, A^B inherits a vector space structure via pointwise operations. Throughout, we let $n \in \mathbb{N}$ with $n \geq 2$. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{M}_n(\mathbb{F})$ denote the associative algebra of $n \times n$ matrices over $\mathbb{F}$. As usual, we write a^* for the conjugate transpose, or adjoint, of a matrix $a \in \mathcal{M}_n(\mathbb{F})$ (which is simply the transpose in the real case). Let $\mathcal{U}_n(\mathbb{F})$, $\mathcal{H}_n(\mathbb{F})$ and $\mathrm{Skew}_n(\mathbb{F})$ denote respectively the sets of unitary, hermitian and skew-hermitian matrices in $\mathcal{M}_n(\mathbb{F})$ (which in the real case are simply the orthogonal, symmetric and skew-symmetric matrices). We also write $\mathrm{Skew}_n^0(\mathbb{F})$ for the set of skew-hermitian matrices with a zero in the $(1,1)$ entry; note that

$\mathrm{Skew}_n^0(\mathbb{R}) = \mathrm{Skew}_n(\mathbb{R})$ and $\mathrm{Skew}_n^0(\mathbb{C}) \subsetneq \mathrm{Skew}_n(\mathbb{C})$. Recall that the commutant S' of a set $S \subseteq \mathcal{M}_n(\mathbb{F})$ is the unital algebra

$$S' = \{y \in \mathcal{M}_n(\mathbb{F}) \colon \forall\, x \in S,\ xy = yx\}.$$

For $x, y \in \mathcal{M}_n(\mathbb{F})$, the commutator of x and y is $[x, y] = xy - yx$. If $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ then $\mathrm{diag}(x)$ denotes the diagonal matrix in $\mathcal{M}_n(\mathbb{F})$ whose ith diagonal entry is x_i.

A norm $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{F})$ is *unitarily invariant* if

$$\|a\| = \|uaw\| \quad \forall\, a \in \mathcal{M}_n(\mathbb{F}),\ \forall\, u, w \in \mathcal{U}_n(\mathbb{F}).$$

A norm $\|\cdot\|_s$ on $\mathbb{R}^n$ is *symmetric* if $\|(x_1, \ldots, x_n)\|_s = \|(|x_{\pi(1)}|, \ldots, |x_{\pi(n)}|)\|_s$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and all permutations $\pi \in S(n)$. Von Neumann [27] characterised unitarily invariant matrix norms on $\mathcal{M}_n(\mathbb{F})$ as those obtained by applying a symmetric norm $\|\cdot\|_s$ to the vector

$$\sigma(a) = (\sigma_1(a), \ldots, \sigma_n(a)),$$

where $\sigma_i(a)$ is the ith largest singular value of the matrix $a \in \mathcal{M}_n(\mathbb{F})$. The correspondence is given by

$$\|a\| := \|\sigma(a)\|_s, \quad \|x\|_s := \|\,\mathrm{diag}(x)\|.$$

Standard examples of unitarily invariant norms are provided by the Schatten p-norms

$$\|a\|_{c_p} := \|\sigma(a)\|_{\ell_p}, \quad \forall\, 1 \le p \le \infty,$$

and the Ky-Fan k-norms

$$\|a\|_k := \sum_{i=1}^{k} \sigma_i(a), \quad \forall\, 1 \le k \le n.$$

The Schatten 1-norm, 2-norm and ∞-norm are known as the trace norm, the Frobenius norm and the spectral norm, respectively. The Frobenius norm is Euclidean in the sense that it is derived from an inner product. The spectral norm is an operator norm with matrices viewed as linear operators on $\mathbb{F}^n$ with the usual Euclidean norm.

2. Rigid motions for admissible matrix spaces

The aim of this section is to describe the linear space of infinitesimal rigid motions for a rich class of normed matrix spaces. Explicit characterisations are obtained for suitable norms in the cases of $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{H}_n(\mathbb{F})$.

2.1. Admissible matrix spaces

Let Γ be a finite set of real-linear maps $\mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ which contains the identity map id, and has the property that $\gamma(I) = I$ for all $\gamma \in \Gamma$; we call such a set Γ a *test set* on $\mathcal{M}_n(\mathbb{F})$. Let X be a real-linear subspace of $\mathcal{M}_n(\mathbb{F})$. If $\gamma \in \Gamma$, then the γ-*commutant* of X is the real-linear subspace

$$X^\gamma = \{y \in \mathcal{M}_n(\mathbb{F}) \colon \forall\, x \in X,\ xy = y\gamma(x)\},$$

and we define

$$X^\Gamma = \bigcup_{\gamma\in\Gamma} X^\gamma.$$

Note that X^Γ decreases as X increases, and

$$X^\Gamma \supseteq X^{\mathrm{id}} = X' \supseteq \mathbb{F}I = \{\lambda I\colon \lambda \in \mathbb{F}\}.$$

Definition 1. If $I \in X$ and $X^\Gamma = \mathbb{F}I$ is as small as possible, then we say that X is Γ-*large* in $\mathcal{M}_n(\mathbb{F})$.

Remark 2. Let $\mathrm{Fix}(X;\Gamma)$ be the set of matrices in X fixed by a test set Γ:

$$\mathrm{Fix}(X;\Gamma) = \{x \in X\colon \forall\, \gamma \in \Gamma,\ \gamma(x) = x\}.$$

Plainly, $X^\Gamma \subseteq \mathrm{Fix}(X;\Gamma)'$. In particular, if e_{ij} denotes the (i,j) matrix unit in $\mathcal{M}_n(\mathbb{F})$ and

$$S := \{e_{ij} + e_{ji}\colon 1 \le i \le j \le n\} \subseteq \mathrm{Fix}(X;\Gamma),$$

then $X^\Gamma \subseteq S' = \mathbb{F}I$, so X is Γ-large in $\mathcal{M}_n(\mathbb{F})$.

Example 3. Consider

$$\Gamma_{\mathbb{R}} = \{\text{identity}, \text{transpose}\} \quad \text{and} \quad \Gamma_{\mathbb{C}} = \Gamma_{\mathbb{R}} \cup \{\text{adjoint}, \text{conjugation}\}.$$

Plainly, $\Gamma_{\mathbb{F}}$ is then a test set on $\mathcal{M}_n(\mathbb{F})$. It is easy to check using Remark 2 that the real-linear spaces $\mathcal{H}_n(\mathbb{R})$, $\mathcal{H}_n(\mathbb{C})$ and $\mathcal{M}_n(\mathbb{R})$ are $\Gamma_{\mathbb{R}}$-large, and $\mathcal{M}_n(\mathbb{C})$ is $\Gamma_{\mathbb{C}}$-large, in the corresponding $\mathcal{M}_n(\mathbb{F})$.

Definition 4.

1. Let Γ be a test set on $\mathcal{M}_n(\mathbb{F})$ and let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{F})$. A real-linear subspace $(X, \|\cdot\|)$ of $\mathcal{M}_n(\mathbb{F})$ has the Γ-*isometry property* if every real-linear isometry $A\colon X \to X$ is of the form

$$A(x) = u\,\gamma(x)\,w, \quad x \in X$$

   for some $u, w \in \mathcal{U}_n(\mathbb{F})$, and some $\gamma \in \Gamma$.
2. Given a real-linear space $X \subseteq \mathcal{M}_n(\mathbb{F})$ and a unitarily invariant norm $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{F})$, we call $(X, \|\cdot\|)$ an *admissible matrix space* (in $\mathcal{M}_n(\mathbb{F})$) if

   (a) there exists a test set Γ such that X is Γ-large in $\mathcal{M}_n(\mathbb{F})$ and $(X, \|\cdot\|)$ has the Γ-isometry property; and
   (b) there exist scalars $\lambda_i \in \mathbb{F}$ for $1 \le i \le n$ so that $e_{ii} \in X$ and $e_{1i} + \lambda_i e_{i1} \in X$; and
   (c) for every $x \in X$, we also have $x^* \in X$.

   We will also say that $(X, \|\cdot\|)$ is *admissible with respect to* Γ.
3. We say that a (unitarily invariant) norm $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{F})$ is *admissible* if $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|)$ is admissible in $\mathcal{M}_n(\mathbb{F})$.

Example 5 *($\mathcal{M}_n(\mathbb{F})$).* Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{F})$ which is not a multiple of the Frobenius norm and, in the case $(\mathbb{F}, n) = (\mathbb{R}, 4)$, is not the Ky-Fan 2-norm. The $\Gamma_{\mathbb{F}}$-isometry property holds by [17, Theorem 4.1] and [24] in the real and complex cases, respectively. Thus $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|)$ is an admissible matrix space.

Example 6 *($\mathcal{H}_n(\mathbb{R})$).* Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{R})$ which is not a multiple of the Frobenius norm. Suppose one of the following conditions holds:

(a) $n \neq 4$, or,
(b) $\|x\| \neq \|\frac{1}{2}(\mathrm{trace}(x))I - x\|$ for some $x \in \mathcal{H}_n(\mathbb{R})$.

Then the subspace $(\mathcal{H}_n(\mathbb{R}), \|\cdot\|)$ has the $\Gamma_{\mathbb{R}}$-isometry property by [19, Theorem 6.3] and so $(\mathcal{H}_n(\mathbb{R}), \|\cdot\|)$ is an admissible matrix space in $\mathcal{M}_n(\mathbb{R})$.

Example 7 *($\mathcal{H}_n(\mathbb{C})$).* Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{C})$ which is not induced by an inner product. Suppose the following conditions hold:

(a) There does not exist $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\|x\| = f(|\mathrm{trace}(x)|, \mathrm{trace}(x^2))$ for all $x \in \mathcal{H}_n(\mathbb{C})$; and
(b) $\|x\| \neq \|\frac{2}{n}(\mathrm{trace}(x))I - x\|$ for some $x \in \mathcal{H}_n(\mathbb{C})$.

Then the subspace $(\mathcal{H}_n(\mathbb{C}), \|\cdot\|)$ has the $\Gamma_{\mathbb{R}}$-isometry property by [18, Theorem 2] and so $(\mathcal{H}_n(\mathbb{C}), \|\cdot\|)$ is an admissible matrix space in $\mathcal{M}_n(\mathbb{C})$.

In particular, $(\mathcal{H}_n(\mathbb{C}), \|\cdot\|_{c_p})$ is admissible in $\mathcal{M}_n(\mathbb{C})$ for $n \geq 3$ and $1 \leq p \leq \infty$ with $p \neq 2$; to verify condition (a), consider $x_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus 0$ and $x_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus 0$.

Example 8 *($(\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_p})$).* Consider $\mathcal{H}_2(\mathbb{C})$ with the Schatten p-norm where $p \neq 2$. Condition (a) in Example 7 fails, since in the 2×2 case the two singular values (and hence also the c_p-norm) of any symmetric 2×2 matrix $x \in \mathcal{H}_2(\mathbb{C})$ are determined by $|\mathrm{trace}(x)|$ and $\mathrm{trace}(x^2)$. Following [18, Theorem 2(c)], in addition to the isometries arising from $\Gamma_{\mathbb{R}}$ and multiplication by unitary matrices, we must also consider isometries $A\colon \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$ which preserve the bilinear form $(x, y) \mapsto \mathrm{trace}(xy)$ on $\mathcal{H}_2(\mathbb{C}) \times \mathcal{H}_2(\mathbb{C})$ and have $A(I) = \pm I$. We claim that any such A must be of the form $A(x) = \pm u\, \gamma(x)\, u^*$ for some $u \in \mathcal{U}_2(\mathbb{C})$ and $\gamma \in \Gamma_{\mathbb{R}}$, so we do indeed have the $\Gamma_{\mathbb{R}}$-isometry property. To see this, we may first negate A if necessary to ensure that $A(I) = I$. Note that $\mathrm{trace}(A(x)^2) = \mathrm{trace}(x^2)$ and $|\mathrm{trace}(A(x))| = |\mathrm{trace}(A(x)A(I))| = |\mathrm{trace}(xI)| = |\mathrm{trace}(x)|$. Hence A preserves singular values, and moreover if $\mathrm{trace}(x) = 0$, then $\mathrm{trace}(A(x)) = 0$. Consider $x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The singular values of x, and hence also $A(x)$, are $(1, 1)$. Composing A with a suitable unitary conjugation, we can arrange that $A(x)$ is diagonal with monotonically decreasing diagonal entries; since $A(x)$ has trace 0, we have $A(x) = x$. The subspace spanned by I and x is D, the space of diagonal matrices in $\mathcal{H}_2(\mathbb{C})$, and we have shown that A acts trivially on D. Hence the subspace $E = D^{\perp}$ spanned by $y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $z = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ must have $A(E) = E$. We have $A(y) = \begin{bmatrix} 0 & \alpha \\ \overline{\alpha} & 0 \end{bmatrix}$ for some $\alpha \in \mathbb{T}$, and $\mathrm{trace}(A(y)A(z)) = \mathrm{trace}(yz) = 0$, so it follows that $A(z) = \begin{bmatrix} 0 & \beta \\ \overline{\beta} & 0 \end{bmatrix}$ where $\beta \in \{i\alpha, -i\alpha\}$. Conjugating by the diagonal unitary $\begin{bmatrix} 1 & 0 \\ 0 & \overline{\alpha} \end{bmatrix}$, we may assume that A fixes I, x and y, and $A(z) = \pm z$. So either $A(z)$ or $A(z^{\mathsf{T}}) = -A(z)$ is equal to z. Precomposing with the transpose if necessary, we reduce A to the identity map, verifying the claim above. Hence $(\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_p})$ is admissible in $\mathcal{M}_2(\mathbb{C})$ provided $p \neq 2$.

Remark 9. These examples show that in particular, $\mathcal{H}_n(\mathbb{F})$ and $\mathcal{M}_n(\mathbb{F})$ are admissible in $\mathcal{M}_n(\mathbb{F})$ with respect to the Schatten p-norm for any $n \geq 2$ and $1 \leq p \leq \infty$ with $p \neq 2$. Note that the Schatten 2-norm is not admissible; however, it arises from an inner product and so the accompanying graph rigidity follows that of the Euclidean norm.

2.2. Rigid motions

Recall [12,13] that a *rigid motion* of a normed space $(X, \|\cdot\|)$ is a collection of continuous paths $\alpha = \{\alpha_x : [-1,1] \to X\}_{x\in X}$, with the following properties:

(a) $\alpha_x(0) = x$ for all $x \in X$;
(b) $\alpha_x(t)$ is differentiable at $t = 0$ for all $x \in X$; and
(c) $\|\alpha_x(t) - \alpha_y(t)\| = \|x - y\|$ for all $x, y \in X$ and for all $t \in [-1, 1]$.

Note that formally, α is a map $\alpha\colon X \times [-1,1] \to X$, $\alpha(x,t) = \alpha_x(t)$ which satisfies these conditions; we will routinely interchange the notation $\alpha(x,t)$ with $\alpha_x(t)$ where it eases the exposition. We write $\mathcal{R}(X, \|\cdot\|)$ for the set of all rigid motions of $(X, \|\cdot\|)$. As we will shortly see, in admissible matrix spaces a rigid motion always has a particularly nice form near $t = 0$.

Lemma 10. *Let $(X, \|\cdot\|)$ be a normed space and let $\alpha \in \mathcal{R}(X, \|\cdot\|)$. Then,*

(i) for each $t \in [-1,1]$ there exists a real-linear isometry $A_t : X \to X$ and a vector $c(t) \in X$ such that

$$\alpha_x(t) = A_t(x) + c(t), \quad \forall\, x \in X,$$

(ii) the map $c : [-1,1] \to X$ is continuous on $[-1,1]$ and differentiable at $t = 0$,
(iii) for every $x \in X$, the map $A_*(x) : [-1,1] \to X$, $t \mapsto A_t(x)$, is continuous on $[-1,1]$ and differentiable at $t = 0$, and,
(iv) $A_0 = I$ and $c(0) = 0$.

Proof. By property (c) of the rigid motion α, for every fixed $t \in [-1,1]$, the map $x \mapsto \alpha_x(t)$ is an isometry of $(X, \|\cdot\|)$. Since X is finite dimensional, this isometry is necessarily surjective (see for example [4, p. 500]) so this is a real-affine map by the Mazur-Ulam theorem. Hence there exists a real-linear isometry $A_t : X \to X$ and $c(t) \in X$ such that

$$\alpha_x(t) = A_t(x) + c(t), \quad \forall\, x \in X.$$

Note that $c(t) = \alpha_0(t)$ is a continuous function of t (and is differentiable at $t = 0$), so $A_t(x) = \alpha_x(t) - c(t)$ is also a continuous function of t (and is differentiable at $t = 0$), for every $x \in X$. Finally, $c(0) = \alpha_0(0) = 0$ and $A_0(x) = \alpha_x(0) = x$ for every $x \in X$. $\square$

In the proof of the following proposition, for $X \subseteq \mathcal{M}_n(\mathbb{F})$ we say that a map $A\colon X \to X$ is *implemented by unitaries* if there exist $r, s \in \mathcal{U}_n(\mathbb{F})$ so that $A(x) = rxs$ for every $x \in X$.

Proposition 11. *Let $(X, \|\cdot\|)$ be an admissible matrix space in $\mathcal{M}_n(\mathbb{F})$. For any $\alpha \in \mathcal{R}(X, \|\cdot\|)$, there is a neighbourhood T of 0 in $[-1,1]$, and matrices $u_t, w_t \in \mathcal{U}_n(\mathbb{F})$ and $c(t) \in X$ for each $t \in T$, so that*

(i) $\alpha_x(t) = u_t x w_t + c(t), \quad \forall\, x \in X,\ t \in T$;
(ii) $c(0) = 0$ and $u_0 = w_0 = I$;
(iii) the maps $t \mapsto c(t)$ and $t \mapsto u_t x w_t$ are both differentiable at $t = 0$, for any $x \in X$; and
(iv) the maps $t \mapsto u_t$ and $t \mapsto w_t$ are continuous at $t = 0$.

Proof. Let $\alpha \in \mathcal{R}(X, \|\cdot\|)$. Then for each $t \in [-1,1]$ there exists a real-linear isometry $A_t : X \to X$ and vector $c(t) \in X$ as in Lemma 10. Consider the set

$$T = \{t \in [-1,1]\colon A_t \text{ is implemented by unitaries}\}.$$

Note that $0 \in T$ since A_0 is the identity map on X. Let Γ be a test set with respect to which $(X, \|\cdot\|)$ is admissible. By the Γ-isometry property, for every $t \in [-1,1]$, there exist $r_t, s_t \in \mathcal{U}_n(\mathbb{F})$ and $\gamma_t \in \Gamma$ so that

$$A_t(x) = r_t\, \gamma_t(x)\, s_t, \quad \forall\, x \in X, \tag{1}$$

and for $t \in T$ we may insist that $\gamma_t = \mathrm{id}$. We can also take $r_0 = s_0 = I$.

For $t \in [-1,1]$, let $\theta_t = \arg(\mathrm{trace}(r_t))$, and define u_t, w_t by

$$u_t = e^{-i\theta_t} r_t, \quad w_t = e^{i\theta_t} s_t.$$

Note that $\mathrm{trace}(u_t) \geq 0$ for all $t \in [-1,1]$, and $u_0 = v_0 = I$. Moreover, for each $x \in X$, we have $A_t(x) = u_t\, \gamma_t(x)\, w_t$. In particular, $\alpha_x(t) = u_t x v_t + c(t)$ for every $x \in X$ and $t \in T$.

If u_t is not continuous at $t = 0$, then there exist $\epsilon > 0$ and a sequence $t_n \to 0$ so that $\|u_{t_n} - I\| \geq \epsilon$ for all $n \in \mathbb{N}$. Since $\mathcal{U}_n(\mathbb{F})$ is compact, there is a subsequence (t_{n_k}) such that $(u_{t_{n_k}})$ and $(w_{t_{n_k}})$ are both convergent, say to u and w, respectively. Then $u, w \in \mathcal{U}_n(\mathbb{F})$ and since $\gamma_t(I) = I$ for every t, we have

$$I = A_0(I) = \lim_{k\to\infty} A_{t_{n_k}}(I) = \lim_{k\to\infty} u_{t_{n_k}}\, \gamma_{t_{n_k}}(I)\, w_{t_{n_k}} = uw,$$

so $w = u^*$. Since the test set Γ is finite, passing to a further subsequence if necessary, we can arrange that $\gamma_{t_{n_k}}$ is independent of k, say $\gamma_{t_{n_k}} = \gamma$ for all $k \geq 1$. For every $x \in X$, we have

$$x = A_0(x) = \lim_{k\to\infty} A_{t_{n_k}}(x) = \lim_{k\to\infty} u_{t_{n_k}}\, \gamma_{t_{n_k}}(x)\, w_{t_{n_k}} = u\,\gamma(x)\, u^*,$$

so $xu = u\gamma(x)$, hence $u \in X^\Gamma = \mathbb{F}I$ since X is Γ-large. Now $\mathrm{trace}(u) = \lim_{k\to\infty} \mathrm{trace}(u_{t_{n_k}}) \geq 0$, so $u = I$ and

$$0 = \|u - I\| = \lim_{k\to\infty} \|u_{t_{n_k}} - I\| \geq \epsilon > 0,$$

a contradiction. Hence $t \mapsto u_t$ is continuous at $t = 0$, so $t \mapsto w_t = u_t^* A_t(I)$ is also continuous at $t = 0$.

Finally, if T is not a neighbourhood of 0, then there is sequence $t_n \to 0$ with $t_n \in [-1,1] \setminus T$ for all $n \geq 1$. Passing to an infinite subsequence on which γ_t is constant, we may assume that $\gamma_{t_n} = \gamma$ does not depend on n. Let $x \in X$. Since $t \mapsto A_t(x)$ is continuous at $t = 0$ and we know that $u_{t_n} \to I$ and $w_{t_n} \to I$ as $n \to \infty$, we have

$$x = \lim_{n\to\infty} A_{t_n}(x) = \lim_{n\to\infty} u_{t_n}\, \gamma(x)\, w_{t_n} = \gamma(x),$$

so $x = \gamma(x)$ for all $x \in X$. In particular, setting $t = t_1 \in [-1,1] \setminus T$, we have $A_t(x) = u_t\, \gamma(x)\, w_t = u_t\, x\, w_t$ for all $x \in X$, so $t \in T$, a contradiction. $\square$

2.3. Infinitesimal rigid motions

A vector field $\eta : X \to X$ of the form $\eta(x) = \alpha_x'(0)$ where $\alpha \in \mathcal{R}(X, \|\cdot\|)$ is referred to as an *infinitesimal rigid motion* of $(X, \|\cdot\|)$. We also say that η is *induced* by the rigid motion α. The collection of all infinitesimal rigid motions of a normed space $(X, \|\cdot\|)$ is a real-linear subspace of X^X, denoted $\mathcal{T}(X, \|\cdot\|)$.

Lemma 12. *If $(X, \|\cdot\|)$ is a normed space, then every $\eta \in \mathcal{T}(X, \|\cdot\|)$ is an affine map.*

Proof. Suppose η is induced by $\alpha \in \mathcal{R}(X, \|\cdot\|)$. For $x \in X$ and $t \in [-1,1]$, write $\alpha_x(t) = A_t(x) + c(t)$ where the real-linear maps $A_t : X \to X$ and vectors $c(t) \in X$ are as in Lemma 10. Then $\eta(x) = \alpha_x'(0) = B(x) + c'(0)$ where $B\colon X \to X$ is the real-linear map given by $B(x) = \frac{d}{dt}A_t(x)|_{t=0}$. $\square$

From the viewpoint of infinitesimal rigidity theory, which we consider in Section 3, infinitesimal rigid motions yield trivial deformations of a framework since they arise from a global deformation of X. We will now identify these in our context.

Theorem 13. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{F})$, and suppose that $(X, \|\cdot\|)$ is an admissible matrix space. If $\eta \in \mathcal{T}(X, \|\cdot\|)$, then there exist unique matrices $a, b, c \in \mathcal{M}_n(\mathbb{F})$ with $a \in \mathrm{Skew}_n(\mathbb{F})$, $b \in \mathrm{Skew}_n^0(\mathbb{F})$ and $c \in X$ so that*

$$\eta(x) = ax + xb + c, \quad \forall\, x \in X.$$

Proof. Choose some $\alpha \in \mathcal{R}(X, \|\cdot\|)$ which induces η and consider a neighbourhood T of 0 and maps $u, w : T \to \mathcal{U}_n(\mathbb{F})$, $u(t) = u_t$ and $w(t) = w_t$ and $c : T \to X$ as in Proposition 11. Note in particular that these maps are continuous at $t = 0$, with $u_0 = w_0 = I$ and $c(0) = 0$, and for all $x \in X$, the restriction of α_x to T is given by

$$\alpha_x(t) = u_t x w_t + c(t)$$

and this restriction is differentiable at $t = 0$.

Suppose first that $c(t) = 0$ for all $t \in T$.

Consider the map

$$\delta_r : X \to \mathcal{M}_n(\mathbb{F}), \qquad \delta_r(x) = \alpha_x'(0) - \alpha_I'(0)x.$$

Note that for each $x \in X$, we have

$$\delta_r(x) = \lim_{t\to 0} \frac{u_t x w_t - x - (u_t w_t - I)x}{t} = \lim_{t\to 0} \tfrac{1}{t} u_t [x, w_t].$$

Since $u_t^* \to I$ as $t \to 0$, we have

$$\delta_r(x) = \lim_{t\to 0} \tfrac{1}{t} u_t^* u_t [x, w_t] = \lim_{t\to 0} \tfrac{1}{t}[x, w_t].$$

Observe that if $s \in \mathcal{M}_n(\mathbb{F})$ has $s_{11} = 0$, then for any $\lambda \in \mathbb{F}$ and $1 \le i, j \le n$, the (i,j) entry of s is given by

$$s_{ij} = \begin{cases} [e_{ii}, s]_{ij} & \text{if } i \neq j, \\ [e_{1i} + \lambda e_{i1}, s]_{1i} & \text{if } i = j. \end{cases}$$

For $0 \neq t \in T$, let $b_t = t^{-1}(w_t - (w_t)_{11} I)$, so that $\delta_r(x) = \lim_{t\to 0}[x, b_t]$ for $x \in X$ and the $(1,1)$ entry of b_t is 0. Since X is admissible, the preceding observation shows that b_t is entrywise convergent, say $b_t \to b$ as $t \to 0$, hence $\delta_r(x) = [x, b]$ for each $x \in X$. Note that the $(1,1)$ entry of b is 0. Let $a = \alpha_I'(0) - b$; then

$$\alpha_x'(0) = \alpha_I'(0)x + \delta_r(x) = ax + xb, \quad x \in X.$$

For each $x \in X$, consider the map

$$\beta_x : T \to \mathcal{M}_n(\mathbb{F}), \quad \beta_x(t) = w_t x u_t.$$

For $t \in T$, we have

$$\beta_x(t) - \beta_x(0) = w_t x u_t - x = w_t(x - w_t^* x u_t^*)u_t = w_t(x^* - u_t x^* w_t)^* u_t,$$

so by the continuity of u_t and w_t at $t = 0$, we have

$$\begin{aligned}\beta_x'(0) &= \lim_{t\to 0} \frac{\beta_x(t) - \beta_x(0)}{t} = \lim_{t\to 0} w_t \left(\frac{x^* - u_t x^* w_t}{t}\right)^* u_t \\ &= -\alpha_{x^*}'(0)^* = -b^* x - x a^*.\end{aligned}$$

Now

$$\begin{aligned} xb - bx = \delta_r(x) &= \lim_{t\to 0} \tfrac{1}{t}[x, w_t] = \lim_{t\to 0} \tfrac{1}{t}[x, w_t]u_t \\ &= \lim_{t\to 0} \tfrac{1}{t}(x w_t - w_t x)u_t = \lim_{t\to 0} \tfrac{1}{t} x(w_t u_t - I) - \tfrac{1}{t}(w_t x u_t - x) \\ &= x\beta_I'(0) - \beta_x'(0) = -x(a^* + b^*) + (b^* x + x a^*) \\ &= b^* x - x b^*, \end{aligned}$$

so $x(b + b^*) = (b + b^*)x$ for all $x \in X$, so $b + b^* \in X' = \mathbb{F}I$. Since $b_{11} = 0$, we have $b + b^* = 0$.

Define $\delta_\ell(x) = \alpha_x'(0) - x\alpha_I'(0)$. We know that $\alpha_x'(0) = ax + xb$, so $\delta_\ell(x) = ax + xb - x(a + b) = [a, x]$. A similar computation to the one above for δ_r yields $\delta_\ell(x) = \beta_I'(0)x - \beta_x'(0)$. It follows that $a + a^* \in \mathbb{F}I$, and hence that $a + a^* = \lambda I$ for some $\lambda \in \mathbb{R}$.

Now consider the maps $\varphi_+, \varphi_- : \mathcal{M}_n(\mathbb{F}) \to \mathbb{R}$ given by the one-sided limits

$$\varphi_\pm(x) = \lim_{t\to 0^\pm} \frac{\|I + tx\| - \|I\|}{t}, \qquad x \in X.$$

These limits are well defined (see, for example, [23, Theorem 23.1]); moreover, φ_+ is sub-additive and φ_- is super-additive, and $\varphi_\pm(\alpha I) = \alpha\|I\|$ for any $\alpha \in \mathbb{R}$. Note that

$$\alpha_I'(0) = a + b \quad \text{and} \quad \|I\| = \|\alpha_I(t)\| \text{ for any } t \in \mathbb{R}.$$

It follows that $\varphi_\pm(a + b) = 0$, since

$$\begin{aligned}\left|\frac{\|I + t(a + b)\| - \|I\|}{t}\right| &= \left|\frac{\|\alpha_I(0) + t\alpha_I'(0)\| - \|\alpha_I(t)\|}{t}\right| \\ &\leq \left\|\frac{\alpha_I(t) - \alpha_I(0)}{t} - \alpha_I'(0)\right\| \to 0 \text{ as } t \to 0.\end{aligned}$$

The conjugate transpose is isometric for the unitarily invariant norm $\|\cdot\|$, so $\varphi_\pm(x^*) = \varphi_\pm(x)$ for any $x \in X$. Hence $\varphi_\pm(a^* + b^*) = 0$. Since $b + b^* = 0$, we have

$$\lambda I = a + a^* = a + b + a^* + b^*.$$

Applying φ_+ and using sub-additivity, we obtain

$$\lambda\|I\| = \varphi_+(a + b + a^* + b^*) \leq \varphi_+(a + b) + \varphi_+(a^* + b^*) = 0.$$

Applying φ_- similarly, we obtain the converse inequality, so $\lambda = 0$. Thus a and b are skew-hermitian, with $b \in \mathrm{Skew}_n^0(\mathbb{F})$.

For uniqueness, if $(a', b') \in \mathrm{Skew}_n(\mathbb{F}) \times \mathrm{Skew}_n^0(\mathbb{F})$ with $ax + xb = a'x + xb'$ for every $x \in X$, then $a''x + xb'' = 0$ where $a'' = a - a'$ and $b'' = b - b'$. Setting $x = I$ gives $b'' = -a''$, so $b'' \in \mathrm{Skew}_n^0(\mathbb{F}) \cap X' = \mathrm{Skew}_n^0(\mathbb{F}) \cap \mathbb{F}I = \{0\}$, so $a'' = b'' = 0$.

Finally, if $c(t)$ is not identically zero then applying the above argument to the rigid motion obtained by replacing $\alpha_x(t)$ with $\alpha_x(t) - c(t)$ for each $x \in X$, we obtain $\alpha_x'(0) = ax + xb + c$ for some unique $(a, b) \in \mathrm{Skew}_n(\mathbb{F}) \times \mathrm{Skew}_n^0(\mathbb{F})$ and where $c = c'(0) \in X$. □

In the case of admissible spaces of the form $(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)$ we obtain the following refinement of Theorem 13.

Corollary 14. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{F})$. If $(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)$ is admissible and $\eta \in \mathcal{T}(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)$, then there exist unique matrices $a \in \mathrm{Skew}_n^0(\mathbb{F})$ and $c \in \mathcal{H}_n(\mathbb{F})$ so that*

$$\alpha_x'(0) = ax - xa + c, \quad \forall x \in \mathcal{H}_n(\mathbb{F}).$$

Proof. Applying Theorem 13 with $X = \mathcal{H}_n(\mathbb{F})$ to obtain $a \in \mathrm{Skew}_n(\mathbb{F})$, $b \in \mathrm{Skew}_n^0(\mathbb{F})$ and $c \in \mathcal{H}_n(\mathbb{F})$, we observe that

$$\alpha_I'(0) - c = a + b \in \mathcal{H}_n(\mathbb{F}) \cap \mathrm{Skew}_n(\mathbb{F}) = \{0\},$$

so $b = -a$. □

2.4. The dimension of $\mathcal{T}(X, \|\cdot\|)$

Let $(X, \|\cdot\|)$ be an admissible matrix space. By Theorem 13, there is a well-defined map

$$\Psi_X : \mathcal{T}(X, \|\cdot\|) \to \mathrm{Skew}_n(\mathbb{F}) \oplus \mathrm{Skew}_n^0(\mathbb{F}) \oplus \mathcal{M}_n(\mathbb{F}),$$

with the property that $\Psi_X(\eta) = (a, b, c)$ if and only if $\eta(x) = ax + xb + c$ for all $x \in X$.

Lemma 15. *The map Ψ_X is injective and linear. Moreover, if $X = \mathcal{M}_n(\mathbb{F})$ then Ψ_X is a linear isomorphism.*

Proof. That Ψ_X is injective and linear is a routine verification. Suppose $X = \mathcal{M}_n(\mathbb{F})$. Then it only remains to prove surjectivity. Let (a, b, c) be in the codomain of Ψ, and for each $x \in \mathcal{M}_n(\mathbb{F})$ define

$$\alpha_x : [-1, 1] \to \mathcal{M}_n(\mathbb{F}), \quad \alpha_x(t) = e^{ta} x e^{tb} + tc.$$

Since a and b are skew-hermitian, e^{ta} and e^{tb} are unitary for every $t \in \mathbb{R}$, so the collection of maps $\{\alpha_x : [-1, 1] \to \mathcal{M}_n(\mathbb{F})\}_{x \in \mathcal{M}_n(\mathbb{F})}$ is a rigid motion of $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|)$. Differentiating, we see that the induced infinitesimal rigid motion is the vector field

$$\eta : X \to X, \quad x \mapsto ax + xb + c.$$

Thus $\Psi(\eta) = (a, b, c)$ and so Ψ is surjective. □

Here and below, we write $\dim Z$ for the real-linear dimension of a real-linear vector space Z.

Proposition 16. *If $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|)$ is an admissible matrix space, then*

$$\dim \mathcal{T}(\mathcal{M}_n(\mathbb{F}), \|\cdot\|) = \begin{cases} 2n^2 - n & \text{if } \mathbb{F} = \mathbb{R}, \\ 4n^2 - 1 & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Proof. By Lemma 15, $\Psi_{\mathcal{M}_n(\mathbb{F})}$ is a linear isomorphism. If $\mathbb{F} = \mathbb{R}$, then

$$\begin{aligned}\dim \mathcal{T}(\mathcal{M}_n(\mathbb{R}), \|\cdot\|) &= \dim(\mathrm{Skew}_n(\mathbb{R}) \oplus \mathrm{Skew}_n(\mathbb{R}) \oplus \mathcal{M}_n(\mathbb{R})) \\ &= \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + n^2 \\ &= 2n^2 - n.\end{aligned}$$

If $\mathbb{F} = \mathbb{C}$, then

$$\begin{aligned}\dim \mathcal{T}(\mathcal{M}_n(\mathbb{C}), \|\cdot\|) &= \dim(\mathrm{Skew}_n(\mathbb{C}) \oplus \mathrm{Skew}_n^0(\mathbb{C}) \oplus \mathcal{M}_n(\mathbb{C})) \\ &= n^2 - 1 + n^2 + 2n^2 \\ &= 4n^2 - 1. \quad \square\end{aligned}$$

We now compute the dimension of the space of infinitesimal rigid motions for admissible matrix spaces of the form $(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)$.

Lemma 17. *The range of* $\Psi_{\mathcal{H}_n(\mathbb{F})}$ *is*

$$\operatorname{ran} \Psi_{\mathcal{H}_n(\mathbb{F})} = \{(a, -a, c) \colon (a, c) \in \mathrm{Skew}_n^0(\mathbb{F}) \oplus \mathcal{H}_n(\mathbb{F})\}.$$

Proof. By Corollary 14, if (a, b, c) is an element of the range of $\Psi_{\mathcal{H}_n(\mathbb{F})}$ then $b = -a$ and $c \in \mathcal{H}_n(\mathbb{F})$. For the reverse inclusion, let $a \in \mathrm{Skew}_n^0(\mathbb{F})$, let $c \in \mathcal{H}_n(\mathbb{F})$, and for each $x \in \mathcal{H}_n(\mathbb{F})$ define

$$\alpha_x : [-1, 1] \to \mathcal{H}_n(\mathbb{F}), \quad \alpha_x(t) = e^{ta} x e^{-ta} + tc.$$

Then $\{\alpha_x : [-1, 1] \to \mathcal{H}_n(\mathbb{F})\}_{x \in \mathcal{H}_n(\mathbb{F})}$ is a rigid motion of $(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)$. The induced infinitesimal rigid motion is the vector field

$$\eta : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{F}), \quad x \mapsto ax - xa + c.$$

Thus $\Psi_{\mathcal{H}_n(\mathbb{F})}(\eta) = (a, -a, c)$ and so $(a, -a, c)$ is contained in the range of $\Psi_{\mathcal{H}_n(\mathbb{F})}$. $\square$

Proposition 18. *If* $(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)$ *is an admissible matrix space, then*

$$\dim \mathcal{T}(\mathcal{H}_n(\mathbb{F}), \|\cdot\|) = \begin{cases} n^2 & \text{if } \mathbb{F} = \mathbb{R}, \\ 2n^2 - 1 & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Proof. By Lemma 15, $\Psi_{\mathcal{H}_n(\mathbb{F})}$ is a linear isomorphism onto its range. Thus by Lemma 17 we have $\dim(\mathcal{T}(\mathcal{H}_n(\mathbb{F}), \|\cdot\|)) = \dim(\mathrm{Skew}_n^0(\mathbb{F}) \oplus \mathcal{H}_n(\mathbb{F}))$, which gives the advertised values. $\square$

3. Infinitesimal rigidity for admissible matrix spaces

In this section we develop infinitesimal rigidity theory for admissible matrix spaces. Our primary goal is to obtain necessary counting conditions for graphs which admit an infinitesimally rigid placement in a given admissible matrix space. This is achieved in Theorem 32, where we provide analogues of the Maxwell counting criteria for Euclidean bar-joint frameworks [20], and in Theorem 33, where we show that minimally rigid graphs belong to the matroidal class of (k, l)-sparse graphs for suitable k and l (see [16]).

Throughout this section X will be a finite dimensional real linear space and $G = (V, E)$ will be a finite simple graph. A *bar-joint framework* in X is a pair (G, p) consisting of a graph G and a map $p : V \to X$, $v \mapsto p_v$, called a *placement* of G in X, with the property that $p_v \neq p_w$ for all $vw \in E$. A *subframework* of (G, p) is a bar-joint framework (H, p_H) with $H = (V(H), E(H))$ a subgraph of G and $p_H(v) = p(v)$ for all $v \in V(H)$.

3.1. Support functionals

Recall that if $\|\cdot\|$ is a norm on X, then a *support functional* for a unit vector $x_0 \in X$ is a linear functional $f : X \to \mathbb{R}$ with $\|f\| := \sup\{|f(x)|: x \in X, \|x\| = 1\} \leq 1$, and $f(x_0) = 1$. The norm $\|\cdot\|$ is said to be *smooth* at $x \in X \setminus \{0\}$ if there exists exactly one support functional at $\frac{x}{\|x\|}$, and we say that $\|\cdot\|$ is *smooth* if it is smooth at every $x \in X \setminus \{0\}$.

We will require the following facts (for details see [13, Section 2]).

Lemma 19. *Let (G, p) be a bar-joint framework in a normed linear space $(X, \|\cdot\|)$, let $vw \in E$ and let $p_0 = \frac{p_v - p_w}{\|p_v - p_w\|}$.*

(i) The norm $\|\cdot\|$ is smooth at $p_v - p_w$ if and only if the limit

$$\varphi_{v,w}(x) := \lim_{t \to 0} \frac{1}{t}(\|p_0 + tx\| - \|p_0\|) \tag{2}$$

exists for all $x \in X$.

(ii) If the norm is smooth at $p_v - p_w$, then the map $\varphi_{v,w} : X \to \mathbb{R}$ is the unique support functional for p_0.

Recall from the introduction that every unitarily invariant norm on $\mathcal{M}_n(\mathbb{F})$ arises from a symmetric norm on $\mathbb{R}^n$ and that $\sigma(x) \in \mathbb{R}^n$ denotes the vector of singular values, arranged in decreasing order, for a matrix $x \in \mathcal{M}_n(\mathbb{F})$.

Lemma 20. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_n(\mathbb{F})$, with corresponding symmetric norm $\|\cdot\|_s$ on $\mathbb{R}^n$, and let $x \in \mathcal{M}_n(\mathbb{F})$. Then $\|\cdot\|$ is smooth at x if and only if $\|\cdot\|_s$ is smooth at $\sigma(x)$.*

Proof. The result follows from [28, Theorem 2]. $\square$

Support functionals for the Schatten p-norms are described in [1]. We apply these results below to characterise the support functionals $\varphi_{v,w}$.

Example 21. Let $1 \leq q \leq \infty$ and let (G, p) be a bar-joint framework in $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|_{c_q})$. Let $vw \in E$, suppose the norm is smooth at $p_v - p_w$ and let $p_0 = \frac{p_v - p_w}{\|p_v - p_w\|_{c_q}}$.

(a) If $q < \infty$, then for all $x \in \mathcal{M}_n(\mathbb{F})$,

$$\varphi_{v,w}(x) = \operatorname{trace}(x|p_0|^{q-1}u^*)$$

where $p_0 = u|p_0|$ is the polar decomposition of p_0.

(b) If $q = \infty$, then by Lemma 20, the largest singular value of the matrix p_0 has multiplicity one. Thus p_0 attains its norm at a unit vector $\zeta \in \mathbb{F}^n$ which is unique (up to scalar multiples). It follows that for all $x \in \mathcal{M}_n(\mathbb{F})$, we have

$$\varphi_{v,w}(x) = \langle x\zeta, p_0\zeta \rangle$$

where $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product on $\mathbb{F}^n$.

3.2. Well-positioned frameworks

A bar-joint framework (G,p) is said to be *well-positioned* in $(X,\|\cdot\|)$ if the norm $\|\cdot\|$ is smooth at $p_v - p_w$ for every edge $vw \in E$.

The following criteria apply to well-positioned bar-joint frameworks in the case of Schatten p-norms.

Proposition 22. *Let $1 \leq q \leq \infty$ with $q \neq 2$, and suppose that $(X, \|\cdot\|_{c_q})$ is an admissible matrix space in $\mathcal{M}_n(\mathbb{F})$. Let (G,p) be a bar-joint framework in $(X, \|\cdot\|_{c_q})$.*

(i) If $q \notin \{1,\infty\}$, then (G,p) is well-positioned.
(ii) If $q = 1$ and $p_v - p_w$ is invertible for all $vw \in E$, then (G,p) is well-positioned. For $X = \mathcal{M}_n(\mathbb{F})$, the converse also holds.
(iii) If $q = \infty$ and $\sigma_1(p_v - p_w) > \sigma_2(p_v - p_w)$ for all $vw \in E$, then (G,p) is well-positioned. For $X = \mathcal{M}_n(\mathbb{F})$, the converse also holds.

Proof. Observe first that if (G,p) is well-positioned in $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|_{c_q})$, then (G,p) is necessarily well-positioned in $(X, \|\cdot\|_{c_q})$. Hence it suffices to give a proof in the case $X = \mathcal{M}_n(\mathbb{F})$. Recall that the ℓ_q norm on $\mathbb{R}^n$ is smooth at the following vectors:

(i) at every non-zero vector in $\mathbb{R}^n$ if $q \notin \{1,\infty\}$;
(ii) at every vector with every entry non-zero if $q = 1$; and
(iii) at every vector $\sigma = (\sigma_1, \dots, \sigma_n)$ so that $\max_{1\leq i\leq n} |\sigma_i|$ is attained at precisely one $i \in \{1,2,\dots,n\}$, if $q = \infty$.

It now suffices to apply Lemma 20. □

3.3. The rigidity map

As in [13], we consider the *rigidity map* f_G, given by

$$f_G : X^V \to \mathbb{R}^E, \qquad (x_v)_{v\in V} \mapsto (\|x_v - x_w\|)_{vw\in E}.$$

If the rigidity map is differentiable at $p \in X^V$, then

$$df_G(p) : X^V \to \mathbb{R}^E,$$

is the differential of f_G at p. Here we equip X^V with the norm topology.

We will require the following results.

Lemma 23. *[13, Proposition 6] Let (G,p) be a bar-joint framework in a normed linear space $(X,\|\cdot\|)$.*

(i) (G,p) is well-positioned in $(X,\|\cdot\|)$ if and only if the rigidity map f_G is differentiable at p.
(ii) If (G,p) is well-positioned in $(X,\|\cdot\|)$ then the differential of the rigidity map is given by

$$df_G(p) : X^V \to \mathbb{R}^E, \quad (z_v)_{v\in V} \mapsto (\varphi_{v,w}(z_v - z_w))_{vw\in E}.$$

An *infinitesimal flex* of a bar-joint framework (G,p) is a vector $z \in X^V$ such that

$$\lim_{t\to 0} \frac{1}{t}(f_G(p+tz) - f_G(p)) = 0.$$

The collection of all infinitesimal flexes of (G,p) is denoted $\mathcal{F}(G,p)$. Note that, by Lemma 23, if (G,p) is well-positioned then $\mathcal{F}(G,p) = \ker df_G(p)$.

3.4. Full sets

Given a normed space $(X, \|\cdot\|)$, and a non-empty subset $S \subseteq X$, consider the restriction map,

$$\rho_S : \mathcal{T}(X, \|\cdot\|) \to X^S, \quad \eta \mapsto (\eta(x))_{x\in S}.$$

Definition 24. A non-empty subset $S \subseteq X$ is *full* in $(X, \|\cdot\|)$ if the restriction map ρ_S is injective; that is, if S is a separating set for $\mathcal{T}(X, \|\cdot\|)$.

Recall that S is said to have *full affine span* in X if $[S] = X$, where $[S]$ is the affine span of S, namely the translation by s_0 of the linear span of $\{s - s_0 : s \in S\}$, where s_0 is any fixed vector in S. (Note that $[S]$ is independent of the choice of s_0.)

Lemma 25. *Let $(X, \|\cdot\|)$ be a normed space and let $\emptyset \neq S \subseteq X$. If S has full affine span in X, then S is full in $(X, \|\cdot\|)$.*

Proof. Let $\eta \in \mathcal{T}(X, \|\cdot\|)$ and suppose $\rho_S(\eta) = 0$. By Lemma 12, η is an affine map and so $\eta(X) = \eta([S]) = 0$. $\square$

Remark 26. Note that full affine span is not strictly necessary for S to be full in a normed space $(X, \|\cdot\|)$. For example, if $[S]$ is the set of upper triangular $n \times n$ matrices then it is not difficult to see that S is full in $(\mathcal{M}_n(\mathbb{F}), \|\cdot\|)$ for any admissible norm.

Definition 27. We say that a bar-joint framework (G,p) in a normed space $(X, \|\cdot\|)$ is,

(a) *full* if $\{p_v : v \in V\}$ is full in $(X, \|\cdot\|)$.
(b) *completely full* if (G,p), and every subframework (H, p_H) of (G,p) with $|V(H)| \geq 2\dim(X)$, is full in $(X, \|\cdot\|)$.

Remark 28. We remark that the property of being completely full, which will be required in Theorem 33, is satisfied by almost all bar-joint frameworks. Indeed, if (G,p) is a bar-joint framework in X and $S = \{p_v : v \in V\}$ is in general position in X, then every subset of S containing at least $\dim(X)+1$ points has full affine span in X. Thus, by Lemma 25, (G,p) is completely full in $(X, \|\cdot\|)$, for all norms on X.

3.5. $k(X)$ and $l(X)$ values

For $X \in \{\mathcal{M}_n(\mathbb{F}), \mathcal{H}_n(\mathbb{F})\}$, we define natural numbers $k(X)$ and $l(X)$ according to the formulae in Table 1. Note that $k(X) = \dim X$ and by Propositions 16 and 18, we have $l(X) = \dim(\mathcal{T}(X, \|\cdot\|))$ for any admissible norm $\|\cdot\|$ on X. For ease of reference, the cases $n = 2$ and $n = 3$ are listed in Table 2.

We will require the following result. As usual, we take $n \in \mathbb{N}$ with $n \geq 2$, and for $m \in \mathbb{N}$ we write K_m for the complete graph on m vertices.

Table 1
k and l values for admissible matrix spaces.

X	$k(X)$	$l(X)$
$\mathcal{H}_n(\mathbb{R})$	$\frac{1}{2}n(n+1)$	n^2
$\mathcal{M}_n(\mathbb{R})$	n^2	$2n^2-n$
$\mathcal{H}_n(\mathbb{C})$	n^2	$2n^2-1$
$\mathcal{M}_n(\mathbb{C})$	$2n^2$	$4n^2-1$

Table 2
k and l values for admissible matrix spaces when $n = 2$ and $n = 3$.

X	$k(X)$	$l(X)$
$\mathcal{H}_2(\mathbb{R})$	3	4
$\mathcal{M}_2(\mathbb{R})$	4	6
$\mathcal{H}_2(\mathbb{C})$	4	7
$\mathcal{M}_2(\mathbb{C})$	8	15

X	$k(X)$	$l(X)$
$\mathcal{H}_3(\mathbb{R})$	6	9
$\mathcal{M}_3(\mathbb{R})$	9	15
$\mathcal{H}_3(\mathbb{C})$	9	17
$\mathcal{M}_3(\mathbb{C})$	18	35

Lemma 29. *Let $X \in \{\mathcal{M}_n(\mathbb{F}), \mathcal{H}_n(\mathbb{F})\}$, let $(k,l) = (k(X), l(X))$. Consider $m \in \mathbb{N}$.*

(i) $|E(K_m)| \le km - l$ if and only if $m \in \{2, \dots, 2k-1\}$.
(ii) $|E(K_m)| = km - l$ if and only if $\mathbb{F} = \mathbb{C}$ and $m \in \{2, 2k-1\}$.

Proof. Consider the quadratic function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \frac{1}{2}(t^2 - (2k+1)t + 2l).$$

It is easy to see that $f(m) = |E(K_m)| - (km - l)$, and $f(1) = f(2k) = l - k > 0$. Moreover,

$$f(2) = f(2k-1) = l + 1 - 2k = \begin{cases} -(n-1) & \text{if } \mathbb{F} = \mathbb{R}, \\ 0 & \text{if } \mathbb{F} = \mathbb{C} \end{cases}$$

so $f(2) = f(2k-1) \le 0$, with equality if and only if $\mathbb{F} = \mathbb{C}$. The claims follow immediately. □

3.6. Trivial infinitesimal flexes

Given a bar-joint framework (G,p), we define

$$\mathcal{T}(G,p) = \{\zeta \colon V \to X \mid \zeta = \eta \circ p \text{ for some } \eta \in \mathcal{T}(X, \|\cdot\|)\} \subseteq X^V.$$

Note that $\mathcal{T}(G,p)$ is a subspace of $\mathcal{F}(G,p)$, the space of infinitesimal flexes of (G,p) (see [12, Lemma 2.3]). The elements of $\mathcal{T}(G,p)$ are referred to as the *trivial infinitesimal flexes* of (G,p).

Example 30. Suppose (G,p) is a bar-joint framework in an admissible matrix space $(X, \|\cdot\|)$. If $X = \mathcal{M}_n(\mathbb{F})$, then by Lemma 15 we have

$$\mathcal{T}(G,p) = \{(ap_v + p_v b + c)_{v \in V} : a \in \mathrm{Skew}_n(\mathbb{F}),\ b \in \mathrm{Skew}_n^0(\mathbb{F}),\ c \in \mathcal{M}_n(\mathbb{F})\},$$

and if $X = \mathcal{H}_n(\mathbb{F})$, then by Lemma 17 we have

$$\mathcal{T}(G,p) = \{(ap_v - p_v a + c)_{v \in V} : a \in \mathrm{Skew}_n^0(\mathbb{F}),\ c \in \mathcal{H}_n(\mathbb{F})\}.$$

Lemma 31. *If (G,p) is a full bar-joint framework in a normed linear space $(X, \|\cdot\|)$, then*

$$\dim \mathcal{T}(G,p) = \dim \mathcal{T}(X, \|\cdot\|).$$

In particular, if $X \in \{\mathcal{M}_n(\mathbb{F}), \mathcal{H}_n(\mathbb{F})\}$ and $(X, \|\cdot\|)$ is an admissible matrix space, then $\dim \mathcal{T}(G,p) = l(X)$.

Proof. Observe that the linear map

$$\rho_{(G,p)} : \mathcal{T}(X, \|\cdot\|) \to X^V, \quad \eta \mapsto (\eta(p_v))_{v \in V},$$

has range $\mathcal{T}(G,p)$. Since $\{p_v : v \in V\}$ is full in $(X, \|\cdot\|)$, $\rho_{(G,p)}$ is also injective. Thus, $\dim \mathcal{T}(G,p) = \dim \mathcal{T}(X, \|\cdot\|)$. $\square$

3.7. Infinitesimal rigidity

A bar-joint framework (G,p) is *infinitesimally rigid* if every infinitesimal flex of (G,p) is trivial (i.e., if $\mathcal{F}(G,p) = \mathcal{T}(G,p)$); otherwise, we say that (G,p) is *infinitesimally flexible*. A framework (G,p) is said to be *minimally infinitesimally rigid* if it is infinitesimally rigid and every subframework obtained by removing an edge from G is infinitesimally flexible.

The following results are analogous to Maxwell's counting criteria for bar-joint frameworks in Euclidean space [20].

Theorem 32. *Let (G,p) be a full and well-positioned bar-joint framework in an admissible matrix space $(X, \|\cdot\|)$, where $X \in \{\mathcal{M}_n(\mathbb{F}), \mathcal{H}_n(\mathbb{F})\}$, and let $(k,l) = (k(X), l(X))$.*

(i) If (G,p) is infinitesimally rigid, then $|E| \geq k|V| - l$.
(ii) If (G,p) is minimally infinitesimally rigid, then $|E| = k|V| - l$.
(iii) If (G,p) is minimally infinitesimally rigid and (H, p_H) is a full subframework of (G,p), then $|E(H)| \leq k|V(H)| - l$.

Proof. Apply [13, Theorem 10] and Lemma 31. $\square$

Let $k, l \in \mathbb{N}$ with $l \in \{0, \dots, 2k-1\}$. As is standard in combinatorial rigidity theory, a graph $G = (V,E)$ is said to be *(k,l)-sparse* if every subgraph $H = (V(H), E(H))$ with $|V(H)| \geq 2$ has at most $k|V(H)| - l$ edges. If in addition $|E| = k|V| - l$, then G is said to be *(k,l)-tight*.

Theorem 33. *Let $\|\cdot\|$ be an admissible norm on $X \in \{\mathcal{M}_n(\mathbb{F}), \mathcal{H}_n(\mathbb{F})\}$, and let $(k,l) = (k(X), l(X))$. Let (G,p) be a completely full and well-positioned bar-joint framework in $(X, \|\cdot\|)$. If (G,p) is minimally infinitesimally rigid, then G is (k,l)-tight.*

Proof. By Theorem 32(*ii*), $|E| = k|V| - l$. Let H be a subgraph of G with $m \geq 2$ vertices. If $m \geq 2k$ then, since (G,p) is completely full, the subframework (H, p_H) is full in $(X, \|\cdot\|)$. Thus, by Theorem 32(*iii*), $|E(H)| \leq k|V(H)| - l$. If $2 \leq m \leq 2k-1$, then by Lemma 29, $|E(H)| \leq |E(K_m)| \leq k|V(H)| - l$. Thus G is (k,l)-tight. $\square$

Remark 34. The (k,l)-sparsity of a multi-graph can be determined for the range $l \in \{0, \dots, 2k-1\}$ by a polynomial time algorithm known as a *pebble game* [16]. As such, the (k,l)-tight conditions obtained above can be verified in $O(|V|^2)$ time.

In the case of admissible norms on $\mathcal{H}_2(\mathbb{R})$, the following additional graph properties are necessary for minimal infinitesimal rigidity. Here we regard the simple graph G as a member of the wider class of multi-graphs with no loops. We also recall that a subgraph H of G is said to be *spanning* if every vertex of G is the endpoint of some edge of H.

Corollary 35. *Let $\|\cdot\|$ be an admissible norm on $\mathcal{H}_2(\mathbb{R})$ and let (G,p) be a completely full, well-positioned and minimally infinitesimally rigid bar-joint framework in $(\mathcal{H}_2(\mathbb{R}),\|\cdot\|)$.*

(i) G can be constructed from a single vertex using a sequence of graph moves of the following form:
- *Adjoin a new vertex v which is incident with at most three new edges, at most two of which are parallel.*
- *Remove a set E' of i edges, where $i\in\{1,2\}$, and let V' be the set of vertices for edges in E'. Adjoin a new vertex v which is incident with each vertex in V'. Adjoin $3-i$ additional edges which are each incident with v and a vertex not in V', such that no three edges in the resulting multi-graph are parallel.*

(ii) If a single edge is added to G then the resulting multi-graph is an edge disjoint union of three spanning trees.

Proof. By Theorem 33, G is $(3,4)$-tight and so (i) is an application of [7, Theorem 1.9] whereas (ii) follows by an argument of Nash-Williams [21] applied to $(3,3)$-tight graphs. □

For $G=K_m$, it follows from the Maxwell counting criteria (Theorem 32) and Lemma 29 that a full and well-positioned bar-joint framework (K_m,p) in an admissible matrix space $(X,\|\cdot\|)$ is not infinitesimally rigid in the following cases:

(i) $X=\mathcal{M}_n(\mathbb{R})$ or $\mathcal{H}_n(\mathbb{R})$, $k=\dim X$ and $m\in\{2,\dots,2k-1\}$.
(ii) $X=\mathcal{M}_n(\mathbb{C})$ or $\mathcal{H}_n(\mathbb{C})$, $k=\dim X$ and $m\in\{2,\dots,2k-2\}$.

We make the following conjectures for larger values of m.

Conjecture 36. *Let $\|\cdot\|$ be an admissible norm on $X\in\{\mathcal{M}_n(\mathbb{F}),\mathcal{H}_n(\mathbb{F})\}$ and let $k=\dim X$.*

(i) If $\mathbb{F}=\mathbb{R}$, then there exists $p\in X^V$ such that (K_m,p) is full, well-positioned and infinitesimally rigid in $(X,\|\cdot\|)$ for all $m\geq 2k$.
(ii) If $\mathbb{F}=\mathbb{C}$, then there exists $p\in X^V$ such that (K_m,p) is full, well-positioned and infinitesimally rigid in $(X,\|\cdot\|)$ for all $m\geq 2k-1$.

In Section 5 we will show that these conjectures hold when $X=\mathcal{H}_2(\mathbb{F})$ and the admissible norm is the trace norm. Namely, we show that there exists $p\in\mathcal{H}_2(\mathbb{R})^V$ such that (K_m,p) is full, well-positioned and infinitesimally rigid in $(\mathcal{H}_2(\mathbb{R}),\|\cdot\|_{c_1})$ for all $m\geq 6$, and, that there exists $p\in\mathcal{H}_2(\mathbb{C})^V$ such that (K_m,p) is full, well-positioned and infinitesimally rigid in $(\mathcal{H}_2(\mathbb{C}),\|\cdot\|_{c_1})$ for all $m\geq 7$.

4. Product norms

In this section we extend to the setting of product norms a framework colouring technique which was introduced in [12] to characterise rigidity in $(\mathbb{R}^d,\|\cdot\|_\infty)$. Our main result is Theorem 47, in which we characterise infinitesimal rigidity with respect to a product norm in terms of projected monochrome sub-frameworks. We will apply the results of this section to the admissible matrix space $(\mathcal{H}_2(\mathbb{F}),\|\cdot\|_{c_1})$ in Section 5.

Let $(X_1, \|\cdot\|_1), \dots, (X_n, \|\cdot\|_n)$ be a finite collection of finite dimensional real normed linear spaces and let $X = X_1 \times \cdots \times X_n$ be the product space. The *product norm* $\|\cdot\|_\pi$ on X is defined by

$$\|x\|_\pi = \max_{j=1,2,\dots,n} \|x_j\|_j,$$

for all $x = (x_1, \dots, x_n) \in X$. For each $j = 1, \dots, n$, denote by P_j the projection onto X_j given by

$$P_j : X \to X_j, \quad (x_1, \dots, x_n) \mapsto x_j,$$

and denote by P_j^* the embedding of X_j into X given by

$$P_j^* : X_j \to X, \quad y \mapsto (0, \dots, \overset{j^{\text{th}}}{y}, \dots, 0).$$

Clearly, P_j^* is an isometry and $\|P_j\| := \sup\{\|P_j(x)\|_j : x \in X, \|x\|_\pi \leq 1\} = 1$. Moreover, $\sum_{j=1}^n P_j^* P_j$ and $P_i P_i^*$ are the identity maps on X and X_i, respectively (where, as usual, we write AB for the composition of two linear maps A and B), and $P_j P_i^* = 0$ if $i \neq j$.

Lemma 37. *Let $x \in X$ with $\|x\|_\pi = \|P_j(x)\|_j = 1$. If φ_j is a support functional for $P_j(x)$ in $(X_j, \|\cdot\|_j)$, then $\varphi = \varphi_j \circ P_j$ is a support functional for x in $(X, \|\cdot\|_\pi)$.*

Proof. We have $\|\varphi\| \leq \|\varphi_j\| \, \|P_j\| = \|\varphi_j\| \leq 1$ and $\varphi(x) = 1$. □

4.1. Framework colours

For $x = (x_1, \dots, x_n) \in X$, we write

$$\kappa(x) = \big\{ j \in \{1, 2, \dots, n\} : \|x\|_\pi = \|x_j\|_j \big\}.$$

We think of the non-empty set $\kappa(x)$ as a set of colours assigned to x by the product norm $\|\cdot\|_\pi$.

We leave the proof of the following elementary lemma to the reader.

Lemma 38. *If x is a unit vector in X and $\kappa(x) = \{j\}$ is a singleton, then there exists $\delta > 0$ so that $\kappa(x + y) = \{j\}$ whenever $y \in X$ with $\|y\|_\pi \leq \delta$.*

Let (G, p) be a bar-joint framework in $(X, \|\cdot\|_\pi)$. There is a natural edge-labelling κ_p where for each edge $vw \in E$, we define $\kappa_p(vw) = \kappa(p_v - p_w)$. An edge $vw \in E$ is said to have *framework colour* j if $j \in \kappa_p(vw)$. The set of all edges in G which have framework colour j is denoted E_j, and we have $E = E_1 \cup \cdots \cup E_n$.

Let $G_j = (V, E_j)$ denote the subgraph of G with the same vertex set as G and edge set E_j consisting of all edges with framework colour j. We refer to G_j as a *monochrome subgraph* of G. Note that G_j may contain vertices of degree 0 (even if G does not). The pair (G_j, p) is a bar-joint framework in X and is referred to as the *monochrome subframework of* (G, p) *with framework colour* j.

For each $j = 1, \dots, n$, we write $p_j = P_j \circ p$. If $vw \in E_j$, then

$$\|p_j(v) - p_j(w)\|_j = \|p(v) - p(w)\|_\pi \neq 0,$$

so (G_j, p_j) is a bar-joint framework in X_j. We call (G_j, p_j) the *projected monochrome subframework with framework colour* j.

If (G_j, p_j) is well-positioned in $(X_j, \|\cdot\|_j)$ and $vw \in E_j$, then we write $\varphi^j_{v,w}$ for the support functional at the unit vector $P_j(p_0)$ in X_j, where $p_0 = \frac{p_v - p_w}{\|p_v - p_w\|_\pi}$.

Proposition 39. *A framework (G, p) in $(X, \|\cdot\|_\pi)$ is well-positioned in $(X, \|\cdot\|_\pi)$ if and only if*

(i) each edge $vw \in E$ has exactly one framework colour, and
(ii) (G_j, p_j) is well-positioned in $(X_j, \|\cdot\|_j)$ for each $j = 1, 2, \ldots, n$.

Moreover, in this case we have $\varphi_{v,w} = \varphi^j_{v,w} \circ P_j$ for every edge $vw \in E_j$.

Proof. Let $vw \in E$ and write $p_0 = \frac{p_v - p_w}{\|p_v - p_w\|_\pi}$. If (G, p) is well-positioned in $(X, \|\cdot\|_\pi)$ then by Lemma 23, the product norm is smooth at p_0 and so p_0 has exactly one support functional. Suppose i and j are two distinct framework colours for vw. Then $\|p_v - p_w\|_\pi = \|P_i(p_v - p_w)\|_i = \|P_j(p_v - p_w)\|_j$ and so $1 = \|p_0\|_\pi = \|P_i(p_0)\|_i = \|P_j(p_0)\|_j$. Choose support functionals φ_i and φ_j for $P_i(p_0)$ and $P_j(p_0)$ in $(X_i, \|\cdot\|_i)$ and $(X_j, \|\cdot\|_j)$ respectively. By Lemma 37, both $\varphi_i \circ P_i$ and $\varphi_j \circ P_j$ are support functionals for p_0, so by smoothness, $\varphi_i \circ P_i = \varphi_j \circ P_j$. Now $\varphi_i = \varphi_i \circ P_i \circ P_i^* = \varphi_j \circ P_j \circ P_i^* = 0$. This is a contradiction since $\varphi_i(P_i(p_0)) = 1$ and so (i) holds.

Suppose $vw \in E_j$. By Lemma 37, if ψ_1 and ψ_2 are two support functionals for $P_j(p_0)$, then $\psi_1 \circ P_j$ and $\psi_2 \circ P_j$ are both support functionals for p_0, hence are equal. Now $\psi_1 = \psi_1 \circ P_j \circ P_j^* = \psi_2 \circ P_j \circ P_j^* = \psi_2$ and so $P_j(p_0)$ has exactly one support functional. Thus the norm $\|\cdot\|_j$ is smooth at $P_j(p_0)$ and so (G_j, p_j) is well-positioned in $(X_j, \|\cdot\|_j)$ by Lemma 23. This proves (ii).

For the converse, if (i) and (ii) hold then consider an edge $vw \in E$ and again write $p_0 = \frac{p_v - p_w}{\|p_v - p_w\|_\pi}$. By (i), vw has a unique framework colour, say j. By Lemma 38, for any $z \in X$ we have

$$\lim_{t\to 0} \frac{1}{t}(\|p_0 + tz\|_\pi - \|p_0\|_\pi) = \lim_{t\to 0} \frac{1}{t}(\|P_j(p_0) + tP_j(z)\|_j - \|P_j(p_0)\|_j),$$

where, by (ii) and Lemma 23, the latter limit exists (and is in fact equal to $\varphi^j_{v,w}(P_j(p_0))$). Thus the product norm is smooth at p_0 and so (G, p) is well-positioned in $(X, \|\cdot\|_\pi)$.

The final claim follows directly from Lemma 37 and the uniqueness of the support functional $\varphi_{v,w}$. □

If $z : V \to X$, then we write $z_j : V \to X_j$, $v \mapsto P_j(z(v))$, and define the linear isomorphism

$$\Phi_V : X^V \to \bigoplus_{j=1}^n X_j^V, \qquad z \mapsto (z_1, \ldots, z_n).$$

By Proposition 39, the monochrome edge sets $E_1, \ldots, E_n$ arising from a well-positioned bar-joint framework (G, p) partition E. Hence, writing $\lambda_j \colon E_j \to \mathbb{R}$ for the restriction to E_j of a map $\lambda : E \to \mathbb{R}$, we have a linear isomorphism

$$\Phi_E : \mathbb{R}^E \to \bigoplus_{j=1}^n \mathbb{R}^{E_j}, \qquad \lambda \mapsto (\lambda_1, \ldots, \lambda_n).$$

Corollary 40. *If (G, p) is well-positioned in $(X, \|\cdot\|_\pi)$, then*

$$df_G(p) = \Phi_E^{-1} \circ (df_{G_1}(p_1) \oplus \cdots \oplus df_{G_n}(p_n)) \circ \Phi_V.$$

Proof. Apply Lemma 23 and Proposition 39. □

Corollary 41. *Let (G, p) be a well-positioned framework in $(X, \|\cdot\|_\pi)$.*

(i) $z \in \mathcal{F}(G, p)$ if and only if $z_j \in \mathcal{F}(G_j, p_j)$ for each $j = 1, \ldots, n$.

(ii) The map

$$\Phi_{(G,p)} : \mathcal{F}(G,p) \to \bigoplus_{j=1}^{n} \mathcal{F}(G_j, p_j), \quad z \mapsto (z_1, \dots, z_n),$$

is a linear isomorphism.

(iii) $\dim \mathcal{F}(G,p) = \sum_{j=1}^{n} \dim \mathcal{F}(G_j, p_j)$.

Proof. The statements follow immediately from Corollary 40 and the observation that $\Phi_{(G,p)}$ is the restriction of Φ_V to the kernel of $df_G(p)$. □

4.2. Rigid motions of product spaces

We will now see that the infinitesimal rigid motions of $(X, \|\cdot\|_\pi)$ coincide with direct sums of infinitesimal rigid motions of the factor spaces $(X_j, \|\cdot\|_j)$.

Given $\alpha_j \in \mathcal{R}(X_j, \|\cdot\|_j)$ and $\eta_j \in \mathcal{T}(X_j, \|\cdot\|_j)$ for $j = 1, \dots, n$, let us define

$$\bigoplus_{j=1}^{n} \alpha_j \colon X \times [-1,1] \to X, \quad (x,t) \mapsto \sum_{j=1}^{n} P_j^* \alpha_j(P_j(x), t)$$

and

$$\bigoplus_{j=1}^{n} \eta_j \colon X \to X, \quad x \mapsto \sum_{j=1}^{n} P_j^* \eta_j(P_j(x)).$$

Lemma 42. *For $j = 1, \dots, n$, let $\alpha_j \in \mathcal{R}(X_j, \|\cdot\|_j)$ and let $\eta_j \in \mathcal{T}(X_j, \|\cdot\|_j)$ be the infinitesimal rigid motion induced by α_j, and consider $\alpha = \bigoplus_{j=1}^{n} \alpha_j$. We have*

(i) $\alpha \in \mathcal{R}(X, \|\cdot\|_\pi)$; and
(ii) the infinitesimal rigid motion induced by α is $\eta := \bigoplus_{j=1}^{n} \eta_j$; and
(iii) $\eta_j = P_j \circ \eta \circ P_j^*$ for each $j = 1, \dots, n$.

Proof. It is clear that $\alpha_x \colon [-1,1] \to X$ is continuous for each $x \in X$ and

$$\alpha_x(0) = \sum_{j=1}^{n} P_j^* \alpha_j(P_j(x), 0) = \sum_{j=1}^{n} P_j^* P_j(x) = x.$$

Also, for any $x, y \in X$ and $t \in [-1,1]$,

$$\begin{aligned}
\|\alpha_x(t) - \alpha_y(t)\|_\pi &= \|\sum_{j=1}^{n} P_j^* \alpha_j(P_j(x), t) - \sum_{j=1}^{n} P_j^* \alpha_j(P_j(y), t)\|_\pi \\
&= \max_{j=1,\dots,n} \|\alpha_j(P_j(x), t) - \alpha_j(P_j(y), t)\|_j \\
&= \max_{j=1,\dots,n} \|P_j(x) - P_j(y)\|_j \\
&= \|x - y\|_\pi.
\end{aligned}$$

Note that for each $x \in X$,

$$\alpha_x'(0) = \sum_{j=1}^{n} P_j^*((\alpha_j)_{P_j(x)})'(0) = \sum_{j=1}^{n} P_j^* \eta_j(P_j(x)).$$

Thus $\alpha \in \mathcal{R}(X, \|\cdot\|_\pi)$ and $\eta = \sum_{j=1}^{n} P_j^* \circ \eta_j \circ P_j$ is its induced infinitesimal rigid motion. Finally, for each $j = 1, \ldots, n$,

$$P_j \circ \eta \circ P_j^* = \sum_{k=1}^{n} (P_j P_k^*) \circ \eta_k \circ (P_k P_j^*) = \eta_j. \quad \square$$

For a finite dimensional normed vector space Y, we write $\mathrm{Isom}(Y)$ for the set of linear isometries of Y, equipped with the norm topology.

Proposition 43. *If $A\colon [-1,1] \to \mathrm{Isom}(X)$ is continuous and $A(0) = I_X$, then there exists $\delta > 0$ so that $A(t) \in \bigoplus_{i=1}^{n} \mathrm{Isom}(X_i)$ for every $t \in [-\delta, \delta]$.*

Proof. To simplify notation, we consider the case $n = 2$; the general case is similar. Write

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$$

where each $A_{ij}(t)$ is a linear map from X_j to X_i. It is then easy to see that $\|A_{ij}(t)\| \le \|A(t)\| = 1$ for every i, j and t. We first claim that $A_{jj}(t) \in \mathrm{Isom}(X_j)$ for $j = 1, 2$ and $|t|$ sufficiently small. If not, taking $j = 1$ without loss of generality, there exist $t_n \to 0$ and $x_n \in X_1$ with $\|x_n\|_1 = 1$ so that $\|A_{11}(t_n)x_n\|_1 < 1$ for every $n \in \mathbb{N}$. Now

$$1 = \left\| A(t_n) \begin{bmatrix} x_n \\ 0 \end{bmatrix} \right\|_\pi = \left\| \begin{bmatrix} A_{11}(t_n)x_n \\ A_{21}(t_n)x_n \end{bmatrix} \right\|_\pi = \max\{\|A_{11}(t_n)x_n\|_1, \|A_{21}(t_n)x_n\|_2\}$$

so we necessarily have $\|A_{21}(t_n)x_n\|_2 = 1$ for every n. However,

$$\|A_{21}(t_n)x_n\|_2 \le \|A_{21}(t_n)\| \to 0 \text{ as } n \to \infty,$$

a contradiction which establishes the claim.

Hence there exists $\delta > 0$ so that $A_{jj}(t) \in \mathrm{Isom}(X_j)$ for $j = 1, 2$ and $|t| \le \delta$. We now claim that $A_{ij}(t) = 0$ for $|t| \le \delta$ and $i \ne j$. We show this for $(i, j) = (1, 2)$, and the other case follows by symmetry. Suppose instead that this claim fails at some $t \in [-\delta, \delta]$, and write $A = A(t)$ and $A_{ij} = A_{ij}(t)$. Since $A_{12} \ne 0$, we can find a unit vector $y \in X_2$ with $A_{12}y \ne 0$. Since A_{11} is an isometry, it is invertible; let $x = \|A_{12}y\|_1^{-1} A_{11}^{-1} A_{12}y$. Since A_{11} is an isometry, we have $\|x\|_1 = 1$, so

$$\left\| A \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\pi = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\pi = \max\{\|x\|_1, \|y\|_2\} = 1.$$

On the other hand,

$$\begin{aligned} \left\| A \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\pi &= \left\| \begin{bmatrix} (\|A_{12}y\|_1^{-1} + 1)A_{12}y \\ * \end{bmatrix} \right\|_\pi \\ &\ge \|(\|A_{12}y\|_1^{-1} + 1)A_{12}y\|_1 \\ &= 1 + \|A_{12}y\|_1 > 1 \end{aligned}$$

where $*$ denotes an unimportant matrix entry. This contradiction shows that $A_{12} = 0$. Hence $A(t) = A_{11}(t) \oplus A_{22}(t)$ for $|t| \le \delta$, as desired. $\square$

Theorem 44. $\mathcal{T}(X, \|\cdot\|_\pi) = \bigoplus_{i=1}^n \mathcal{T}(X_i, \|\cdot\|_i)$.

Proof. The inclusion "$\supseteq$" follows from Lemma 42. For the reverse inclusion, let $\eta \in \mathcal{T}(X, \|\cdot\|_\pi)$ and choose $\alpha \in \mathcal{R}(X, \|\cdot\|_\pi)$ which induces η. By the Mazur-Ulam theorem, we may write

$$\alpha(x,t) = A(t)x + c(t)$$

where $c\colon [-1,1] \to X$ and $A\colon [-1,1] \to \mathrm{Isom}(X)$ are continuous and differentiable at $t = 0$ with $c(0) = 0$ and $A(0) = I_X$. Choose $\delta > 0$ by Proposition 43, so that $A(t) = \bigoplus_{i=1}^n A_i(t)$ for $t \in [-\delta, \delta]$ where $A_i\colon [-\delta, \delta] \to \mathrm{Isom}(X_i)$ are continuous and differentiable at $t = 0$. Consider the map

$$\alpha_i\colon X_i \times [-1,1] \to X_i, \quad \alpha_i(y,t) = P_i\alpha(P_i^* y, \tau) \quad \text{where } \tau = \min\{\delta, t\}.$$

This map is plainly continuous in t and differentiable at $t = 0$, and it is isometric in y since

$$\begin{aligned} \|\alpha_i(y,t) - \alpha_i(z,t)\|_i &= \|A_i(\tau)y + P_ic(\tau) - (A_i(\tau)z + P_ic(\tau))\|_i \\ &= \|A_i(y-z)\|_i = \|y-z\|_i. \end{aligned}$$

Hence $\alpha_i \in \mathcal{R}(X_i, \|\cdot\|_i)$. Moreover, for $|t| \le \delta$ and $x \in X$, we have

$$\begin{aligned} \sum_{j=1}^n P_j^*\alpha_j(P_j(x), t) = \sum_{j=1}^n P_j^* P_j \alpha(P_j^* P_j x, t) &= \sum_{j=1}^n P_j^*(A_j(t)P_jx + P_jc(t)) \\ &= A(t)x + c(t) = \alpha_x(t). \end{aligned}$$

This shows that on a neighbourhood of $t = 0$, the rigid motion α coincides with $\bigoplus_{j=1}^n \alpha_j$. Hence α and $\bigoplus_{j=1}^n \alpha_j$ induce the same infinitesimal rigid motion, namely η. Hence $\eta \in \bigoplus_{j=1}^n \mathcal{T}(X_j, \|\cdot\|_j)$ by Lemma 42. $\square$

As a corollary we obtain the following characterisation for full sets in product spaces.

Corollary 45. *A set $S \subset X$ is full in $(X, \|\cdot\|_\pi)$ if and only if $P_j(S)$ is full in $(X_j, \|\cdot\|_j)$ for each $j = 1, \dots, n$.*

4.3. Trivial infinitesimal flexes of frameworks

Let (G, p) be a bar-joint framework in $(X, \|\cdot\|_\pi)$. For $j = 1, \dots, n$ and $z \in \mathcal{T}(G, p)$ (so that $z\colon V \to X$), we define (as before) $z_j = P_j \circ z\colon V \to X_j$.

Proposition 46. *Let (G, p) be a bar-joint framework in $(X, \|\cdot\|_\pi)$.*

(i) *If $z \in \mathcal{T}(G,p)$, then $z_j \in \mathcal{T}(G_j, p_j)$ for each $j = 1, \dots, n$.*
(ii) *The map*

$$\tilde{\Phi}_{(G,p)} : \mathcal{T}(G,p) \to \bigoplus_{j=1}^n \mathcal{T}(G_j, p_j), \quad z \mapsto (z_1, \dots, z_n)$$

is a linear isomorphism.
(iii) $\dim \mathcal{T}(G,p) = \sum_{j=1}^n \dim \mathcal{T}(G_j, p_j)$.

Proof. (i) Since $z \in \mathcal{T}(G,p)$, there exists an infinitesimal rigid motion $\eta \in \mathcal{T}(X, \|\cdot\|_\pi)$ with $z(v) = \eta(p_v)$ for each $v \in V$. By Theorem 44, $\eta = \bigoplus_{i=1}^n \eta_i$ where $\eta_i \in \mathcal{T}(X_i, \|\cdot\|_i)$ for $i = 1, \dots, n$, so

$$z_j(v) = P_j\left(\bigoplus_{i=1}^n \eta_i\right)(p_v) = \eta_j(P_j(p_v)) = \eta_j(p_j(v)).$$

Thus z_j is the trivial infinitesimal flex of (G_j, p_j) induced by the infinitesimal rigid motion η_j, so $z_j \in \mathcal{T}(G_j, p_j)$.

(ii) By (i) this map is well defined, and it is easily seen to be linear. Since $z = \sum_{j=1}^n P_j^* \circ P_j \circ z = \sum_{j=1}^n P_j^* \circ z_j$ for any $z \in \mathcal{T}(G,p)$, we see immediately that $\tilde{\Phi}_{(G,p)}$ is injective. For surjectivity, observe that if $w = (w_1, \dots, w_n) \in \oplus_{j=1}^n \mathcal{T}(G_j, p_j)$, then $w_j = \eta_j \circ p_j$ for some $\eta_j \in \mathcal{T}(X_j, \|\cdot\|_j)$, hence $w_j = \eta_j \circ P_j \circ p$. We have $\eta := \bigoplus_{j=1}^n \eta_j \in \mathcal{T}(X, \|\cdot\|_\pi)$ by Theorem 44, so $\eta \circ p \in \mathcal{T}(G,p)$. Let $j \in \{1, \dots, n\}$. We have $P_j \circ \eta = \eta_j \circ P_j$, so the jth component of $\tilde{\Phi}_{(G,p)}(\eta \circ p)$ is $P_j \circ \eta \circ p = \eta_j \circ P_j \circ p = w_j$, hence $\tilde{\Phi}_{(G,p)}(\eta \circ p) = w$. Assertion (iii) follows immediately. □

4.4. A characterisation of infinitesimal rigidity

We can now characterise infinitesimal rigidity for well-positioned bar-joint frameworks in terms of their projected monochrome subframeworks.

Theorem 47. *If (G,p) is a well-positioned bar-joint framework in $(X, \|\cdot\|_\pi)$, then the following statements are equivalent.*

(i) *(G,p) is (minimally) infinitesimally rigid in $(X, \|\cdot\|_\pi)$.*
(ii) *The projected monochrome subframeworks (G_j, p_j) are (minimally) infinitesimally rigid in $(X_j, \|\cdot\|_j)$ for each $j = 1, 2, \dots, n$.*

Proof. The statement follows from Corollary 41 and Proposition 46. Indeed, if (G,p) is infinitesimally rigid then

$$\sum_{j=1}^n \dim \mathcal{F}(G_j, p_j) = \dim \mathcal{F}(G,p) = \dim \mathcal{T}(G,p) = \sum_{j=1}^n \dim \mathcal{T}(G_j, p_j)$$

and, since $\mathcal{T}(G_j, p_j)$ is a subspace of $\mathcal{F}(G_j, p_j)$ for each j, condition (ii) follows. Conversely, if (ii) holds then

$$\dim \mathcal{F}(G,p) = \sum_{j=1}^n \dim \mathcal{F}(G_j, p_j) = \sum_{j=1}^n \dim \mathcal{T}(G_j, p_j) = \dim \mathcal{T}(G,p)$$

and so condition (i) follows. □

The following result was obtained by different methods in [12].

Corollary 48. *Let (G,p) be a well-positioned framework in $(\mathbb{R}^d, \|\cdot\|_\infty)$. The following statements are equivalent.*

(i) *(G,p) is minimally infinitesimally rigid in $(\mathbb{R}^d, \|\cdot\|_\infty)$.*
(ii) *The monochrome subgraphs $G_1, \dots, G_d$ are spanning trees in G.*

Proof. By Theorem 47, (G, p) is infinitesimally rigid if and only if each (G_j, p_j) is infinitesimally rigid. The result now follows from the observation that a framework in $\mathbb{R}$ is (minimally) infinitesimally rigid if and only if the underlying graph is connected (respectively, a tree). $\square$

5. Application to $(\mathcal{H}_2(\mathbb{F}), \|\cdot\|_{c_1})$

In this section we apply Theorem 47 to characterise infinitesimal rigidity in the matrix space $\mathcal{H}_2(\mathbb{F})$ endowed with the trace norm, for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. These normed spaces can be identified, under a suitable isometric isomorphism, with a product norm on $\mathbb{R}^3$ or $\mathbb{R}^4$ respectively. We also show how to construct an infinitesimally rigid placement of the complete graph K_m in $(\mathcal{H}_2(\mathbb{F}), \|\cdot\|_{c_1})$ for sufficiently large values of m. Recall that in Euclidean space $\mathbb{R}^d$, the set of infinitesimally rigid placements for a graph $G = (V, E)$ is either empty, or, an open and dense subset of $(\mathbb{R}^d)^V$. This is no longer true in general normed spaces and so the construction of infinitesimally rigid placements is a non-trivial problem, even for complete graphs.

5.1. Symmetric matrices

Denote by $\|\cdot\|_{\mathrm{cyl}}$ the product norm on $\mathbb{R}^3 = X_1 \times X_2$, where $X_1 = \mathbb{R}^2$ and $X_2 = \mathbb{R}$, given by

$$\|(x, y, z)\|_{\mathrm{cyl}} = \max\{\sqrt{x^2 + y^2}, |z|\}.$$

Note that the closed unit ball in $(\mathbb{R}^3, \|\cdot\|_{\mathrm{cyl}})$ is a cylinder $D \times [-1, 1]$ where D is the closed unit disk in the Euclidean plane. We refer to a normed linear space which is isometrically isomorphic to $(\mathbb{R}^3, \|\cdot\|_{\mathrm{cyl}})$ as a *cylindrical normed space.*

Lemma 49.

(i) $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$ *is a cylindrical normed space.*
(ii) Every cylindrical normed space $(X, \|\cdot\|)$ *satisfies* $\dim \mathcal{T}(X, \|\cdot\|) = 4$.
(iii) A bar-joint framework in $(\mathbb{R}^3, \|\cdot\|_{\mathrm{cyl}})$ *is full if and only if its projection onto* $X_1 = \mathbb{R}^2$ *contains at least two distinct points.*

Proof. The map

$$\Psi : (\mathbb{R}^3, \|\cdot\|_{\mathrm{cyl}}) \to (\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1}), \quad (x, y, z) \mapsto \frac{1}{2}\begin{pmatrix} z + y & x \\ x & z - y \end{pmatrix}$$

is an isometric isomorphism. Indeed, the eigenvalues of $\Psi(x, y, z)$ are $\lambda_\pm = \frac{1}{2}(z \pm \sqrt{x^2 + y^2})$, hence $\|\Psi(x, y, z)\|_{c_1} = |\lambda_+| + |\lambda_-| = \|(x, y, z)\|_{\mathrm{cyl}}$. Statement (ii) follows from the corresponding property of $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$, established in Proposition 18. Statement (iii) follows from Corollary 45 and the easily verified fact that every bar-joint framework in the Euclidean plane containing two distinct points, and every bar-joint framework in $\mathbb{R}$, is full. $\square$

Lemma 50. *Let* (G, p) *be a bar-joint framework in* $(\mathbb{R}^3, \|\cdot\|_{\mathrm{cyl}})$.

(i) For $p_v - p_w = (x, y, z)$ *where* $vw \in E$, *we have* $1 \in \kappa_p(vw)$ *if and only if* $x^2 + y^2 \geq z^2$, *and* $2 \in \kappa_p(vw)$ *if and only if* $x^2 + y^2 \leq z^2$.
(ii) (G, p) *is well-positioned in* $(\mathbb{R}^3, \|\cdot\|_{\mathrm{cyl}})$ *if and only if* $p_v - p_w$ *does not lie in the cone* $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ *for each edge* $vw \in E$.

Proof. Part (i) follows immediately from the definitions in Section 4.1. Since the Euclidean norm is smooth, every bar-joint framework in the Euclidean plane and every bar-joint framework in $\mathbb{R}$ is well-positioned. Also note that an edge $vw \in E$ has exactly one framework colour if and only if $p_v - p_w \notin C$. Thus (ii) follows from Proposition 39. $\square$

Let $\omega = \omega(G, X, \|\cdot\|) \subset X^V$ denote the set of all well-positioned placements of a graph G in a normed space $(X, \|\cdot\|)$. A placement $p \in \omega$ is said to be *regular* if the function,

$$\omega \to \{1, \dots, |E|\}, \quad x \mapsto \operatorname{rank} df_G(x),$$

achieves its maximum value at p.

Remark 51. The set $\omega(G, \mathbb{R}^d, \|\cdot\|_2)$ of regular placements for a graph $G = (V, E)$ in Euclidean space is an open and dense subset of $(\mathbb{R}^d)^V$. Moreover, if G admits an infinitesimally rigid placement in $(\mathbb{R}^d, \|\cdot\|_2)$ then all regular placements of G in $(\mathbb{R}^d, \|\cdot\|_2)$ are infinitesimally rigid. (See [2, p. 283 and Corollary 2] for example.) In this case, G is said to be *generically rigid* in $(\mathbb{R}^d, \|\cdot\|_2)$.

A graph is said to be a *Laman graph* if it is $(2, 3)$-tight.

Theorem 52. *Let (G, p) be a well-positioned bar-joint framework in the cylindrical normed space $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$. The following statements are equivalent.*

(i) (G, p) is minimally infinitesimally rigid in $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$.
(ii) The projected monochrome subframeworks (G_1, p_1) and (G_2, p_2) are minimally infinitesimally rigid in the Euclidean plane and the real line respectively.
(iii) The monochrome subgraphs G_1 and G_2 are respectively a Laman graph and a tree, and p_1 is a regular placement of G_1 in the Euclidean plane.

Proof. Theorem 47 shows that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is an application of standard results on infinitesimal rigidity for bar-joint frameworks in Euclidean space. See for example [2, §3-4], Laman [14, Theorem 6.5] and [29, Propositions 2.4 and 2.5]. $\square$

The following theorem shows that $K_6 - e$, the complete graph K_6 with a single edge removed, is the smallest graph which admits a well-positioned and minimally infinitesimally rigid bar-joint framework in a cylindrical normed space.

Theorem 53. *Let $(X, \|\cdot\|)$ be a cylindrical normed space.*

(i) If (G, p) is a full, well-positioned and minimally infinitesimally rigid bar-joint framework in the space $(X, \|\cdot\|)$, then either $G = K_6 - e$ or $|V(G)| \geq 7$.
(ii) There is a placement p of $K_6 - e$ in X so that $(K_6 - e, p)$ is full, well-positioned and minimally infinitesimally rigid in $(X, \|\cdot\|)$.

Proof. (i) We may assume, without loss of generality, that $(X, \|\cdot\|) = (\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$. By the Maxwell condition for $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$ given in Theorem 32(ii), we have $|E| = 3|V| - 4$. If $|V| \in \{2, 3, 4, 5\}$ then, by Lemma 29, $|E| < 3|V| - 4$. Thus $|V| \geq 6$. If $|V| = 6$ then $|E| = 3|V| - 4 = 14 = \binom{6}{2} - 1$ edges. Thus $G = K_6 - e$.

(ii) It is sufficient to construct such a placement of $K_6 - e$ in $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$. Let $V = \{v_i : 1 \leq i \leq 6\}$ be the vertex set of $G = K_6 - e$ where $e = v_5v_6$. Let $\epsilon, \delta \in (0, \frac{1}{2})$ and consider the placement $p : V \to \mathbb{R}^3$ with

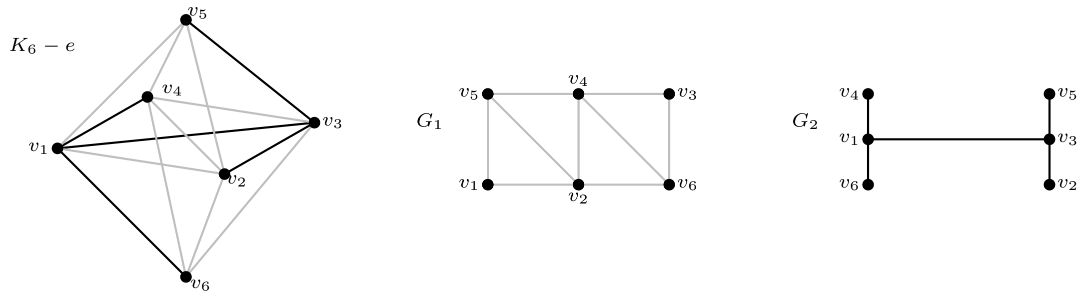


Fig. 1. The framework colouring of $K_6 - e$ in the proof of Theorem 53 with monochrome subgraphs G_1 and G_2 indicated in grey and black respectively. The monochrome subgraphs G_1 and G_2 are respectively a Laman graph and a tree.

$$\begin{aligned} v_1 &\mapsto (0,-1,-1), & v_2 &\mapsto (0,1,-1) \\ v_3 &\mapsto (0,1,1+2\epsilon), & v_4 &\mapsto (0,-1,1-2\epsilon) \\ v_5 &\mapsto (2\delta,1,-1), & v_6 &\mapsto (2\delta,-1,1-2\epsilon). \end{aligned}$$

By Lemma 49(iii), (G,p) is full in $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$. A calculation using Lemma 50 shows that (G,p) is a well-positioned bar-joint framework, with monochrome subgraphs $G_1 = \kappa_p^{-1}(\{1\})$ and $G_2 = \kappa_p^{-1}(\{2\})$ as indicated in Fig. 1. Note that G_1 is a Laman graph and G_2 is a tree. We claim that p_1 is a regular placement of G_1 in the Euclidean plane. This is an exercise in elementary planar rigidity. The rank of the differential $df_{G_1}(p_1)$ may be computed as the rank of an associated (Euclidean) *rigidity matrix* $R(G_1,p_1)$ with rows indexed by $E(G_1)$ and block columns indexed by V. The (v_iv_j, v_i)-entry, for each edge v_iv_j, is the row vector $(p_1(v_i) - p_1(v_j))^t$. All remaining entries are zero. (See [8, Chapter 2].) In this case, ordering the edges of G_1 as $(15, 45, 25, 12, 46, 26, 24, 34, 36)$ we obtain

$$R(G_1,p_1) = 2\begin{bmatrix} -\delta & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta & -1 & \delta & 1 & 0 & 0 \\ 0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & -\delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta & 1 & 0 & 0 & 0 & 0 & \delta & -1 \end{bmatrix}$$

Note that each row contains a nonzero entry with only zeros below. Hence, the rows of $R(G_1,p_1)$ are linearly independent and the rank of the differential $df_{G_1}(x)$ at p_1 is maximal. Thus p_1 is a regular placement of G_1 and so, by Theorem 52, (G,p) is minimally infinitesimally rigid in $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$. $\square$

Remark 54. Applying the isometric isomorphism Ψ from the proof of Lemma 49 to the framework constructed in Theorem 53, we obtain the following matrices which, for $\epsilon, \delta \in (0, \frac{1}{2})$ form a minimally infinitesimally rigid framework $(K_6 - e, p)$ in $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$, where $e = v_5v_6$.

$$p_{v_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_{v_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$p_{v_3} = \begin{pmatrix} 1+\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \qquad p_{v_4} = \begin{pmatrix} -\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix},$$

$$p_{v_5} = \begin{pmatrix} 0 & \delta \\ \delta & -1 \end{pmatrix}, \qquad p_{v_6} = \begin{pmatrix} -\epsilon & \delta \\ \delta & 1-\epsilon \end{pmatrix}.$$

Theorem 55. *Let $(X, \|\cdot\|)$ be a cylindrical normed space. If $m \geq 6$, then there is a placement p of the complete graph K_m in X so that (K_m, p) is full, well-positioned and infinitesimally rigid in $(X, \|\cdot\|)$.*

Proof. Again, it is sufficient to construct such a placement of K_m in $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$. Consider the full, well-positioned and minimally infinitesimally rigid framework $(K_6 - e, p)$ constructed in Theorem 53, with corresponding induced monochrome subgraphs G_1 and G_2 of $G = K_6 - e$. Since $p(v_5) \neq p(v_6)$, the placement p also yields a bar-joint framework (K_6, p), with respect to which $\kappa_p(v_5v_6) = \{1\}$. Thus (K_6, p) is full, well-positioned and infinitesimally rigid in $(\mathbb{R}^3, \|\cdot\|_{\text{cyl}})$.

Now consider the complete graph K_7 obtained by adjoining a vertex v_7 to K_6. We will show that we can extend p to a suitable placement of K_7 by choosing $p(v_7)$ to be a small perturbation of $p(v_5)$. By Lemma 50, there is an open neighbourhood U of $p(v_5)$ which does not contain $p(v_i)$ for $1 \leq i \leq 6$ with $i \neq 5$, such that for any choice of $p(v_7)$ in $U \setminus \{p(v_5)\}$, the extended bar-joint framework (K_7, p) is well-positioned and satisfies $\kappa_p(v_iv_7) = \kappa_p(v_iv_5)$ for $i = 1, 2, 3, 4, 6$. Let G_1' and G_2' be the induced monochrome subgraphs of K_7 with framework colours 1 and 2 respectively. Note that G_2' contains a spanning tree obtained by adjoining the vertex v_7 and the edge v_3v_7 to G_2. Also observe that G_1' has a spanning subgraph, obtained by adjoining the vertex v_7 and the edges v_1v_7, v_2v_7 to G_1, which is also a Laman graph, hence is minimally infinitesimally rigid in $(\mathbb{R}^2, \|\cdot\|_2)$. By Remark 51, every regular placement of G_1' in $(\mathbb{R}^2, \|\cdot\|_2)$ is infinitesimally rigid and the set of regular placements for G_1' is dense in $(\mathbb{R}^2)^V$, so we may choose $p(v_7)$ in $U \setminus \{p(v_5)\}$ such that p_1 is a regular placement of G_1'. Thus, by Theorem 52, (K_7, p) has a minimally infinitesimally rigid subframework and so is itself infinitesimally rigid.

We can now apply this method iteratively to obtain a full, well-positioned and infinitesimally rigid placement of K_m for any $m > 6$. □

5.2. Hermitian matrices

Similar methods may be applied in the case of $(\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_1})$. Denote by $\|\cdot\|_{\text{hcyl}}$ the product norm on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ given by

$$\|(w, x, y, z)\|_{\text{hcyl}} = \max\{\sqrt{w^2 + x^2 + y^2}, |z|\}.$$

A normed space which is isometrically isomorphic to $(\mathbb{R}^4, \|\cdot\|_{\text{hcyl}})$ will be referred to as a *hyper-cylindrical normed space.*

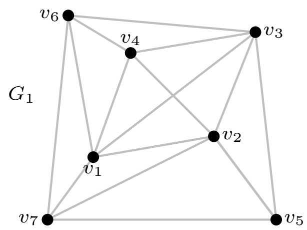
Lemma 56.

(i) $(\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_1})$ *is a hyper-cylindrical normed space.*
(ii) Every hyper-cylindrical normed space $(X, \|\cdot\|)$ *satisfies* $\dim \mathcal{T}(X, \|\cdot\|) = 7$.

Proof. The map

$$\Psi : (\mathbb{R}^4, \|\cdot\|_{\text{hcyl}}) \to (\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_1}), \quad (w, x, y, z) \mapsto \frac{1}{2}\begin{pmatrix} z + y & x - wi \\ x + wi & z - y \end{pmatrix}$$

is an isometric isomorphism. Statement *(ii)* now follows from the corresponding property of $(\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_1})$, established in Proposition 18. □



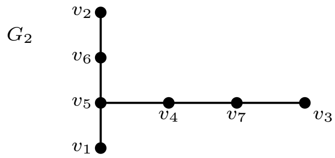


Fig. 2. The monochrome subgraphs G_1 and G_2 of K_7 constructed in Theorem 58 are respectively a spanning $(3, 6)$-tight, generically 3-rigid, block-and-hole graph and a spanning tree.

Lemma 57. *Let (G, p) be a bar-joint framework in $(\mathbb{R}^4, \|\cdot\|_{\mathrm{hcyl}})$.*

(i) For $p_v - p_w = (u, x, y, z)$ where $vw \in E$, we have $1 \in \kappa_p(vw)$ if and only if $u^2 + x^2 + y^2 \geq z^2$, and $2 \in \kappa_p(vw)$ if and only if $u^2 + x^2 + y^2 \leq z^2$.
(ii) (G, p) is well-positioned in $(\mathbb{R}^4, \|\cdot\|_{\mathrm{hcyl}})$ if and only if $p_v - p_w$ does not lie in the cone $C = \{(u, x, y, z) \in \mathbb{R}^4 : u^2 + x^2 + y^2 = z^2\}$ for each edge $vw \in E$.

Proof. The proof is analogous to Lemma 50. □

We can now show that K_7 is the smallest graph which admits a full, well-positioned rigid and infinitesimally rigid bar-joint framework in hyper-cylindrical normed spaces.

Theorem 58. *Let $(X, \|\cdot\|)$ be a hyper-cylindrical normed space.*

(i) If (G, p) is a full, well-positioned and infinitesimally rigid bar-joint framework in $(X, \|\cdot\|)$, then either $G = K_7$ or $|V| \geq 8$.
(ii) For every $G \in \{K_m : m \geq 7\}$, there is a placement p in X so that (G, p) is full, well-positioned and infinitesimally rigid in $(X, \|\cdot\|)$.

Proof. (i) By Lemma 56, we may assume, without loss of generality, that $(X, \|\cdot\|) = (\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_1})$. By the Maxwell condition for $(\mathcal{H}_2(\mathbb{C}), \|\cdot\|_{c_1})$ given in Theorem 32(*ii*), we have $|E| = 4|V| - 7$. If $|V| \in \{3, 4, 5, 6\}$ then, by Lemma 29, $|E| < 4|V| - 7$. The complete graph K_2 does not admit a full bar-joint framework in a hyper-cylindrical space. Thus $|V| \geq 7$. If $|V| = 7$ then $|E| = 4|V| - 7 = 21 = \binom{7}{2}$ edges and so $G = K_7$.

(*ii*) It is again sufficient to construct a suitable placement of K_m in $(\mathbb{R}^4, \|\cdot\|_{\mathrm{hcyl}})$. First consider the case $m = 7$. Let $\delta \in (1, \frac{6}{5})$ and $\epsilon \in (\frac{\delta}{3}, 1 - \frac{\delta}{2})$, and consider the placement $p : V \to \mathbb{R}^4$ with

$$\begin{aligned}
&v_1 \mapsto (0, -1, -1, 0), && v_2 \mapsto (0, 1, -1, 0) \\
&v_3 \mapsto (0, 1, 1, 2\epsilon), && v_4 \mapsto (0, -1, 1, -\delta) \\
&v_5 \mapsto (0, -1, 1, 2 + \epsilon), && v_6 \mapsto (0, 1, -1, -2 + 3\epsilon) \\
&v_7 \mapsto (0, 1, 1, \delta).
\end{aligned}$$

A calculation using Lemma 57 shows that (G, p) is a well-positioned bar-joint framework, with monochrome subgraphs $G_1 = \kappa_p^{-1}(\{1\})$ and $G_2 = \kappa_p^{-1}(\{2\})$ as indicated in Fig. 2. The graph G_1 is an example of a block-and-hole graph, with one quadrilateral block and one quadrilateral hole and it follows from [30, Theorem 4.1] that G_1 is generically minimally rigid in $(\mathbb{R}^3, \|\cdot\|_2)$. Alternatively, note that G_1 can be constructed from K_4 by successively adjoining vertices of degree three. It is a standard result that K_4 is generically minimally rigid in $(\mathbb{R}^3, \|\cdot\|_2)$ (see for example [29, Theorem 3.1]), and that the graph operation of adjoining degree three vertices preserves generic minimal rigidity in $(\mathbb{R}^3, \|\cdot\|_2)$ ([29, Corollary 2.2]). Also note that the

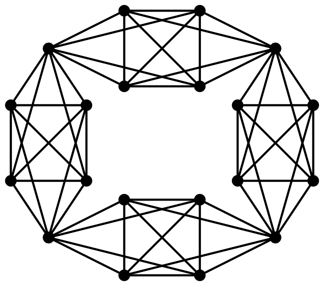


Fig. 3. A $(3,4)$-tight graph which does not admit an edge-disjoint packing consisting of a spanning Laman graph and a spanning tree and hence does not admit an infinitesimally rigid placement in the cylindrical normed space $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$.

monochrome subgraph G_2 is a spanning tree. By perturbing the vertices of G, we may assume that the projected monochrome subframework (G_1, p_1) is regular, and hence minimally infinitesimally rigid, and also that (G,p) is full. By Theorem 47, (G,p) is minimally infinitesimally rigid. The argument from the proof of Theorem 55 can now be adapted to show that K_m admits a full, well-positioned and infinitesimally rigid placement in $(\mathbb{R}^4, \|\cdot\|_{\text{hcyl}})$ for any $m > 7$. □

6. Remarks on sufficient conditions

The (k,l)-sparsity conditions obtained in Theorem 33 are in general not sufficient for the existence of an infinitesimally rigid placement in an admissible matrix space. An example of this, due to Shin-ichi Tanigawa, is the $(3,4)$-tight graph G in Fig. 3. Note that G is composed of four copies of $K_6 - e$. This graph satisfies the necessary Maxwell count for $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$ but fails to admit an edge-disjoint packing consisting of a spanning Laman graph and a spanning tree. Thus, by Theorem 52, G does not have an infinitesimally rigid placement in the cylindrical normed space $(\mathcal{H}_2(\mathbb{R}), \|\cdot\|_{c_1})$.

As with Euclidean 3-space, characterising 3-dimensional rigidity in cylindrical normed spaces is likely to be difficult. We conjecture below, in part (a), that every graph which admits an edge-disjoint packing consisting of a spanning Laman graph and a spanning tree can be realised as an infinitesimally rigid bar-joint framework in a cylindrical normed space. This result, if proven, and Theorem 52 would together provide a complete combinatorial characterisation of rigidity for cylindrical norms. One possible line of attack is to use a multigraph construction scheme, based on the graph moves described in Corollary 35(i), and with $K_6 - e$ as the base graph. With this approach it is sufficient to show that each multigraph in the construction admits a placement with a framework colouring that induces the required packing property. However, it is not currently known whether such placements exist.

In 1982, Lovász and Yemini [15] proved that every 6-vertex-connected graph is generically rigid in the Euclidean plane and conjectured that every 12-vertex-connected graph is generically rigid in Euclidean 3-space. This conjecture is still open. Sufficient connectivity conditions for graphs which contain a packing consisting of spanning Laman graphs and spanning trees have recently been considered by Cheriyan et al. [6] and by Gu [9]. Their results suggest the analogous connectivity conjectures (b) and (c) below for 3-dimensional cylindrical normed spaces.

Conjecture 59. *Let $(X, \|\cdot\|)$ be a cylindrical normed space and let G be a simple graph which has one of the following properties.*

(a) G admits an edge-disjoint packing consisting of a spanning Laman graph and a spanning tree.
(b) G is $(8,2)$-connected in the sense of [6].
(c) G is 6-edge-connected and essentially 8-edge-connected in the sense of [9].

Then G admits an infinitesimally rigid placement in $(X, \|\cdot\|)$.

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