The Stability Space of the Derived Category of Holomorphic Triples and Further Investigations

Arne Rüffer

Doctoral Dissertation

Mary Immaculate College, Universitiy of Limerick

Submitted to the University of Limerick

July 2020

Abstract

In this thesis we give a complete description of the Bridgeland stability space for the bounded derived category of holomorphic triples over a smooth projective curve of genus one as a connected, four dimensional complex manifold.

We will then prove a number of helpful facts that characterise the bounded derived category of holomorphic triples and will subsequently generalise some of the results on the stability space of the bounded derived category of holomorphic triples to that of holomorphic chains.

Author's Declaration:

I hereby declare that this thesis represents my own work and has not been submitted, in whole or in part, by me or any other person, for the purpose of obtaining any other qualification with the exception of the results in chapters 3 and 4 which are based on joint work with Eva Martínez and Alejandra Rincón ([49]) who contributed equally to these results and who used them in their Ph.D. theses ([48] and [50]).

Signed: Arne Rüffer   Date: 7. of October 2020

Acknowledgement

Firstly, I would like to express my sincerest gratitude to Bernd Kreussler who supervised me over the last couple of years. Without his dedication and patience the realisation of this thesis would have been virtually impossible. Special thanks also go to Eva Martínez and Alejandra Rincón for their invaluable contribution to this project.

Furthermore I would like to thank the Department of Mathematics and Computer Studies of Mary Immaculate College Limerick for their support. Particular thank goes to Diarmuid O'Driscoll for the material support he was able to provide and which in turn made this project possible – and to Norbert Hoffmann who helped me over the course of many discussions to complete my goal academically. Thanks also to the Institute für Mathematik of the Freie Universität Berlin for their help in funding research trips to the benefit of this thesis.

My deepest gratitude goes to my grandmother, Anna Schröfel whose importance in my life was of the highest order and whose memory I would like to honour with this thesis. Equally deeply I would like to thank my wife, Ciara Nash, who has been an irreplaceable support to me throughout the duration of this project. Very special thanks I would like to extend to my own parents, Ursula Rüffer-Schröfel and Wolfgang Rüffer, to Ciara's parents, Eileen Nash and Tadhg Nash and to Nora Desmond for their moral and material support over the past years.

I would like to thank all my friends whose company helped me through what was not always an easy time. My very special thanks go to Harvey.

Contents

1 Introduction

This thesis attempts to contribute to the research on stability spaces of a given derived category. The derived category to be investigated is that of holomorphic triples on an elliptic curve.

Bridgeland introduced stability conditions on triangulated categories in [18], formalising ideas from physics by Douglas (see [25] and [26]). As main result of [18], Bridgeland asserted that the set of stability conditions has the structure of a complex manifold, usually referred to as a stability space. These have several applications in algebraic geometry. They serve for example as an important invariant of and are therefore essential for the understanding of derived categories. There are – moreover – applications in birational geometry as well ([35], [6]).

Describing stability spaces of a certain derived category is generally not an easy endeavour and hence not many examples of derived categories are known where the entire stability space has been computed. Bridgeland gave a complete description of the stability manifold of non-singular projective curves of genus one that was subsequently generalised by Burban and Kreussler to singular irreducible projective curves of genus one in [20]. On the other hand, Macrí generalised the smooth case to higher genus in [45] while Okada found the stability manifold for $\mathbb{P}^1$ in [53]. In general, not as much is known and one has to settle for less comprehensive results such as the computation of a connected component as it was done in [51], [17], [61], [8] and in [7]. In other cases, the results are even weaker, such as whether the stability space is non-empty or stability conditions are simply used in other applications (for example in birational geometry) without even the attempt to compute the stability space.

However, assuming C to be a complex projective non-singular curve of genus 1, we will provide a complete description of the stability space of the bounded derived category of the (abelian) category of holomorphic triples, sometimes denoted $\mathcal{D}^b(\mathcal{T}\,\mathrm{Coh}(C))$. The concept of a holomorphic triple was introduced by Bradlow and García-Prada in [14] and [15]. A holomorphic triple consists of coherent sheaves E_1, E_2 on C and a morphism $\varphi : E_1 \to E_2$. We will subsequently discuss what of these results can be generalised to holomorphic chains and also investigate related questions that aim at understanding the category $\mathcal{D}^b(\mathcal{T}\,\mathrm{Coh}(C))$ better.

As far as the organisation of this thesis is concerned, we will proceed as follows. Section 2 will present the underlying basic notation and briefly discuss the categories involved. Subsequently, an introduction to the theory of stability spaces will be provided.

In section 3, we will discuss a technique which we call CP-gluing, that was introduced by Collins and Polishchuk in [21] and that uses semiorthogonal decompositions to generate the hearts of t-structures – as a key-ingredient of a stability condition – on the generalised version of the derived category of holomorphic triples from those of the derived category of the underlying abelian category. This technique is in a very basic way quite broadly applicable and we obtain some preliminarily results (theorems 3.2.34 and 3.2.39) without imposing any restrictions on the abelian category that we start with.

In section 4, we impose the extra condition on the abelian category we are starting with, that its derived category should have a Serre functor – and for extensive parts even that it should be equal to the (abelian) category of coherent sheaves on an elliptic curve – meaning the genus of the curve should be 1. The condition of the existence of a Serre functor immediately unlocks a variety of new possibilities since it provides us with new functors and therefore with more semiorthogonal decompositions than in the more basic situation of section 3. Additionally it is now possible to apply an (older) technique from which CP-gluing derives and that is known as recollement. This technique which was introduced by Beilinson, Bernstein and Deligne in [10] allows one to compute more t-structures than the CP-gluing technique provided that one has enough functors available. Hence, we will apply recollement to our situation, however, will obtain the following important theorem, clarifying the situation between CP-gluing and recollement in the situation we are in (theorem 4.4.6).

Theorem 1 (Jealousy Lemma)**.** *If $H_{i,\alpha,\beta}$ for $i \in \{1,2,3\}$ is the heart of a t-structure as in definition 4.4.1 that is not obtained by CP-gluing via either of the three semiorthogonal decompositions $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ or $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ then there is no stability condition with heart $H_{i,\alpha,\beta}$.*

We then continue by developing the theory of the stability space and have the following – crucial – theorem that provides the first characterisation on the stability space as a whole (theorem 4.5.29).

Theorem 2. *Assume that $\mathcal{D}$ is the derived category of $\mathrm{Coh}(C)$, where C is an elliptic curve, then*

$$\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) = \Theta_{12} \cup \Theta_{31} \cup \Theta_{23}.$$

with Θ_{ij} like in definition 4.5.27.

Note that $\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow})$ are stability conditions that do not necessarily fulfil a condition added later which is that they need to satisfy what is called the support property – however, we obtain the analogous result as a direct implication anyway (corollary 4.5.30).

Corollary 3. *Let Θ'_{ij} be like Θ_{ij} where now we assume $\sigma \in \operatorname{Stab}(\mathcal{D}^{\uparrow})$ instead of* $\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow})$*, then we have*

$$\operatorname{Stab}(\mathcal{D}^{\uparrow}) = \Theta'_{12} \cup \Theta'_{31} \cup \Theta'_{23}.$$

Prior to continuing with our investigation of the stability space we are now also able to provide a description of the Serre functor on the level of objects (theorem 4.6.4). We conjecture that the derived category of holomorphic triples is fractional Calabi-Yau with the fraction in question being $\frac{3}{4}$.

After this, we introduce another technique named tilting (see [1] for details) and use it to complete the picture of the stability space from a constructive point of view. We have the following (theorem 4.8.36).

Theorem 4. *Let $\mathcal{A} = \operatorname{Coh}(C)$. We have*

$$\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) = \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Gamma.$$

where Θ_i are stability conditions that are up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action (lemma 2.5.50 and definition 2.5.47) obtained by CP-gluing and Γ stability conditions that are up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action obtained by tilting (see definition 4.8.20).

Finally, we investigate the support property mentioned above by distinguishing between different cases of possible discriminants of matrices given as part of the data of a stability condition, which gives $\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) = \operatorname{Stab}(\mathcal{D}^{\uparrow})$ (theorem 4.9.37). We are then able to investigate the topology of $\operatorname{Stab}(\mathcal{D}^{\uparrow})$ and to provide a description given by the main theorem (4.10.31) on $\operatorname{Stab}(\mathcal{D}^{\uparrow})$.

Theorem 5. *The space of stability conditions of the derived category of holomorphic triples on an elliptic curve is a connected, four dimensional complex manifold.*

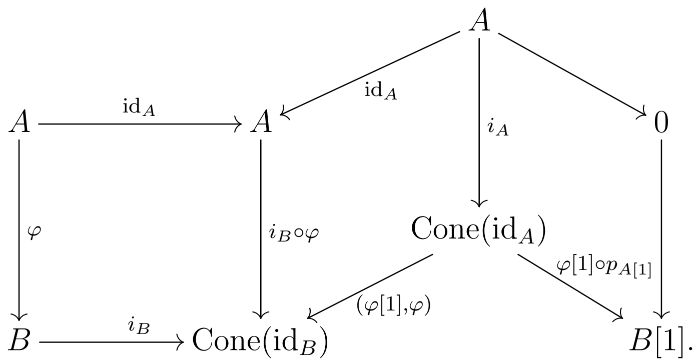
Both sections 3 and 4 are joint work with Eva Martínez and Alejandra Rincón. The results in it appear in [49].

As part of the investigation of the stability space – that is after all first and foremost a tool to understand its associated derived category better – certain questions have come up which are worth considering in order to achieve a better understanding of the derived category of holomorphic triples and that will be dealt with in section 5. The investigation of recollement in section 4 provides the question of the shape of a t-structure that is obtained by recollement and that usually looks a lot more complicated then one obtained by CP-gluing, be it part of a stability condition or not. Since this is a rather constructive approach we will provide a manifold of examples to illustrate our findings (theorems 5.1.18 and 5.1.23). Section 5 additionally investigates certain connecting-homomorphisms which are a key-feature for the understanding of the derived category of holomorphic triples. We have the following (theorem 5.2.8).

Theorem 6. *Let* $X = (A \xrightarrow{\varphi} B)$. *The connecting morphism* $\xrightarrow{+}$ *of the exact triangle*

$$i_2(\rho_2(X)) \to X \to i_1(\lambda_1(X)) \xrightarrow{+}$$

is – up to isomorphisms – given by the chain-complex homomorphism φ *via the roof*



Finally we conclude section 5 and therefore our investigation of the derived category of holomorphic triples by altering our perspective to a situation where we have an exceptional collection available – which allows us to apply theory by Macrí developed in [45] to obtain results on the stability space. The most obvious example one would have in mind in this regard is that of coherent sheaves on $\mathbb{P}^N$. As well as the more general proposition 5.3.11, we obtain the following result (corollary 5.3.13).

Corollary 7. *There is an open, connected and also simply connected* N*-dimensional submanifold* $\Theta_{\mathcal{E}} \subset \operatorname{pre}\operatorname{Stab}(\mathcal{D}^b((\mathbb{P}^N)^{\uparrow}))$.

Section 6 is dedicated to the generalisation of findings from section 4 on the stability space of the derived category of holomorphic triples. We generalise to holomorphic chains of length two initially and subsequently of length n. For the case of $n = 2$, we will provide the basic data involved by giving a description of the semiorthogonal decompositions that generalise those of the $n = 1$ case and also describe the resulting recollement data. We will demonstrate that stability conditions exist in this case (proposition 6.1.24). After providing a generalisation of the Jealousy Lemma (6.2.5), we will give a short description of what the basic data should look like in the case of a holomorphic chain of length n. In this case we too will demonstrate how to obtain a stability condition (proposition 6.3.7).

Section 4 is heavily based on the availability of a Serre functor on the derived category of holomorphic triples and its existence is provided by the fact that the derived category of coherent sheaves on a smooth projective curve is equipped with one. We achieve this implication from a theorem of Bondal and Kapranov ([12, Proposition 3.8]) who also provide a short proof. Because of the importance of [12, Proposition 3.8] we provide a detailed proof of it in the appendix (section A).

This thesis can be seen as a continuation of [22] on the research-level. Therefore, the reader unfamiliar with the general concepts discussed here is recommended to read [22] in addition to the standard literature.

2 Basics: notation and framework

This chapter is deemed to provide a common ground on which the findings that this thesis discusses can be presented. We will therefore provide the crucial notation and framework of stability conditions as it was introduced by Bridgeland in [18].

2.1 Notation

This very brief subsection introduces the notation and explains the basic concepts that we will use. We will, throughout this thesis, assume familiarity with triangulated, and in particular with derived categories – we refer to [32] for the definition and for basic properties of triangulated and to [58] for basic properties of derived categories.

Since the motivating examples for our involved categories are those of coherent sheaves over a variety we will start with the following.

Definition 2.1.1. We assume all varieties (hence in particular curves) to be defined over $\mathbb{C}$.

The following assumption is a condition for theorem 2.5.51 to hold.

Definition 2.1.2. A smooth projective curve C will always be assumed to have genus ≥ 1 throughout this thesis.

Definition 2.1.3. Let $\mathcal{TR}$ denote an arbitrary triangulated category.

Remark 2.1.4. Note that for $F, E \in \mathcal{TR}$, one defines

$$\mathrm{Hom}^n(E, F) = \mathrm{Hom}(E, F[n]).$$

Accordingly $\mathrm{Hom}^{<0}(E,F) = \bigoplus_{n<0} \mathrm{Hom}(E,F[n])$ and analogously to that, the sets $\mathrm{Hom}^{>0}(E,F)$, $\mathrm{Hom}^{\leq 0}(E,F)$ and $\mathrm{Hom}^{\geq 0}(E,F)$ are defined.

Definition 2.1.5. Let $\mathcal{B}$ be a subcategory of a category $\mathcal{B}$. We say that "$\mathcal{B}$" is strictly full if $\mathcal{B}$ is full and closed under isomorphisms.

Definition 2.1.6. We say that a triangulated category $\mathcal{TR}$ has a semiorthogonal decomposition $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$, if

- $\mathcal{TR}^1$ and $\mathcal{TR}^2$ are triangulated categories and i_1, i_2 strictly full embeddings of $\mathcal{TR}^1$, $\mathcal{TR}^2$ respectively into $\mathcal{TR}$ that commute with the shift-functor and send exact triangles to exact triangles,
- we have that $\mathrm{Hom}(i_2(E_2), i_1(E_1)) = 0$ for every $E_1 \in \mathcal{TR}^1$ and $E_2 \in \mathcal{TR}^2$,
- for every $E \in \mathcal{TR}$ there is an exact triangle

  $$i_2(E_2) \to E \to i_1(E_1) \xrightarrow{+}$$

  with $E_1 \in \mathcal{TR}^1$ and $E_2 \in \mathcal{TR}^2$.

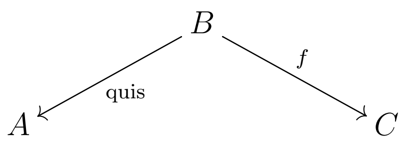
Remark 2.1.7. Note that definition 2.1.6 is often given more generally for n embedded subcategories that generate $\mathcal{TR}$ by extensions and for which $\mathrm{Hom}(i_o(\mathcal{TR}^o), i_p(\mathcal{TR}^p)) = 0$ if $1 \leq p < o \leq n$. However we will – for the most part – not require this level of generality.

Definition 2.1.8. Define

1. - $\lambda_1(E) = E_1$ for an object $E \in \mathcal{TR}$ and E_1 the corresponding object of definition 2.1.6,
   - $\lambda_1(f) = f_1$ for a morphism $f \in \mathcal{TR}$, where $f_1 \in \mathcal{TR}^1$ is the morphism on E_1 induced by f,
2. - $\rho_2(E) = E_2$ for an object $E \in \mathcal{TR}$ and E_2 the corresponding object of definition 2.1.6,
   - $\rho_2(f) = f_2$ for a morphism $f \in \mathcal{TR}$, where $f_2 \in \mathcal{TR}^2$ is the morphism on E_2 induced by f.

Notation 2.1.9. Let $\mathcal{A}$ denote an arbitrary abelian category. In the case that $\mathcal{A}$ has the additional property that its bounded derived category exists, we denote the bounded derived category of $\mathcal{A}$ (usually denoted by $\mathcal{D}^b(\mathcal{A})$) by $\mathcal{D}$.

Remark 2.1.10. Although familiarity with derived categories will generally assumed throughout this thesis, we include this quick reminder for the purpose of convenience. The derived category of an abelian category $\mathcal{A}$ has (bounded) chain-complexes of objects in $\mathcal{A}$ as objects – for more details on the category $\mathcal{C}(\mathcal{A})$ of (bounded) chain-complexes see for example [62] or [55] (or many other sources). The objects of $\mathcal{C}(\mathcal{A})$ also form the objects of the homotopy category $\mathcal{K}(\mathcal{A})$. The morphisms in $\mathcal{K}(\mathcal{A})$ are those of $\mathcal{C}(\mathcal{A})$ modulo an equivalence relation called homotopy equivalence – again we refer to [62], [55] or other suitable literature for more details. The morphisms of $\mathcal{D}^b(\mathcal{A})$ are defined as the localisation of $\mathcal{K}(\mathcal{A})$ with regard to quasi-isomorphisms. These are elements in $\mathrm{mor}(\mathcal{K}(\mathcal{A}))$ that induce isomorphisms on the cohomology objects of the objects. A morphism in $\mathcal{D}^b(\mathcal{A})$ hence is usually denoted as a "roof":



where quis denotes a quasi-isomorphism. This, on the other hand, implies that any $(E \xrightarrow{f} F) \in \mathrm{mor}(\mathcal{K}(\mathcal{A}))$ can be considered a morphism in $\mathcal{D}^b(\mathcal{A})$, simply by denoting it as $E \xleftarrow{=} E \xrightarrow{f} F$, since the identity induces the identity mapping on the cohomology objects, which, in particular, is an isomorphism.

It is now time to define the important concept of the arrow category, which if defined over an abelian category is also abelian. We include a short outline of the proof of this fact as part of the next subsection.

Definition 2.1.11. Let $\mathcal{I}_n, n \in \mathbb{N}_{\geq 1}$ be the category given by the graph

$$\underbrace{\cdot \to \cdots \to \cdot}_{n-arrows}$$

with $n+1$ dots and n arrows. Define $\mathcal{A}^{n\uparrow} := \mathrm{Func}(\mathcal{I}_n, \mathcal{A})$ for an abelian category $\mathcal{A}$.

We will express this concept in a less theoretical language. The following definition coincides with the previous for $n = 1$.

Definition 2.1.12. For an abelian category $\mathcal{A}$ denote by $\mathcal{A}^{\uparrow}$ the category for which $\mathrm{obj}(\mathcal{A}^{\uparrow})$ is the set of all arrows

$$A \to B$$

between objects $A, B \in \mathcal{A}$. For $(A \xrightarrow{f} B), (A' \xrightarrow{f'} B') \in \text{obj}(\mathcal{A}^\uparrow)$ denote by $\text{Hom}((A \xrightarrow{f} B), (A' \xrightarrow{f'} B'))$ the set of all pairs (ϕ, ϕ') of arrows such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{\phi'} & B' \end{array}$$

commutes.

The following example is that of "holomorphic triples" introduced by Bradlow and García-Prada (see [14], [15] and [27]). We replace "holomorphic vector bundles" in the original definition by coherent sheaves since however unnecessary at this stage, we want to work with an abelian category in the long run.

Example 2.1.13. *Let C be a smooth projective curve and let $\mathcal{A}$ be the category of coherent sheaves on C. Then $\mathcal{A}^\uparrow$ is called the category of "holomorphic triples" over C.*

Remark 2.1.14. Note that we could make definition 2.1.12 for any category, abelian or else – the reason for us not to do so is that we want to use the notation $\mathcal{D}^\uparrow$ in a different sense, as expressed by 2.2.4.

Notation 2.1.15. For functors L, R we write

$$L \dashv R$$

if L is left adjoint to R (and hence R right adjoint to L).

2.2 First properties of $\mathcal{A}^\uparrow$ and $\mathcal{D}^\uparrow$

This thesis is based on the theory of "Bridgeland stability", a concept that was introduced by Bridgeland in [18]. The concepts, which Bridgeland establishes in [18] and that will be discussed throughout the thesis will now be introduced throughout this and the following subsections.

The first question that arises from the previous chapter is, if the nice property of being abelian transfers from $\mathcal{A}$ to $\mathcal{A}^\uparrow$. This – seen by the following proposition that can already be found in [22, Proposition 5.3.1] – is indeed the case.

Proposition 2.2.1. *If $\mathcal{A}$ is abelian, $\mathcal{A}^\uparrow$ is also abelian.*

Proof. The proof of this is a lengthy yet straightforward exercise in such a way that as a key point of the proof, the existence of all required concepts in $\mathcal{A}^{\uparrow}$ extends from that in $\mathcal{A}$. The requirement of a kernel in $\mathcal{A}^{\uparrow}$ for any morphism $(\beta, \beta') \in \mathcal{A}^{\uparrow}$ as in 2.1.12 for example, is met in the obvious way – one simply takes the kernels of β and β' in $\mathcal{A}$ and uses the morphism $\widetilde{f}$, induced by f as the connecting arrow:

$$\begin{array}{ccc}
\mathrm{K}(\beta) & \xrightarrow{\widetilde{f}} & \mathrm{K}(\beta') \\
\Big\downarrow{\scriptstyle \ker(\beta)} & & \Big\downarrow{\scriptstyle \ker(\beta')} \\
B & \xrightarrow{f} & B' \\
\Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta'} \\
C & \xrightarrow{g} & C'
\end{array}$$

To prove this one needs to verify at first that there is an $\widetilde{f}$ that does indeed make the upper square commutative. We have $\beta' \circ f \circ \ker(\beta) = g \circ \beta \circ \ker(\beta)$ since the lower square in the above diagram is a morphism in $\mathcal{A}^{\uparrow}$ and hence commutes. But $\beta \circ \ker(\beta) = 0$ and we obtain $\beta' \circ f \circ \ker(\beta) = g \circ \beta \circ \ker(\beta) = g \circ 0 = 0$, which means – due to the universal property of the kernel (of β') – that there is a unique $\widetilde{f}$ such that $f \circ \ker(\beta) = \ker(\beta') \circ \widetilde{f}$.

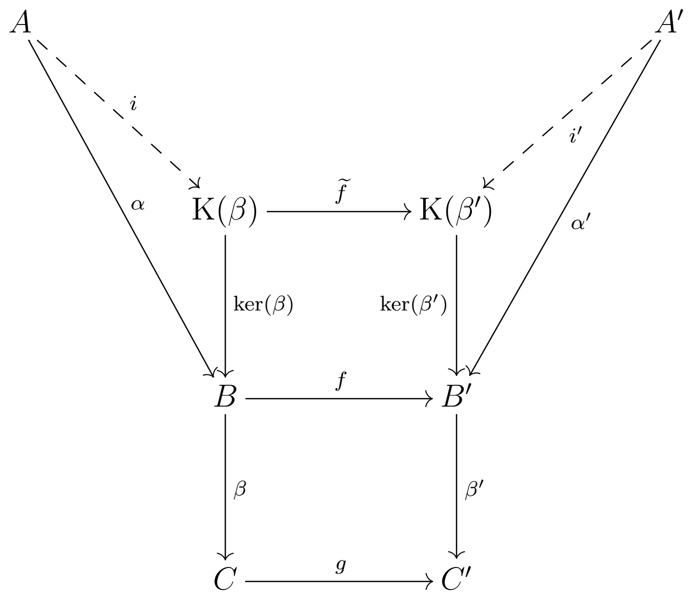
We now have to prove that $\ker(\beta) \xrightarrow{\widetilde{f}} \ker(\beta')$ fulfils the universal property by which kernels in abelian categories are defined. Assume that for a morphism $(\alpha, \alpha') \in \mathcal{A}^{\uparrow}$,

$$\begin{array}{ccc}
A & \xrightarrow{e} & A' \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha'} \\
B & \xrightarrow{f} & B'
\end{array}$$

we have that the composition

$$\begin{array}{ccc} A & \xrightarrow{e} & A' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ B & \xrightarrow{f} & B' \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ C & \xrightarrow{g} & C' \end{array}$$

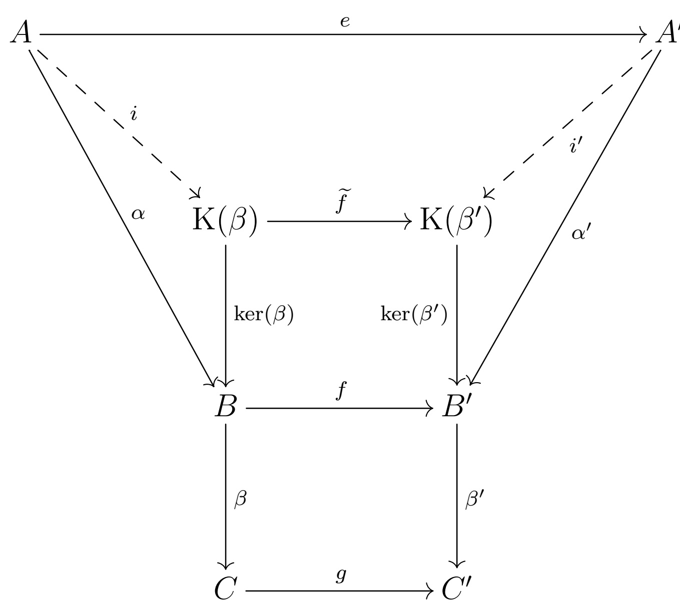
equals to the zero-morphism in $\mathcal{A}^{\uparrow}$. Since $\mathcal{A}$ is abelian, we obtain unique morphisms i and i' such that the diagram



commutes. We hence need to show that the diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & A' \\ {\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow \\ \mathrm{K}(\beta) & \xrightarrow{\widetilde{f}} & \mathrm{K}(\beta') \end{array}$$

commutes, in order to prove that it defines a morphism from the object $A \xrightarrow{e} A'$ to the object $\mathrm{K}(\beta) \xrightarrow{\widetilde{f}} \mathrm{K}(\beta')$ (Note that the uniqueness of this morphism is due to the uniqueness of the morphisms i and i' in $\mathcal{A}$). We observe that $\beta' \circ f \circ \ker(\beta) \circ i = g \circ \beta \circ \ker(\beta) \circ i = g \circ 0 \circ i = 0$ which, by the universal property of a kernel in the (abelian) category $\mathcal{A}$, implies that there is a unique morphism m from A to $\ker(\beta')$ such that $f \circ \ker(\beta) \circ i = \ker(\beta') \circ m$. We have – on the other hand – that $\ker(\beta') \circ i' \circ e = \alpha' \circ e = f \circ \alpha = f \circ \ker(\beta) \circ i$ which implies $m = i' \circ e$ and hence we obtain that the diagram



commutes, which finishes our proof. Similarly one extends the other parts of the definition from $\mathcal{A}$ to $\mathcal{A}^{\uparrow}$ in order to prove that the latter is indeed an abelian category – the cokernel of (β, β') in

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} \\ C & \xrightarrow{g} & C' \end{array}$$

for instance is – in analogy to the concept of the kernel – given by the arrow $\mathrm{C}(\beta) \xrightarrow{\widetilde{g}} \mathrm{C}(\beta')$, completing the previous diagram to

$$\begin{array}{ccc} B & \xrightarrow{f} & B' \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} \\ C & \xrightarrow{g} & C' \\ \downarrow{\scriptstyle\mathrm{coker}(\beta)} & & \downarrow{\scriptstyle\mathrm{coker}(\beta')} \\ \mathrm{C}(\beta) & \xrightarrow{\widetilde{g}} & \mathrm{C}(\beta'). \end{array}$$

□

Before we continue with the – particularly important – implication of the previous, we will take the opportunity to introduce another nice property that $\mathcal{A}^{\uparrow}$ inherits from $\mathcal{A}$.

Lemma 2.2.2. *If $\mathcal{A}$ is noetherian then so is $\mathcal{A}^{\uparrow}$.*

Proof. Assume $\mathcal{A}$ noetherian and consider a chain

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & \dots \\ \downarrow & & \downarrow & & \\ B_1 & \xrightarrow{j_1} & B_2 & \xrightarrow{j_2} & \dots \end{array} \tag{2.1}$$

of embeddings $(i_x, j_x), x \in \mathbb{Z}$ in $\mathcal{A}^{\uparrow}$. Then both

$$A_1 \overset{i_1}{\hookrightarrow} A_2 \overset{i_1}{\hookrightarrow} \dots \text{ and } B_1 \overset{j_1}{\hookrightarrow} B_2 \overset{j_2}{\hookrightarrow} \dots$$

are chains of embeddings in $\mathcal{A}$ and – therefore – become stationary after finite numbers of steps. Denote these by n, m respectively. Let $d = \max\{n, m\}$, then (2.1) becomes stationary after d steps, which finishes the proof. □

We now continue with the implication that proposition 2.2.1 provides.

Corollary 2.2.3. *The category $\mathcal{D}^b(\mathcal{A}^{\uparrow})$ exists.*

Proof. We work with the – common – notion of a derived category being obtained from an abelian category. Since the derived category can be defined for any given abelian category, proposition 2.2.1 does indeed guarantee that $\mathcal{D}^b(\mathcal{A}^{\uparrow})$ is a derived category. Note that it is possible to define a derived category also when relaxing certain conditions on $\mathcal{A}$. □

Notation 2.2.4. Let $\mathcal{D}^{\uparrow} = \mathcal{D}^b(\mathcal{A}^{\uparrow})$.

The important difference of the previous definition to 2.1.12 is that $\mathcal{D}^{\uparrow} \neq (\mathcal{D}^b(\mathcal{A}))^{\uparrow}$. Here, $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ is defined in analogy to definition 2.1.12 for a derived category.

Remark 2.2.5. Note that by 2.1.9 we require $\mathcal{A}^{\uparrow}$ to be abelian in order for 2.2.4 to make sense – this is guaranteed by corollary 2.2.3.

The following definition will naturally integrate itself into language that will be introduced later.

Definition 2.2.6. For $(X \to Y) \in \text{obj}(\mathcal{C}(\mathcal{A}^{\uparrow}))$ and $(f,g) \in \text{mor}(\mathcal{C}(\mathcal{A}^{\uparrow}))$, define functors $\lambda_1^{\mathcal{C}(\mathcal{A})}$ and $\rho_2^{\mathcal{C}(\mathcal{A})}$ by

- $$\lambda_1^{\mathcal{C}(\mathcal{A})}(X \to Y) = X$$
  $$\lambda_1^{\mathcal{C}(\mathcal{A})}(f,g) = f$$

  and

- $$\rho_2^{\mathcal{C}(\mathcal{A})}(X \to Y) = Y$$
  $$\rho_2^{\mathcal{C}(\mathcal{A})}(f,g) = g.$$

Corollary 2.2.7. *Let $(X \to Y) \in \mathcal{C}(\mathcal{A}^{\uparrow})$ then $(X \to Y)$ is an exact complex if and only if $\lambda_1^{\mathcal{C}(\mathcal{A})}(X \to Y)$ and $\rho_2^{\mathcal{C}(\mathcal{A})}(X \to Y)$ are exact complexes.*

Proof. Since exactness depends on taking kernels and images and, as seen in the proof of proposition 2.2.1, both operations commute with $\lambda_1^{\mathcal{C}(\mathcal{A})}$ and $\rho_2^{\mathcal{C}(\mathcal{A})}$, the proof is finished. □

Corollary 2.2.8. *Let $A \xrightarrow{f} B \in \mathcal{D}^{\uparrow}$ and $H^n_{\mathcal{D}^{\uparrow}}(A \xrightarrow{f} B) \in \mathcal{A}^{\uparrow}$ be the cohomology object of $A \xrightarrow{f} B$ at position n. Let $H^n_{\mathcal{D}}(A)$ denote the cohomology object of $A \in \mathcal{D}$ and $H^n_{\mathcal{D}}(f)$ be the morphism on the cohomology object at position n induced by f. Then*

$$H^n_{\mathcal{D}^{\uparrow}}(A \xrightarrow{f} B) = H^n_{\mathcal{D}}(A) \xrightarrow{H^n_{\mathcal{D}}(f)} H^n_{\mathcal{D}}(B).$$

Proof. This is an implication of the proof of proposition 2.2.1. Since kernels (and similarly images) are being taken componentwise, so is cohomology. □

One might – somewhat intuitively assume – that the category $\mathcal{D}^{\uparrow}$ should be naturally related to the category $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$. The following shows that there is no reason to assume equivalence of both categories. Since this is a crucial fact we will borrow theory from section 3 to illustrate our point better.

Definition 2.2.9. Define a functor $T : \mathcal{D}^b(\mathcal{A}^{\uparrow}) \to (\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ by $T(A \to B) = (A \xleftarrow{\text{id}} A \to B)$ for objects in $\mathcal{D}^b(\mathcal{A}^{\uparrow})$. The functor T acts on morphisms by $T(f) = (\lambda_1(f), \rho_2(f))$ with λ_1, ρ_2 as in lemma 3.2.8.

Remark 2.2.10. One easily checks that T is – indeed – a functor.

Lemma 2.2.11. *There exists no triangulated structure on $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ such that T is an exact functor.*

Proof. Assume for a contradiction that $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ does carry the triangulated structure induced by T. Then let $A \in \mathcal{A}$ and observe that the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{2.2}$$

in $\mathcal{C}(\mathcal{A}^{\uparrow})$ induces exact triangles

$$(0 \to A) \to (A \stackrel{\mathrm{id}_A}{\to} A) \stackrel{f}{\to} (A \to 0) \stackrel{+}{\to} \tag{2.3}$$

and

$$(A \stackrel{\mathrm{id}_A}{\to} A) \to (A \to 0) \stackrel{g}{\to} (0 \to A[1]) \stackrel{+}{\to} \tag{2.4}$$

in $\mathcal{D}^{\uparrow}$ and by our assumption hence also in $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$. Therefore we obtain from the axioms of triangulated categories that $\mathrm{Cone}(f) = (0 \to A[1])$, making the morphism $+$ in (2.3) a pair $(g_1, g_2) \in (\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ for which $g_1 \in \mathrm{Hom}_{\mathcal{D}}(A, 0) = 0$ and $g_2 \in \mathrm{Hom}_{\mathcal{D}}(0, A) = 0$. We have $(g_1, g_2) = (0, 0)$ which is the 0-morphism in $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$. By [59, Tag 05QT] we obtain from (2.3) that

$$(A \stackrel{\mathrm{id}_A}{\to} A) = (A \to 0) \oplus (0 \to A). \tag{2.5}$$

Observe now that $\mathrm{Hom}_{(\mathcal{D}^b(\mathcal{A}))^{\uparrow}}((A \stackrel{\mathrm{id}_A}{\to} A), (0 \to A)) = 0$ by commutativity (see definition 2.1.12). Combining this with (2.5) this provides us with

$$0 = \mathrm{Hom}_{(\mathcal{D}^b(\mathcal{A}))^{\uparrow}}((A \stackrel{\mathrm{id}_A}{\to} A), (0 \to A)) =$$
$$\mathrm{Hom}_{(\mathcal{D}^b(\mathcal{A}))^{\uparrow}}((A \to 0), (0 \to A)) \oplus \mathrm{Hom}_{(\mathcal{D}^b(\mathcal{A}))^{\uparrow}}((0 \to A), (0 \to A)) \neq 0$$

whenever $A \neq 0$ since that implies $\mathrm{Hom}_{(\mathcal{D}^b(\mathcal{A}))^{\uparrow}}((0 \to A), (0 \to A)) = 0$. This is the contradiction we wanted and therefore the proof is finished. □

Similarly we prove another enlightening result on $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$.

Lemma 2.2.12. *The functor T is not an equivalence of categories.*

Proof. Since $\mathcal{D}^{\uparrow}$ is triangulated and (2.2) an exact sequence in $\mathcal{C}(\mathcal{A}^{\uparrow})$, (2.3) is an exact triangle – and hence $\stackrel{+}{\to}$ non-zero in general. Therefore, g of (2.4), which equals to $\stackrel{+}{\to}$ is generally non-zero. However,

$$T(g) \in \mathrm{Hom}_{(\mathcal{D}^b(\mathcal{A}))^{\uparrow}}(A \to 0, 0 \to A[1]) = 0$$

by commutativity in $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ and therefore T is not faithful. □

Remark 2.2.13. Note that the finding of lemmas 2.2.11 and 2.2.12 are based on the exactness of the triangle

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \mathrm{id}_A & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

which is via that on $\mathcal{D}^{\uparrow}$ essentially inherited from the triangulated structure of $\mathcal{D}$. Therefore, one might generalise lemma 2.2.11 to the analogous statement for a triangulated category in general – leading us to the conjecture that taking the arrow category of a given triangulated category will generally not give a triangulated category back.

2.3 Motivation

The motivation to introduce stability spaces is based on the theory of stability of vector bundles and, more general, coherent sheaves. The idea is to compare the sheaf to its subsheaves in terms of the invariants rank and degree. Classically this was only done for vector bundles (note that on a smooth projective curve any torsion-free sheaf is a vector bundle), for reasons that will become obvious with the definition of μ in 2.3.2. Note also that there are concepts available that generalise this definition to all non-zero objects in the yet to be defined category $\mathcal{C}$. We will however omit this theory altogether.

Notation 2.3.1. Let $\mathcal{C} = \mathrm{Coh}(C)$ denote the (abelian) category of coherent sheaves on a smooth projective curve C.

Definition 2.3.2. Let $E \in \mathcal{C}$ be non-zero torsion-free and $\deg(E)$ and $\mathrm{rank}(E)$ denote degree and rank of E. We define $\mu(E) = \deg(E)/\mathrm{rank}(E)$.

Definition 2.3.3. A torsion-free sheaf $F \in \mathcal{C}$ is called "μ-stable" if for any non-zero subsheaf E of F with $E \neq F$,

$$\mu(E) < \mu(F).$$

Analogously μ-semistability is defined as

Definition 2.3.4. A torsion-free sheaf $F \in \mathcal{C}$ is called "μ-semistable" if for any non-zero subsheaf E of F

$$\mu(E) \leq \mu(F).$$

Definition 2.3.5. Let $E \in \mathcal{C}$ be non-zero and torsion-free. A filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E, n \in \mathbb{N}$$

where all $F_i = E_i/E_{i-1}$ are μ-semistable objects of $\mathcal{C}$ and $\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n)$ is called a "Harder-Narashiman filtration".

Theorem 2.3.6. *There is a unique Harder-Narashiman filtration for any non-zero torsion-free $F \in \mathcal{C}$.*

Proof. [42, Proposition 5.4.2] We will prove the existence of the filtration first. For F μ-semistable the result is trivial. Hence we assume F not μ-semistable which implies the existence of subsheaves of higher μ. Consider the subset of subsheaves among these which have maximal μ. This is possible as both rank and degree are bounded above. These subsheaves are μ-semistable, as by assumption they cannot have a subsheaf of higher μ. We narrow this set down further by choosing those subsheaves amongst them that have maximal rank and choose a subsheaf F_1 of this set of subsheaves with maximal μ and rank. This F_1 is the first non-zero object of the required chain. If we now consider the quotient F/F_1, we obtain a non-zero torsion-free sheaf that is either μ-semistable or has – by the same construction we used above – a semistable subsheaf of maximal rank and μ. We denote this subsheaf by F_2/F_1 which we can do, since $F_1 \subset F_2$. Repeating this procedure with F_2 now assuming the place of F_1, we obtain an ascending chain of subobjects $0 = F_0 \subset F_1 \subset \ldots$ with μ-semistable quotients F_i/F_{i-1}.

If we now consider quotients $F_{d-1}/F_{d-2} \subset F/F_{d-2}$ and $F_d/F_{d-2} \subset F/F_{d-2}$ for $d \in \mathbb{N}_{\geq 2}$ with all F_i taken from this filtration, we observe that since without loss of generality $F_d/F_{d-1} \neq 0$ can be assumed, we also obtain $F_{d-1}/F_{d-2} \subsetneq F_d/F_{d-2}$. This means on the other hand that $\mu(F_{d-1}/F_{d-2}) < \mu(F_d/F_{d-2})$ because F_{d-1}/F_{d-2} was chosen to be the maximal rank subsheaf of F/F_{d-2}. If we now consider the exact sequence

$$0 \to F_{d-1}/F_{d-2} \to F_d/F_{d-2} \to F_d/F_{d-1} \to 0,$$

we obtain $\mu(F_{d-1}/F_{d-2}) > \mu(F_d/F_{d-2}) > \mu(F_d/F_{d-1})$, which implies that $\mu(F_{d-1}/F_{d-2}) > \mu(F_d/F_{d-1})$. This is the required condition on the quotients. Moreover, it implies the (also required) finiteness as – due to the fact that rank and degree of the sheaves are bounded – the set of potential μs is finite. Hence, the sequence $\mu(F_1) > \mu(F_2/F_1) > \ldots$ must eventually stabilise and therefore with it the sequence $0 = F_0 \subset F_1 \subset \ldots$ which finishes the existence-proof.

To prove the uniqueness of the Harder-Narashiman filtration for a non-zero torsion-free $F \in \mathcal{C}$ we start with the observation that if $0 = F_0 \subset$

$F_1 \subset \cdots \subset F_n = F$ is a Harder-Narashiman filtration of F and $F' \subset F$ a subbundle, then $\mu(F') \leq \mu(F_1)$ with $F' = F_1$ if equality holds. To prove this we define a filtration of F' setting $F'_i = F' \cap F_i$ for $i \in \{1, \ldots, n\}$, which implies $F'_{i+1}/F'_i \subset F_{i+1}/F_i$. It follows from the semistability of F_{i+1}/F_i that, provided that F'_{i+1}/F'_i is not zero, $\mu(F'_{i+1}/F'_i \leq \mu(F_{i+1}/F_i)$. From the canonical exact sequence $0 \to F_i \to F_{i+1} \to F_{i+1}/F_i \to 0$ we obtain $\mu(F_{i+1}) \leq \mu(F_{i+1}/F_i)$. Hence $\mu(F') \leq \mu(F'_1) = \mu(F'_1/F'_0) \leq \mu(F_1/F_0) = \mu(F_1)$ which means that $\mu(F') \leq \mu(F_1)$. If, on the other hand, equality holds in this equation, this would also imply $\mu(F') = \mu(F'_1)$ and all quotients hence had to be 0. In other words, $F = F'_1 \subset F_1$, proving that $F \subset F_1$ if $\mu(F) = \mu(F')$.

We can now set up a classic uniqueness-proof. Assume we have two Harder-Narashiman filtrations $E_i, i \in \{1, \ldots, n\}$ and $F_i, i \in \{1, \ldots, m\}$ of a non-zero torsion-free $F \in \mathcal{C}$. Defining F' from above as $F' = F_1$ we obtain $\mu(F_1) \leq \mu(E_1)$ when similarly letting $F' = E_1$ provides $\mu(E_1) \leq \mu(F_1)$ resulting in $\mu(E_1) = \mu(F_1)$. This results in $E_1 \subset F_1$ and at the same time $F_1 \subset E_1$ giving $E_1 = F_1$. Repeating the procedure with F/F_1 and F/E_1 we obtain the required equality. □

2.4 Stability functions

The Grothendieck group is one of the key tools in defining stability conditions.

Definition 2.4.1. For an abelian category $\mathcal{A}$ let G be the free abelian group that is generated by isomorphism classes of objects in $\mathcal{A}$. Let $E_1, E_2, E_3 \in \mathcal{A}$ and let $\widetilde{G}$ be the subgroup generated by all elements of the form $E_1 - E_2 - E_3$ and there exists an exact sequence $0 \to E_2 \to E_1 \to E_3 \to 0$. We define the "Grothendieck group" $\mathcal{K}(\mathcal{A})$ of $\mathcal{A}$ as the quotient $G/\widetilde{G}$.

The motivation of Bridgeland stability is based on the concept of μ-stability as defined in 2.3.3. To be able to define Bridgeland stability it is hence very important to generalise this concept – in [18], Bridgeland does this by introducing the notion of a stability function Z. The role that μ had to play in the previous will now be assumed by this Z. To introduce it we need the notion of the strict upper half plane.

Definition 2.4.2. We define

- The "upper half plane" $\mathbb{H}_0$ is defined as

$$\mathbb{H}_0 = \{r \exp(i\pi\phi) \mid 0 < \phi \leq 1, r \geq 0\} \subset \mathbb{C}.$$

- The "strict upper half plane" $\mathbb{H}$ is defined as

$$\mathbb{H} = \{r\exp(i\pi\phi) \mid 0 < \phi \leq 1, r > 0\} \subset \mathbb{C}.$$

Definition 2.4.3. A "weak stability function" on an abelian category $\mathcal{A}$ is a group homomorphism $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ such that for any $E \in \mathcal{A}$, we have $Z(E) \in \mathbb{H}_0$.

Definition 2.4.4. A "stability function" is a weak stability function such that for any $E \in \mathcal{A}_{\neq 0}$, we have $Z(E) \in \mathbb{H}$.

Example 2.4.5. *Related to the concept that was described in definition 2.3.2 we can, for example, define a stability function Z on $\mathcal{K}(\mathcal{C})$ as $Z(E) = -\deg(E) + i\,\mathrm{rank}(E)$. One could think of $\mu(E) = \deg(E)/\mathrm{rank}(E)$ as the function that sends $Z(E)$, regarded as a vector in the complex plane to it's slope.*

More generally we define

Definition 2.4.6. Let $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a stability function. The "phase" of a non-zero object $E \in \mathcal{A}$ is defined to be

$$\phi(E) = (1/\pi)\arg(Z(E)) \in (0, 1].$$

Following we introduce the slope phase correspondence.

Definition 2.4.7. Let $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a stability function. For $E \in \mathcal{A}$ define the "slope" of Z by

$$\mu_Z(E) := \frac{-\Re(Z(E))}{\Im(Z(E))}.$$

Lemma 2.4.8. *Let $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a stability function and $E \in \mathcal{A}$. There is a correspondence between phase and slope provided by*

$$\mu_Z(E) = -\cot(\pi\phi(E))$$

Proof. This is seen from

$$\mu_Z(E) = \frac{-\Re(Z(E))}{\Im(Z(E))} = -\cot(\pi\phi(E)).$$

□

Generalising the concept of a μ-(semi)stable object in definitions 2.3.3 and 2.3.4 Bridgeland has introduced the crucial concept of a semistable object:

Definition 2.4.9. Let $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a stability function. A non-zero object $E \in \mathcal{A}$ is called "semistable" (with respect to Z), if for every non-zero subobject $A \subset E$ we have $\phi(A) \leq \phi(E)$.

Remark 2.4.10. Note that this definition is equivalent to saying that for any non-zero quotient B of E we have $\phi(B) \geq \phi(E)$. Once lemma 2.4.22 is established it will follow as an immediate consequence.

Remark 2.4.11. If $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve and

$$Z(E) = -\deg(E) + i\,\mathrm{rank}(E) \text{ for } E \in \mathcal{A}$$

then for vector bundles on C we obtain the usual μ-stability and additionally all torsion sheaves are semistable since $\phi(T) = 1$ for any torsion sheaf T on C.

In the context of stability conditions – the one in which we are interested – a very important feature of a stability function is the so-called Harder-Narashiman property. Since it has proved to be an important feature, the existence of Harder-Narashiman filtrations will be included in the definition of a stability condition. We will now generalise the definition of μ-semistability, given in 2.3.4 and that of the Harder-Narashiman filtration defined in 2.3.5 in order to provide a definition of the Harder-Narashiman property.

Definition 2.4.12. For a given stability function $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ on an abelian category, a "Harder-Narashiman filtration" of a non-zero object $E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E, n \geq 1$$

such that $F_i = E_i/E_{i-1}$ for $i \in \{1, \ldots, n\}$ are semistable objects of $\mathcal{A}$ and $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$.

Lemma 2.4.13. *If for a stability function $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ and an object $E \in \mathcal{A}$ a Harder-Narashiman filtration exists, then it is unique up to isomorphisms.*

Proof. Assume that, using a suitable change of notation

$$0 = E_n \subset E_{n-1} \subset \cdots \subset E_1 \subset E_0 = E, n \geq 1$$

and

$$0 = E'_m \subset E'_{n-1} \subset \cdots \subset E'_1 \subset E'_0 = E, n \geq 1$$

are Harder-Narashiman filtrations of a non-zero object $E \in \mathcal{A}$ with regard to the stability function Z. We proceed by induction and observe at first that

$E'_0 = E = E_0$. Assume now that for an $i \leq n, i \leq m$, we have $E_j \cong E'_j$ for every $j \leq i$. Consider the exact sequences

$$0 \to E_{i+1} \to E_i \to F_i \to 0$$

and

$$0 \to E'_{i+1} \to E'_i \to F'_i \to 0$$

where F_i and F'_i are the quotients defined in 2.4.12 (recall that the notation has been reversed). Without loss of generality we assume $\phi(F_i) \geq \phi(F'_i)$. Consider now the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & F_i & \longrightarrow & 0 \\ & & & & \cong\big\downarrow & & & & \\ 0 & \longrightarrow & E'_{i+1} & \longrightarrow & E'_i & \longrightarrow & F'_i & \longrightarrow & 0 \end{array}$$

and apply the functor $\mathrm{Hom}(E_{i+1}, -)$ to the second row – we obtain the exact sequence

$$0 \to \mathrm{Hom}(E_{i+1}, E'_{i+1}) \to \mathrm{Hom}(E_{i+1}, E'_i) \to \mathrm{Hom}(E_{i+1}, F'_i).$$

It can be proved by induction that $\phi(E_{i+1}) > \phi(F_i)$ and since $\phi(F_i) \geq \phi(F'_i)$ we obtain $\phi(E_{i+1}) > \phi(F'_i)$ implying $\mathrm{Hom}(E_{i+1}, F'_i) = 0$. Hence we now have $\mathrm{Hom}(E_{i+1}, E'_{i+1}) \cong \mathrm{Hom}(E_{i+1}, E'_i)$ and conclude that we get an arrow $E_{i+1} \to E'_{i+1}$ and via that an arrow $F_i \to F'_i$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & F_i & \longrightarrow & 0 \\ & & \big\downarrow & & \cong\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & E'_{i+1} & \longrightarrow & E'_i & \longrightarrow & F'_i & \longrightarrow & 0 \end{array}$$

commutes.

The arrow $F_i \to F'_i$ is not zero as otherwise $E'_i \to F'_i$ would be zero implying F'_i being zero. This means, on the other hand, that $\phi(F_i) \leq \phi(F'_i)$ and as we have $\phi(F_i) \geq \phi(F'_i)$ we obtain $\phi(F_i) \geq \phi(F'_i)$. Similar to what we did before, we can therefore apply the functor $\mathrm{Hom}(E'_{i+1}, -)$ to get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_{i+1} & \longrightarrow & E'_i & \longrightarrow & F'_i & \longrightarrow & 0 \\ & & \big\downarrow & & =\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & F_i & \longrightarrow & 0, \end{array}$$

in other words we have arrows in both directions. Combining both diagrams we get a new diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & F_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow & & \\ 0 & \longrightarrow & E_{i+1} & \longrightarrow & E_i & \longrightarrow & F_i & \longrightarrow & 0 \end{array}$$

and, as before, we have $\mathrm{Hom}(E_{i+1}, E_{i+1}) \cong \mathrm{Hom}(E_{i+1}, E_i)$. This means, that the identity as a possible choice for the arrow $E_{i+1} \to E_{i+1}$ is in fact the only possible choice.

We proceed in the same manner with the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_{i+1} & \longrightarrow & E'_i & \longrightarrow & F'_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow & & \\ 0 & \longrightarrow & E'_{i+1} & \longrightarrow & E'_i & \longrightarrow & F'_i & \longrightarrow & 0, \end{array}$$

to obtain that $E'_{i+1} \to E'_{i+1}$ from the diagram has to be the identity. We obtain that

$$(E'_{i+1} \to E_{i+1}) \circ (E_{i+1} \to E'_{i+1}) = \mathrm{id}_E$$

and that

$$(E_{i+1} \to E'_{i+1}) \circ (E'_{i+1} \to E_{i+1}) = \mathrm{id}_{E'}\,.$$

This means $E_{i+1} \cong E'_{i+1}$ which finishes the proof.

□

Definition 2.4.14. A stability function $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ on an abelian category $\mathcal{A}$ has the "Harder-Narashiman property" if every non-zero object $E \in \mathcal{A}$ has a Harder-Narashiman filtration with regard to Z.

This definition however is somewhat hard to handle – we do, on the other hand, have an easier approach available, that Bridgeland introduced in [18] and that is based on certain chain-conditions that the stability function needs to satisfy. Our aim is to establish theorem 2.4.23, in order to do that, we will start with stating the chain conditions that by theorem 2.4.23 will provide a criterion to decide whether a stability function has the Harder-Narashiman property.

Definition 2.4.15. Let $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a stability function.

(a) We say that Z satisfies the "subobject-chain condition" if there are no infinite sequences $\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$ of subobjects in $\mathcal{A}$ with $\phi(E_{j+1}) > \phi(E_j)$ for all j.

(b) We say that Z satisfies the "quotient-chain condition" if there are no infinite sequences $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \ldots$ of quotients in $\mathcal{A}$ with $\phi(E_j) > \phi(E_{j+1})$ for all j.

We will now, after introducing the concept of a maximal destabilising quotient, prove the theorem 2.4.23. To do so, we need a series of lemmas which we have worked out from a somewhat less extensive version by Bridgeland.

Definition 2.4.16. We define an "mdq" (maximal destabilising quotient) of a non-zero $E \in \mathcal{A}$ to be a non-zero quotient $E \twoheadrightarrow B$ with the property that every non-zero quotient $E \twoheadrightarrow B'$ satisfies $\phi(B') \geq \phi(B)$ and in case of equality $E \twoheadrightarrow B'$ factors via $E \twoheadrightarrow B$.

Lemma 2.4.17. *Assume that the quotient-chain condition holds. A quotient $E \twoheadrightarrow B$ is an mdq of a non-zero $E \in \mathcal{A}$ if every non-zero semistable quotient $E \twoheadrightarrow B'$ satisfies $\phi(B') \geq \phi(B)$ and in case of equality $E \twoheadrightarrow B'$ factors via $E \twoheadrightarrow B$.*

Proof. Every $E \in \mathcal{A}$ has a semistable quotient as otherwise we would get an infinite chain of non-semistable quotients with descending ϕ (the dual statement is proved more detailed in theorem 2.4.23). Let $E \twoheadrightarrow B''$ be a quotient of E, then there is a semistable quotient $B'' \twoheadrightarrow B'$ of B'' with $\phi(B'') \geq \phi(B')$. If B'' is not semistable, then we have indeed $\phi(B'') > \phi(B')$. Semistability implies, that B'' is its own semistable quotient – in other words $\phi(B'') = \phi(B')$ implies that $B'' = B'$ meaning then that $E \twoheadrightarrow B''$ factors via B. If, on the other hand we assume $\phi(B'') > \phi(B')$ and observe that B' is also a quotient of E and since B' is semistable, we obtain by assumption that $\phi(B') \geq \phi(B)$ which in turn implies that $\phi(B'') > \phi(B') \geq \phi(B)$ and hence $\phi(B'') > \phi(B)$. □

Remark 2.4.18. Note that if B is an mdq $E \twoheadrightarrow B$ of $E \in \mathcal{A}$, B must be semistable and $\phi(E) \geq \phi(B)$ as non-semistability of B would imply the existence of a non-zero quotient B' of B with $\phi(B') < \phi(B)$, which is impossible by definition since via

$$E \twoheadrightarrow B \twoheadrightarrow B'$$

we obtain that B' is a quotient of E as well. Since we also have $E \twoheadrightarrow E$ it follows from the definition of an mdq that $\phi(E) \geq \phi(B)$.

If, furthermore, E is not semistable, we can observe that there is a quotient $E \twoheadrightarrow E'$ with $\phi(E) > \phi(E')$. We obtain $B \twoheadrightarrow E \twoheadrightarrow E'$ and conclude that $B \twoheadrightarrow E'$ is a quotient with $\phi(E) > \phi(E')$, which contradicts E being and mdq..

Lemma 2.4.19. *If $E \in \mathcal{A}$ is semistable it is its own mdq.*

Proof. Since E is semistable, we obtain $\phi(E') \geq \phi(E)$ for any non-zero quotient $E \to E'$ of E. As any quotient $E \to E'$ factors via the quotient $E \twoheadrightarrow E$, so do those quotients E' where $\phi(E') = \phi(E)$. □

Lemma 2.4.20. *Let a stability function Z fulfil the quotient-chain condition then every non-zero $E \in \mathcal{A}$ has an mdq.*

Proof. If E is semistable it is its own mdq by lemma 2.4.19 – otherwise, if we assume E not semistable, there are objects $A, E' \in \mathcal{A}$ and a short exact sequence: $0 \to A \to E \to E' \to 0$ with A semistable and $\phi(A) > \phi(E) > \phi(E')$, as being not semistable means that E has a subobject A with bigger ϕ then itself, E' is the corresponding quotient.

Assume that an mdq of E', $E' \twoheadrightarrow B$, exists. We will prove that under this condition the quotient $E \twoheadrightarrow B$ is an mdq for E. If $E \twoheadrightarrow B'$ is a quotient with B' semistable and $\phi(B') \leq \phi(B)$ (recall that by lemma 2.4.17 it is good enough to conduct this proof under the assumption that B' is semistable) then $\phi(B') \leq \phi(B) \leq \phi(E') < \phi(E) < \phi(A)$, implying $\phi(B') < \phi(A)$. Since both A and B' are semistable, this implies that $\mathrm{Hom}(A, B') = 0$. So we deduce from the exact sequence $\mathrm{Hom}(E', B') \to \mathrm{Hom}(E, B') \to \mathrm{Hom}(A, B')$ which hence equals to $\mathrm{Hom}(E', B') \to \mathrm{Hom}(E, B') \to 0$ that $E \twoheadrightarrow B'$ factors via E'. The map $E \twoheadrightarrow B'$ on the other hand factors as $E \twoheadrightarrow E' \twoheadrightarrow B'$. We have $\phi(B') \leq \phi(B)$ and $E' \twoheadrightarrow B$ is an mdq for E', which means that $\phi(B') \geq \phi(B)$, hence $\phi(B') = \phi(B)$ implying that the quotient $E' \twoheadrightarrow B'$ factors via B. This means that the quotient $E \twoheadrightarrow B'$ factors via B as well. We have proved that B is indeed an mdq for E.

If, on the other hand, there is no mdq for E', we apply the same procedure as before, where now E' is assuming the role of E. If we keep on repeating this process, we will eventually find an mdq for E, as otherwise we would get an infinite sequences $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \ldots$ of quotients in $\mathcal{A}$ with $\phi(E_j) > \phi(E_{j+1})$ for all j. This would violate the quotient-chain condition. We have proved that E has indeed an mdq. □

Lemma 2.4.21. *Let $z_1, z_2 \in \mathbb{C}^*$ such that $z_1 = r\exp(i\alpha)$ and $z_2 = s\exp(i\beta)$ and $t\exp(i\gamma) = z_1 + z_2$ for $r, s, t \in \mathbb{R}_{>0}$ and assume $|\alpha - \beta| < \pi$. Then either $\alpha = \beta = \gamma$ or γ lies strictly between α and β.*

Proof. If $\alpha = \gamma$, then $r\exp(i\alpha) + s\exp(i\beta) = 2rs\exp(i\beta)$. This implies $\beta = \gamma$.

Otherwise, assuming without loss of generality that $\alpha < \beta$, we obtain $r\exp(i\alpha) + s\exp(i\beta) = \exp(i\alpha)(r + s\exp(i(\beta - \alpha)))$ and as $\exp(i\beta)$ is a rotation we may – again without loss of generality – assume that $\beta = 0$ and additionally that $\beta \in (0, \pi)$. We then have $\exp(i\alpha)(r + s\exp(i(\beta - \alpha))) = \exp(i\alpha)(r + s\exp(i(\beta - \alpha))) = \exp(i0)(r + s\exp(i(\beta - 0))) = r + s\exp(i\beta) = r + s(\cos(\beta) + i\sin(\beta))$ and at the same time $t\exp(i\gamma) = t(\cos(\gamma) + i\sin(\gamma)$. This means that $r + s(\cos(\beta) + i\sin(\beta)) = z_1 + z_2 = t(\cos(\gamma) + i\sin(\gamma))$ implying $r + s\cos(\beta) = t\cos(\gamma)$ and $s\sin\beta = t\sin(\gamma)$.

If we let $\beta \in (0, \frac{\pi}{2})$ we obtain $t\cos(\gamma) > 0 < t\sin(\gamma)$ and since $t > 0$ this means that $\cos(\gamma) > 0 < \sin(\gamma)$ which gives $\gamma \in (0, \frac{\pi}{2})$. We have $t^2 = |r+s\exp(i\beta)|^2 = r^2+s^2+2rs\cos(\beta) > s^2$ since $\cos(\beta) > 0$ for $\beta \in (0, \frac{\pi}{2})$ and $r, s > 0$ by assumption. Therefore $t^2 > s^2$ providing $t > s$ and hence $\frac{s}{t} \in (0, 1)$. Now we obtain from $s\sin\beta = t\sin(\gamma)$ that $\sin(\gamma) = \frac{s}{t}\sin(\beta)$ which means that $\sin(\gamma) < \sin(\beta)$ and as $\sin(x)$ is strictly increasing for $x \in (0, \frac{\pi}{2})$ we have $\gamma < \beta$.

If, on the other hand, $\beta \in (\frac{\pi}{2}, \pi)$, we have $\cos(\beta) < 0$ and hence we must distinguish between $r + s\cos(\beta) < 0$ and $r + s\cos(\beta) \geq 0$. If indeed $r+s\cos(\beta) < 0$ we have $\gamma \in (\frac{\pi}{2}, \pi)$ and additionally – similar to the previous case – that $t^2 = |r+s\exp(i\beta)|^2 = r^2+s^2+2rs\cos(\beta) = s^2+r(r+2s\cos(\beta) < s^2$ because $r + 2s\cos(\beta) < r + s\cos(\beta) < 0$ by assumption. Hence $\frac{s}{t} > 1$, implying that $\sin(\gamma) > \sin(\beta)$ and as $\sin(x)$ is strictly decreasing for $x \in (\frac{\pi}{2}, \pi)$ we must have $\gamma < \beta$. If $r + s\cos(\beta) > 0$, we obtain $\cos(\gamma) > 0$ and hence $\gamma \in (0, \frac{\pi}{2}]$. Therefore $\gamma < \beta$. □

Lemma 2.4.22. *Suppose $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ is a weak stability function and let $Z(E') = r\exp(i\alpha), Z(E'') = r\exp(i\beta)$ such that $|\alpha - \beta| < \pi$. If $0 \to E'' \to E \to E' \to 0$ is exact then exactly one of the following (in)equalities holds.*

1. $\phi(E'') > \phi(E) > \phi(E')$;
2. $\phi(E'') = \phi(E) = \phi(E')$;
3. $\phi(E'') < \phi(E) < \phi(E')$.

Proof. Recall that by 2.4.4, a stability function is defined on the Grothendieck group of $\mathcal{A}$ and therefore $E = E' + E''$. As the stability function is a homomorphism of groups we obtain $Z(E) = Z(E') + Z(E'')$. Recalling the definition of ϕ provided in 2.4.6, the result now follows from lemma 2.4.21. □

Theorem 2.4.23. *Let $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a stability function that satisfies the subobject-chain condition and the quotient-chain condition of definition 2.4.15. Then Z has the Harder-Narashiman property.*

Proof. [18, Proposition 2.4] A non-zero object $E \in \mathcal{A}$ is either semistable or has a non-zero subobject E' with $\phi(E') > \phi(E)$. Replacing E by E' we can deduce from the first chain condition that every non-zero object of $\mathcal{A}$ has a semistable subobject $A \subset E$ with $\phi(A) \geq \phi(E)$. This is easily seen by induction if we let $E_0 = E$ and $E_1 = E'$. For some n and E_{n-1} not semistable E_n has to be semistable in order to avoid an infinite chain – therefore we obtain the desired result. If we now take a non-zero object $E \in \mathcal{A}$, then E either is its own filtration if it is semistable or by lemma

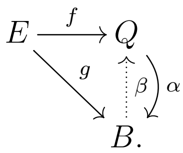
2.4.20 there exists a subobject E' such that $0 \to E' \to E \to B \to 0$ where $E \twoheadrightarrow B$ is an mdq and $\phi(E') > \phi(E)$. For an mdq $E' \twoheadrightarrow B'$ of E' we now construct a diagram of short exact sequences, starting with:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & E & & & & \\ & & & & \downarrow{\scriptstyle g} & & & & \\ & & & & B & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

where $K \to E'$ is the kernel of the map $E' \to B'$. Dually to this idea we now define the map $f : E \to Q$ as the quotient of the map $K \to E$ and obtain the following diagram of short exact sequences:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \xrightarrow{f} & Q & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} & & \\ & & & & B & = & B & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

By the snake lemma (we refer to [60, Lemma 1.3.2] or to [41, Section 9] for details) we obtain that the sequence $0 \to B' \to Q \to B \to 0$ is exact. We now deduce from the diagram that $\phi(Q) = \phi(B)$ leads to a contradiction. According to the definition of an mdq the equality would give us a factorisation β of f via B, that is: $f = \beta g$:



Additionally we know from the diagram that $\alpha f = g$ (commutativity). Therefore $\beta\alpha f = f = \operatorname{id} f$ and as f is an epimorphism we obtain $\beta\alpha = \operatorname{id}$. Moreover $\beta g = f$, implying $\alpha\beta g = \alpha f = g = \operatorname{id} g$, giving $\alpha\beta = \operatorname{id}$ since g is an epimorphism which means that Q and B are isomorphic. We therefore obtain $B' = 0$. But B' was assumed to be an mdq of E' and so we have $E' = 0$ which gives the contradiction. Then, as $f : E \twoheadrightarrow Q$ is a quotient of E it follows that $\phi(Q) > \phi(B)$ and this implies $\phi(B') > \phi(B)$ using lemma 2.4.22.

We now rename E into E_0 and E' into E_1. This provides the $n = 0$ case of an induction proof: If E_{n-1} is not automatically semistable, as before, when we investigated the object E, E_{n-1} if not semistable has a destabilizing subobject E_n, analogous to E' in the previous proof. Then again, E_n is either semistable or has a destabilizing subobject. To all of these we can apply the previous proof in order to show that over the descending chain the ϕ strictly increases. That means, we obtain a – due to the first chain condition eventually terminating sequence of subobjects $\cdots \subset E_i \subset E_{i-1} \subset \cdots \subset E_1 \subset E_0 = E$ with $\phi(E_i) > \phi(E_{i-1})$. It has semistable factors $F_i = E_i/E_{i-1}$ of ascending phase (the mdq's). This is the Harder-Narashiman Filtration.

□

2.5 Stability conditions

We are now ready to discuss the most important concept of this thesis. The definition of a stability condition uses so called slicings and central charges – we will now define the necessary terms and provide the connection to stability functions.

Definition 2.5.1. A slicing $\mathcal{P}$ of a triangulated category $\mathcal{TR}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{TR}$ for each $\phi \in \mathbb{R}$ satisfying:

1. for all $\phi \in \mathbb{R}, \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
2. if $\phi_1 > \phi_2, A_1 \in \mathcal{P}_1, A_2 \in \mathcal{P}_2$ then $\operatorname{Hom}_{\mathcal{TR}}(A_1, A_2) = 0$;
3. for each non-zero object $E \in \mathcal{TR}$ there is an $n \in \mathbb{N}_{\geq 1}$, a sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n,$$

and a collection of exact triangles

$$\begin{array}{ccccccccccc} E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots & \longrightarrow & E_{n-1} & \longrightarrow & E_n \\ {\scriptstyle +}\uparrow & \swarrow & {\scriptstyle +}\uparrow & \swarrow & & & & & {\scriptstyle +}\uparrow & \swarrow & \\ A_1 & & A_2 & & & & & & A_n & & \end{array}$$

with $E_0 = 0, E_n = E$ and $A_j \in \mathcal{P}(\phi_j)$ for all $j \in \{1, \dots, n\}$.

Definition 2.5.2. For a non-zero object $E \in \mathcal{TR}$ and a slicing $\mathcal{P}$ of $\mathcal{TR}$ let $\phi^+ = \phi_1$ and $\phi^- = \phi_n$, where $\phi_1 > \phi_2 > \cdots > \phi_n, n \geq 1$ is the series of definition 2.5.1.

Lemma 2.5.3. *If $E \in \mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ we have $\phi^+(E) = \phi^-(E)$.*

Proof. If $E \in \mathcal{P}(\phi)$ then $A_1 = A_n = E$. □

Definition 2.5.4. Let $E \in \mathcal{P}(\xi)$ for $\xi \in \mathbb{R}$, for the slicing $\mathcal{P}$ define

$$\phi(E) := \xi.$$

Remark 2.5.5. We will refer to the collection of exact triangles of definition 2.5.1 as a "filtration" in the style of a Harder-Narashiman filtration.

Before we can use the term slicing as part of the definition of a stability condition we need – in order to define the second part, the (pre-)stability condition consists of – to introduce the notion of the Grothendieck group of a triangulated category. Since the concept of a triangulated category adapts the idea of the concept of an abelian category with regard to exact sequences, the following definition is only a slight alteration of definition 2.4.1.

Definition 2.5.6. For a triangulated category $\mathcal{TR}$ let G be the free abelian group generated by isomorphism classes of objects in $\mathcal{TR}$. Let $\widetilde{G}$ be the subgroup generated by all elements of the form $E_1 - E_2 - E_3$ where E_1, E_2 and E_3 are objects in $\mathcal{TR}$ and there exists an exact triangle $E_2 \to E_1 \to E_3 \xrightarrow{+}$. We define the "Grothendieck group" $\mathcal{K}(\mathcal{TR})$ of $\mathcal{TR}$ as the quotient $G/\widetilde{G}$.

We need to somewhat strengthen this definition.

Definition 2.5.7. We define the "numerical Grothendieck group" $\mathcal{N}(\mathcal{TR})$ as the quotient $\mathcal{K}(\mathcal{TR})/\mathcal{K}(\mathcal{TR})^\perp$, where $\mathcal{K}(\mathcal{TR})^\perp$ denotes the right orthogonal with respect to the Euler form, that is

$$\mathcal{K}(\mathcal{TR})^\perp = \{E \in \mathcal{K}(\mathcal{TR}) \mid \chi(E, F) = 0 \text{ for all } F \in \mathcal{K}(\mathcal{TR})\}.$$

Moreover, if $\mathcal{N}(\mathcal{TR})$ has finite rank then $\mathcal{TR}$ is called "numerically finite".

Using the data that we have now available, Bridgeland has defined the concept of a pre-stability condition in [18]. This is a pair consisting of a group homomorphism and a slicing that fulfils a certain compatibility condition provided in the following definition which now introduces stability conditions. Note that the concept of a pre-stability condition was not used in [18] yet and pre-stability conditions simply referred to as stability conditions.

Definition 2.5.8. Let Λ to be a finite-rank $\mathbb{Z}$-lattice and a (fixed) surjective homomorphism of groups $v : \mathcal{K}(\mathcal{TR}) \twoheadrightarrow \Lambda$.

A quasi-stability condition on a triangulated category $\mathcal{TR}$ is a pair $(\mathcal{P}, Z)$ that consists of a slicing $\mathcal{P}$ on a triangulated category $\mathcal{TR}$ together with a group homomorphism $Z : \mathcal{K}(\mathcal{TR}) \to \mathbb{C}$, such that for any non-zero object $E \in \mathcal{P}(\phi)$ we have $Z(v(E)) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$. We then call Z the "central charge" of the quasi-stability condition $(\mathcal{P}, Z)$.

We need the notion of a category of finite length.

Definition 2.5.9. We define

1. A non-zero object E in an abelian category $\mathcal{A}$ is called simple if for any non-zero subobject F of E one has $F = E$;

2. A "composition series" for $X \in \mathcal{A}$ is a series

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X, n \geq 0$$

   where X_{i+1}/X_i is simple for any $i \in \{0, \ldots, n-1\}$.

Definition 2.5.10. An abelian category $\mathcal{A}$ is of "finite length" if any non-zero $X \in \mathcal{A}$ has a composition series.

Definition 2.5.11. We define

1. An object E in an abelian category $\mathcal{A}$ is said to be "noetherian" if any increasing chain $E_1 \subset E_2 \subset \ldots$ of subobjects of E becomes stationary after finitely many steps. An abelian category $\mathcal{A}$ is noetherian if every object in it is noetherian.

2. An abelian category $\mathcal{A}$ is said to be "artinian" if any infinite chain of subobjects $\cdots \subset E_2 \subset E_1$ becomes stationary.

Lemma 2.5.12. *An abelian category $\mathcal{A}$ is of finite length if and only if it is artinian and noetherian.*

Proof. Assume first that $\mathcal{A}$ is of finite length. First we prove that it is noetherian. For an $E \in \mathcal{A}$ let

$$E_1 \subset E_2 \subset \cdots \subset E$$

be a series of subobjects in $\mathcal{A}$. We have that E has a JHF

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = E$$

where X_{i+1}/X_i simple. Hence $X_1 = X_1/0 = X_1/X_0$ too is simple. Hence, if we consider $E_i \cap X_1 \subset X_1$, then we must have $E_i \cap X_1 = X_1$ or $E_i \cap X_1 = 0$.

1. If there is j such that $E_j \cap X_1 = X_1$, then $X_1 \subset E_j \subset E_{j+1} \subset \cdots \subset E$ and hence $E_j/X_1 \subset E_{j+1}/X_1 \subset \cdots \subset E/X_1$. We set up a prove by induction over n starting with E simple for $n = 1$. Now E/X_1 has JHF of length $n-1$ that hence becomes stationary such that $E_j/X_1 = E_{j+1}/X_1$ which implies $E_j = E_{j+1}$ and so on.

2. If $E_i \cap X_1 = 0$ for all i then $E_i = E_i/(E_i \cap X_1) \cong (E_i + X_1)/X_1 \subset E/X_1$ which has JHF of length $n-1$, similar to before we are finished.

To prove that $\mathcal{A}$ is artinian we use

$$\cdots \subset E_2 \subset E_1 \subset E$$

and, distinguishing $X_1 \cap E_j = 0$ for an existing j and $X_1 \cap E_i = X_1$ for all i we argue similar to the noetherian case that $\mathcal{A}$ is artinian.

Assume now that $\mathcal{A}$ is artinian and noetherian. Any $E_1 \in \mathcal{A}$ has a subobject E_2 such that E_1/E_2 is simple. Otherwise we would obtain that E_1/E_2 is not simple for any choice of E_2 which would mean that there is an object E_3 in $\mathcal{A}$ such that $0 \subsetneq E_3 \subset E_1/E_2$. Hence there is a non-zero $F \in \mathcal{A}$ such that we have $E_3 = F/E_2$. We obtain $0 = E_2/E_2 \subsetneq F/E_2 \subset E_1/E_2$ which implies $E_2 \subsetneq F \subsetneq E_1$. We could now find an object G with $F \subsetneq G \subsetneq E_1$ in the same manner and keep on repeating this process to obtain an infinite chain. This violates the ascending chain condition which implies that it is always possible to choose an object E_2 such that E_1/E_2 is simple. For E_2 we can now choose a subobject E_3 such that E_2/E_3 is simple. We obtain a descending chain of subobjects that will – due to the descending chain condition – become stationary after finitely many steps.

□

Definition 2.5.13. For an interval $I \subset \mathbb{R}$ let $\mathcal{P}(I)$ be the extension closed subcategory of $\mathcal{TR}$ that is generated by all subcategories $\mathcal{P}(\phi)$ for which $\phi \in I$.

Definition 2.5.14. A Slicing $\mathcal{P}$ of a triangulated category $\mathcal{TR}$ is called "locally finite" if there is a real number $\nu > 0$ such that for all $t \in \mathbb{R}$ the category $\mathcal{P}(t-\nu, t+\nu) \in \mathcal{TR}$ is of finite length.

Remark 2.5.15. Note that $\mathcal{P}(t-\nu, t+\nu)$ of definition 2.5.14 is not generally abelian, but "quasi-abelian" which is a slightly weaker concept. In the theoretical framework of quasi-abelian categories one would have to restate definition 2.5.11 using the term "strict subobject". This derives from the fact, that the concept of strictness as a feature of morphisms plays an important role in the context of quasi-abelian categories, in so far as while in an abelian category every morphism is strict that is not necessarily the case in a quasi-abelian category. Strictness of a morphism means, as we already pointed at in the proof of theorem 2.5.33 that the canonical map $\zeta : \operatorname{coim} f \to \operatorname{im} f$ between the coimage and the image of a morphism f is an isomorphism. We will however omit this terminology and refer to [57] for more information on quasi-abelian categories.

Definition 2.5.16. A quasi-stability condition $(\mathcal{P}, Z)$ will be called locally finite if the corresponding slicing $\mathcal{P}$ is.

Finally we approach the definition of the concept of a "stability condition" which nowadays usually includes the stability condition to fulfil the support property which we will define later (definition 2.5.44).

Definition 2.5.17. A quasi-stability condition will be referred to as a "pre-stability condition" if it is locally finite.

Notation 2.5.18. Objects $E \in \mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ will be referred to as "semistable". If additionally E simple in $\mathcal{P}$ then E will be referred to as "stable".

Remark 2.5.19. Throughout this thesis we will work with $\Lambda = \mathcal{N}(\mathcal{TR})$, since the triangulated category that we are working with is numerically finite. We will – in abuse of notation – write $Z(E)$ instead of $Z(v(E))$ from now on.

We will now explain the connection between stability functions on $\mathcal{A}$ and pre-stability conditions on $\mathcal{D}$. To provide this important connection we need the notion of a bounded t-structure. Beilinson, Bernstein and Delingne introduced t-structures in [10] and they have since been subject to many research articles.

Definition 2.5.20. A "t-structure" on a triangulated category $\mathcal{TR}$ is a pair of strictly full subcategories $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ of $\mathcal{TR}$ such that for $\mathcal{TR}^{\leq n} = \mathcal{TR}^{\leq 0}[-n]$ and $\mathcal{TR}^{\geq n+1} = \mathcal{TR}^{\geq 1}[-n], n \in \mathbb{Z}$ the following conditions hold:

1. For $X \in \mathcal{TR}^{\leq 0}$ and $Y \in \mathcal{TR}^{\geq 1}$, $\mathrm{Hom}_{\mathcal{TR}}(X, Y) = 0$;
2. $\mathcal{TR}^{\leq 0} \subset \mathcal{TR}^{\leq 1}$ and $\mathcal{TR}^{\geq 1} \subset \mathcal{TR}^{\geq 0}$;
3. For each $X \in \mathcal{TR}$ there is an exact triangle

$$A \to X \to B \xrightarrow{+}$$

where $A \in \mathcal{TR}^{\leq 0}$ and $B \in \mathcal{TR}^{\geq 1}$.

Lemma 2.5.21. *Let $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ be an arbitrary t-structure. The inclusion $i_{\mathcal{TR}^{\leq n}} : \mathcal{TR}^{\leq n} \to \mathcal{TR}$ admits a right adjoint functor $\tau_{\leq n}$ and the inclusion $i_{\mathcal{TR}^{\geq n}} : \mathcal{TR}^{\geq n} \to \mathcal{TR}$ admits a left adjoint functor $\tau_{\geq n}$. For any $E \in \mathcal{TR}$ there is an exact triangle*

$$\tau_{\leq 0}E \to E \to \tau_{\geq 1}E \xrightarrow{+} . \tag{2.6}$$

Proof. See [10, Proposition 1.3.3]. □

Lemma 2.5.22. *Let $E \in \mathcal{TR}$, then $(\tau_{\leq n+m}E)[m] = \tau_{\leq n}(E[m])$ and additionally $(\tau_{\geq n+m}E)[m] = \tau_{\geq n}(E[m])$.*

Proof. Let $X \in \mathcal{TR}^{\leq n}$. We have

$$\mathrm{Hom}(\tau_{\leq n+m}E)[m], X) = \mathrm{Hom}(\tau_{\leq n+m}E), X[-m]) =$$
$$\mathrm{Hom}(E, i_{\mathcal{TR}^{n+m}}(X[-m])) = \mathrm{Hom}(E, X[-m]) = \mathrm{Hom}(E[m], X),$$

providing $(\tau_{\leq n+m}E)[m] = \tau_{\leq n}(E[m])$. Similarly one proves that $(\tau_{\geq n+m}E)[m] = \tau_{\geq n}(E[m])$. □

Lemma 2.5.23. *For any $E \in \mathcal{TR}$ there is an exact triangle*

$$\tau_{\leq n}E \to E \to \tau_{\geq n+1}E \xrightarrow{+} . \tag{2.7}$$

Proof. From lemma 2.5.21 we obtain that for $E \in \mathcal{TR}$, $E[n]$ fits into the exact triangle

$$\tau_{\leq 0}(E[n]) \to E[n] \to \tau_{\geq 1}(E[n]) \xrightarrow{+}$$

which, by lemma 2.5.22, equals to

$$(\tau_{\leq n}E)[n] \to E[n] \to (\tau_{\geq n+1}E)[n] \xrightarrow{+}$$

Applying $[-n]$ to this we obtain the exact triangle

$$\tau_{\leq n}E \to E \to \tau_{\geq n+1}E \xrightarrow{+},$$

which finishes the proof. □

Lemma 2.5.24. *If* $\mathrm{Hom}(Y,T)=0$ *for all* $T\in\mathcal{TR}^{\geq n+1}$ *then* $Y\in\mathcal{TR}^{\leq n}$. *If* $\mathrm{Hom}(T',X)=0$ *for all* $T'\in\mathcal{TR}^{\leq m}$ *then* $X\in\mathcal{TR}^{\geq m+1}$.

Proof. From $\mathrm{Hom}(Y,T)=0$ for any $T\in\mathcal{TR}^{\geq n+1}$, considering that by definition $\tau_{\geq n+1}Y\in\mathcal{TR}^{\geq n+1}$, we obtain that $\mathrm{Hom}(Y,\tau_{\geq n+1}Y)=0$. Since $\tau_{\leq n}Y[1]\in\mathcal{TR}^{\leq n}$ we additionally obtain that $\mathrm{Hom}((\tau_{\leq n}Y)[1],\tau_{\geq n+1}Y)=0$. Now using the exact triangle (2.7) from 2.5.23 we obtain the exact sequence

$$\mathrm{Hom}((\tau_{\leq n}Y)[1],\tau_{\geq n+1}Y)\to\mathrm{Hom}(\tau_{\geq n+1}Y,\tau_{\geq n+1}Y)\to\mathrm{Hom}(Y,\tau_{\geq n+1}Y)$$

implying $\mathrm{Hom}(\tau_{\geq n+1}Y,\tau_{\geq n+1}Y)=0$ and therefore $\tau_{\geq n+1}Y=0$. The sequence $\tau_{\leq n}Y\to Y\to\tau_{\geq n+1}Y$ must therefore equal to $\tau_{\leq n}Y\to Y\to 0$, implying $\tau_{\leq n}Y\cong Y$ which is what we wanted to prove. Similarly one obtains that if $\mathrm{Hom}(T',X)=0$ for all $T'\in\mathcal{TR}^{\leq m}$ then $X\in\mathcal{TR}^{\geq m+1}$. □

Lemma 2.5.25. *The categories* $\mathcal{TR}^{\leq n}$ *and* $\mathcal{TR}^{\geq m}$ *are extension closed.*

Proof. Let $X\to Y\to Z\xrightarrow{+}$ be an exact triangle. Let $X,Z\in\mathcal{TR}^{\leq n}$, then $\mathrm{Hom}(X,T)=\mathrm{Hom}(Z,T)=0$ for all $T\in\mathcal{TR}^{\geq 1}$. Hence we obtain from the exact sequence

$$\mathrm{Hom}(Z,T)\to\mathrm{Hom}(Y,T)\to\mathrm{Hom}(X,T)$$

that $\mathrm{Hom}(Y,T)=0$. By lemma 2.5.24 this implies $Y\in\mathcal{TR}^{\leq n}$. This means that $\mathcal{TR}^{\leq n}$ is extension closed. Similarly one obtains $\mathcal{TR}^{\geq m}$ extension closed. □

For the particular topic that we are interested in throughout this thesis we will need a slightly stronger term.

Definition 2.5.26. A t-structure is "bounded" if $\cup_{n\in\mathbb{Z}}\mathcal{TR}^{\leq n}=\mathcal{TR}$ and at the same time $\cup_{n\in\mathbb{Z}}\mathcal{TR}^{\geq n}=\mathcal{TR}$.

We also need the notion of a heart of a t-structure.

Definition 2.5.27. The heart of a t-structure $(\mathcal{TR}^{\leq 0},\mathcal{TR}^{\geq 1})$ on a triangulated category $\mathcal{TR}$ is the category $\mathcal{TR}^{\leq 0}\cap\mathcal{TR}^{\geq 0}$.

There is a first – immediate – implication of this definition which will be useful at a later stage.

Lemma 2.5.28. *Let* H *be the heart of a t-structure on* $\mathcal{TR}$. *Then* H *is closed under isomorphisms in* $\mathcal{TR}$.

Proof. This is implied by the strictness of the inclusion of the subcategories out of which H is obtained (see definition 2.5.20) □

In the following we will prove that the heart of a t-structure is actually an abelian category. This fact was proved by Beilinson, Bernstein and Deligne in [10]. The proof is broken down into the following series of lemmas.

Lemma 2.5.29. *Let H be the heart of a t-structure on $\mathcal{TR}$. Then we have* $\mathrm{Hom}^{<0}(E,F)=0$ *for $E,F\in H$.*

Proof. By definition $H=\mathcal{TR}^{\leq 0}\cap\mathcal{TR}^{\geq 0}$ for a t-structure $\mathcal{TR}^{\leq 0},\mathcal{TR}^{\geq 1}$ on $\mathcal{TR}$. Then for $E,F\in H$ we have $E\in\mathcal{TR}^{\leq 0}\cap\mathcal{TR}^{\geq 0}\subset\mathcal{TR}^{\leq 0}$ and $F\in\mathcal{TR}^{\leq 0}\cap\mathcal{TR}^{\geq 0}\subset\mathcal{TR}^{\geq 0}$. Hence $F[-m]\in\mathcal{TR}^{\geq m}\subset\mathcal{TR}^{\geq 1}$ for any positive integer m since $\mathcal{TR}^{\geq m}\subset\mathcal{TR}^{\geq m-1}\subset\ldots\subset\mathcal{TR}^{\geq 1}$. That means $\mathrm{Hom}^{-m}(E,F)=\mathrm{Hom}(E,F[-m])=0$ as $\mathrm{Hom}(\mathcal{TR}^{\leq 0},\mathcal{TR}^{\geq 1})=0$. We obtain $\mathrm{Hom}^{<0}(E,F)=0$. □

Lemma 2.5.30. *Let $a\leq b$ and $X\in\mathcal{TR}$. Then $\tau_{\geq a}\tau_{\leq b}X=\tau_{\leq b}\tau_{\geq a}X$.*

Proof. See [10, Proposition 1.3.5]. □

Remark 2.5.31. Note that the condition $a\leq b$ of lemma 2.5.30 is not necessary for the statement to hold true. This is a somewhat unusual way of stressing that only with the condition $a\leq b$ holding, one actually obtains an interesting result – in cases where the condition does not hold, both compositions of the two functors are just zero and hence trivially equal to each other.

Lemma 2.5.32. *If $X\in\mathcal{TR}^{\leq n}$ then $\tau_{\geq m}X\in\mathcal{TR}^{\leq n}$. If $X\in\mathcal{TR}^{\geq n}$ then $\tau_{\leq m}X\in\mathcal{TR}^{\geq n}$.*

Proof. If $X\in\mathcal{TR}^{\leq n}$ then $X\cong\tau_{\leq n}X$. Hence, using lemma 2.5.30, we obtain

$$\tau_{\geq m}X\cong\tau_{\geq m}\tau_{\leq n}X=\tau_{\leq n}\tau_{\geq m}X.$$

This means that $\tau_{\geq m}X\cong\tau_{\leq n}\tau_{\geq m}X$ implying $\tau_{\geq m}X\in\mathcal{TR}^{\leq n}$. Similarly one proves that $\tau_{\leq m}X\in\mathcal{TR}^{\geq n}$ for $X\in\mathcal{TR}^{\geq n}$. □

With the preparations we have made up to this point we are now able to prove the following theorem that has been introduced by Beilinson, Bernstein and Deligne.

Theorem 2.5.33. *The heart H of a t-structure $(\mathcal{TR}^{\leq 0},\mathcal{TR}^{\geq 1})$ is an abelian category.*

Proof. [10, Theorem 1.3.6] Let $f : A \to B$ be a morphism in H. We will show that it has a kernel and a cokernel in $\mathcal{A}$. For the morphism f there is an object $S \in \mathcal{TR}$ such that $A \xrightarrow{f} B \to S \xrightarrow{+}$ is an exact triangle. We have $S \in \mathcal{TR}^{\leq 0} \cap \mathcal{TR}^{\geq -1}$ by lemma 2.5.25. Let $C = \tau_{\geq 0}S$, then, since by lemma 2.5.32 we have $C = \tau_{\geq 0}S \in \mathcal{TR}^{\leq 0}$, we obtain $C \in \mathcal{A}$. We compose $B \to S$ from the exact triangle $A \xrightarrow{f} B \to S \xrightarrow{+}$ with the canonical map $S \to \tau_{\geq 0}S$ which equals to $S \to C$ and obtain a map $B \to C$. Let $X \in H$ and consider the exact sequence

$$\mathrm{Hom}(A[1], X) \to \mathrm{Hom}(S, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X).$$

We have $\mathrm{Hom}(A[1], X) = \mathrm{Hom}(A, X[-1]) = 0$ due to lemma 2.5.29 and additionally $\mathrm{Hom}(S, X) = \mathrm{Hom}(S, i_{\mathcal{TR}^{\geq 0}}X) = \mathrm{Hom}(\tau_{\geq 0}S, X) = \mathrm{Hom}(C, X)$ which combined provides us with the exact sequence

$$0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X).$$

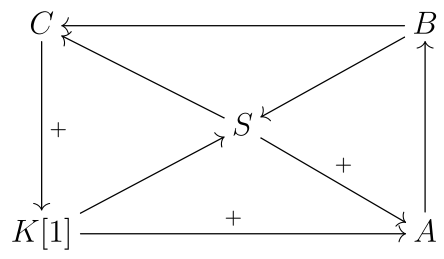
This proves that $B \to C$ is the cokernel of f. Similarly we define $K = (\tau_{\leq -1}S)[-1])$ which, again by lemma 2.5.25 and lemma 2.5.32 is in H. As before we obtain a morphism $K \to A$ this time via the exact triangle $S[-1] \to A \to B \xrightarrow{+}$. Applying lemma 2.5.22 to see that, on one hand we have $\mathrm{Hom}(X, S[-1]) = \mathrm{Hom}(X, (\tau_{\leq -1}S)[-1])$ and lemma 2.5.29 to see that on the other, we have $\mathrm{Hom}(X, B[-1]) = 0$ we obtain that the sequence

$$\mathrm{Hom}(X, B[-1]) \to \mathrm{Hom}(X, S[-1]) \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B),$$

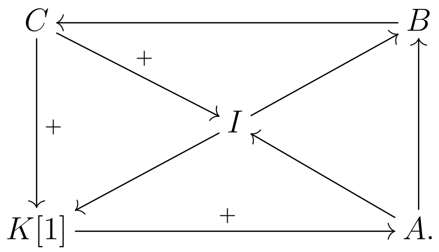
equals to

$$0 \to \mathrm{Hom}(X, K) \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B).$$

Hence $K \to A$ is the kernel of f. To complete the proof we need to show that f is strict – this means that the canonical morphism between the coimage and the image of f is an isomorphism. Consider the following diagram in which the left and the right triangle are exact triangles and the upper and the lower triangle commutative diagrams:



By the octahedron axiom for triangulated categories we obtain an object $I \in \mathcal{TR}$ and the following diagram in which the left and the right triangle are commutative diagrams while the upper and the lower triangle are exact triangles:



This means that we obtain the exact triangles

$$I \to B \to C \xrightarrow{+} \tag{2.8}$$

and

$$K \to A \to I \xrightarrow{+} . \tag{2.9}$$

Applying similar techniques to the ones we used before we see that we now obtain that $\mathrm{coim}(f) = \mathrm{coker}(K \to A) = \tau_{\geq 0}(I)$ and moreover that at the same time we have $\mathrm{im}(f) = \ker(B \to C) = (\tau_{\leq -1} I[1])[-1] = \tau_{\leq 0} I$. From (2.9) we obtain the exact triangle $A \to I \to K[1] \xrightarrow{+}$ and as $A \in \mathcal{TR}^{\leq 0}$ and $K[1] \in \mathcal{TR}^{\leq -1} \subset \mathcal{TR}^{\leq 0}$ we have $I \in \mathcal{TR}^{\leq 0}$ by lemma 2.5.25. Similarly, combining (2.8) with lemma 2.5.25 we obtain $I \in \mathcal{TR}^{\geq 0}$. Hence $\mathrm{im}(f) = \tau_{\leq 0} I \cong \tau_{\geq 0} = \mathrm{coim}(f)$. Then we obtain $f = (I \to B) \circ \mathrm{id}_I \circ (A \to I)$. Let $\widetilde{f}$ be the morphism $\mathrm{coim}\, f \to \mathrm{im}\, f$ induced by f. We obtain $(\tau_{\leq 0} I \xrightarrow{\cong} I) \circ \widetilde{f} \circ (I \xrightarrow{\cong} \tau_{\geq 0} I) = \mathrm{id}_I$ and hence $\widetilde{f}$ has to be an isomorphism which proves that f is strict. □

The motivating example for t-structures is the standard t-structure on the bounded derived category of an abelian category. It is straightforward, that the following definition fulfils the conditions of definition 2.5.20. The t-structure is moreover bounded (definition 2.5.26).

Definition 2.5.34. For a bounded derived category $\mathcal{D}$ of an abelian category $\mathcal{A}$, we define the "standard t-structure" by letting $\mathcal{D}^{\leq 0}$ be the full subcategory of $\mathcal{D}$ such that $obj(\mathcal{D}^{\leq 0}) = \{E \in obj(\mathcal{D}) \mid H^i(E) = 0, i > 0\}$ and $\mathcal{D}^{\geq 0}$ be the full subcategory of $\mathcal{D}$ such that $obj(\mathcal{D}^{\geq 0}) = \{E \in obj(\mathcal{D}) \mid H^i(E) = 0, i < 0\}$ where H^i is the standard cohomology.

Example 2.5.35. *The heart of the standard t-structure of $\mathcal{D}$ is the underlying abelian category $\mathcal{A}$ itself. Therefore, in the context of t-structures, $\mathcal{A}$ is sometimes referred to as the "standard heart".*

In [18], Bridgeland proved the following crucial lemma, that provides the connection between stability functions and pre-stability conditions as defined in 2.5.17.

Lemma 2.5.36. *To give a pre-stability condition on a triangulated category $\mathcal{TR}$ is equivalent to giving a bounded t-structure on $\mathcal{TR}$ and a stability function on its heart with the Harder-Narashiman property.*

Proof. See [18, Proposition 5.3]. □

We can now produce the standard example of a pre-stability condition in the language of t-structures and stability functions. It was introduced by Bridgeland in [18].

Example 2.5.37. *Let $\mathcal{A}$ be the abelian category of coherent $\mathcal{O}_C$-modules on a non-singular projective curve C over an algebraically closed field k of characteristic zero. Let a stability function $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ given by*

$$Z(E) = -\deg(E) + i\,\mathrm{rank}(E)$$

as in example 2.4.5. Then $(Z, \mathcal{A})$ is a pre-stability condition on $\mathcal{D}^b(\mathcal{A})$. (see [18, Example 5.4]).

The stability condition from example 2.5.37 will play an important role in later investigations. Therefore we introduce the following definition.

Definition 2.5.38. The stability function Z from example 2.5.37 will be referred to as Z_μ. The resulting pre-stability condition $(Z_\mu, \mathcal{A})$ will be referred to as σ_μ.

Lemma 2.5.36 provides the connection between pre-stability conditions and stability functions. However, as a tool to study stability conditions, the relation between stability conditions and t-structures is much more essential. This is very important, as the approach to find new stability conditions may consist in finding new t-structures using lemma 2.5.36. In fact we will, throughout this thesis, often make use of the observation provided by lemma 2.5.36.

Having introduced stability conditions as hearts of t-structures with stability functions on them we can extend our definition of ϕ.

Definition 2.5.39. Let $\sigma = (Z, \mathcal{H})$ where $\mathcal{H}$ is the heart of a bounded t-structure on $\mathcal{D}$ and Z a stability function on $\mathcal{H}$, then we define for $E \in \mathcal{H}$

$$\phi_\sigma(E) = \arg(Z(E)).$$

Lemma 2.5.40. *If σ as in definition 2.5.39, $\sigma = (\mathcal{P}, Z)$ and E is semistable with regard to σ then definitions 2.5.39 and 2.5.4 coincide.*

Proof. This is due to the compatibility between the central Z, that the stability function extends to and $\mathcal{P}$. □

The connection between pairs of slicings and central charges on one hand and t-structures and stability functions on the other hand provided by lemma 2.5.36 also allows us to extend concepts from one set of data to the other. We can now link our concept of a standard t-structure into that of a slicing.

Definition 2.5.41. We define a "standard slicing" on $\mathcal{D}$, the bounded derived category of an abelian category $\mathcal{A}$, to be a slicing $\mathcal{P}$ for which we have $\mathcal{P}(0, 1] = \mathcal{A}$.

The connection is now provided by the following remark.

Remark 2.5.42. Note that if $\mathcal{P}$ is the slicing of a pre-stability condition, for which the corresponding t-structure is the standard t-structure defined in 2.5.34, then $\mathcal{P}$ is a standard slicing. In other words, $\mathcal{P}(0, 1]$ is the standard heart of example 2.5.35.

Definition 2.5.43. A pre-stability condition $\sigma = (Z, H)$ satisfies the "support property" if there is a symmetric bilinear form Q on $\Lambda \otimes \mathbb{R} = \Lambda_{\mathbb{R}}$ such that

1. All σ-semistable objects $E \in H$ satisfy $Q(v(E), v(E)) \geq 0$.

2. All non zero vectors $v \in \Lambda_{\mathbb{R}}$ with $Z(v) = 0$ satisfy $Q(v, v) < 0$.

Now we can conclude this section by providing the definition of a stability condition.

Definition 2.5.44. A pre-stability condition $\sigma = (Z, H)$ on a triangulated category $\mathcal{TR}$ that satisfies the support property is called a "stability condition".

Remark 2.5.45. Note that stability conditions as defined in 2.5.44 are also referred to as "Bridgeland stability conditions".

The following lemma provides that the conditions for the support property can be relaxed for its purposes.

Lemma 2.5.46. *It suffices to assume all σ-stable objects $E \in H$ satisfy $Q(v(E), v(E)) \geq 0$ in definition 2.5.43.*

Proof. See [8, Lemma 11.6]. □

Definition 2.5.47. A pre-stability condition is discrete if the image of Z is a discrete subgroup of $\mathbb{C}$.

Finally we have the following important features.

Definition 2.5.48. Define $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ to be the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$.

Remark 2.5.49. In [18] Bridgeland explained that $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ should be thought of as pairs (T, f) where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving linear isomorphism, such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z}$ coincide. More concretely we can say

$$M \exp(i\pi f(t)) \in \mathbb{R}_{>0} \exp(i\pi t)$$

where – in slight abuse of notation – identify the number $\exp(i\pi f(t)) \in \mathbb{C}$ with the vector $(\cos(i\pi f(t)), \sin(i\pi f(t))) \in \mathbb{R}^2$ such that we can multiply it by $M = T^{-1}$.

Lemma 2.5.50. *There is a (right) action of* $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ *on* $\operatorname{pre}\operatorname{Stab}(\mathcal{TR})$ *given by* $\sigma g = (T^{-1} \circ Z, \mathcal{P}(f(\phi)))$*, where* $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(\mathcal{TR})$ *and* $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$*.*

Proof. See [18, lemma 8.2]. □

Using the language we have just introduced, Macrí has – generalising a result by Bridgeland ([18, theorem 9.1]) – proved the following crucial result on the stability space of $\mathcal{D}^b(\mathrm{Coh}(C))$. A lot of the content of this thesis is based on it.

Theorem 2.5.51. *Let* $\mathcal{A} = \mathrm{Coh}(C)$ *where* C *is a smooth projective curve, then the action of* $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ *is free and transitive so that*

$$\operatorname{Stab}(\mathcal{D}) \cong \widetilde{\mathrm{GL}}_2^+(\mathbb{R}).$$

Proof. See [18, Theorem 9.1] and [45, Theorem 2.7]. □

Remark 2.5.52. Note that theorem 2.5.51 implies that for $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve and any $\sigma \in \operatorname{Stab}(\mathcal{D})$ the semistable objects are simply (shifts of) the μ-stable objects in $\mathcal{A}$, as $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ in this situation merely changes the numbering of the slices.

Definition 2.5.53. Define $\operatorname{Aut}(\mathcal{TR})$ to be the group of exact autoequivalences of $\mathcal{TR}$.

Lemma 2.5.54. *There is a (left) action by isometries of the group* $\mathrm{Aut}(\mathcal{TR})$ *on* $\mathrm{pre\,Stab}(\mathcal{TR})$*, given by* $\Phi(\sigma) = (\Phi(\mathcal{P}), Z \circ \Phi^{-1})$*, where* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{TR})$ *and* $\Phi \in \mathrm{Aut}(\mathcal{TR})$*. These two actions commute.*

Proof. See [18, Lemma 8.2]. □

The connection of both actions with the support property is crucial.

Lemma 2.5.55. *Let* $\sigma \in \mathrm{Stab}(\mathcal{TR}), g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ *and* $\Phi \in \mathrm{Aut}(\mathcal{TR})$*, then*

$$\Phi\sigma g \in \mathrm{Stab}(\mathcal{TR})$$

Proof. Let Q be a quadratic form such that σ satisfies the support property with respect to Q. The result now follows from the definition of the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ given in lemma 2.5.50 and $\Phi \in \mathrm{Aut}(\mathcal{TR})$ given in lemma 2.5.54. Then, σg satisfies the support property with respect to the quadratic form Q and $\Phi\sigma$ with respect to $Q \circ \Phi^{-1}$. □

2.6 Topology of Stab

Previously we have established the set of stability conditions on $\mathcal{TR}$. We can, however, say more – in [18], Bridgeland has defined a generalised metric on the set of stability conditions on $\mathcal{TR}$, that gives this set the structure of a topological space and that we will now introduce. In doing this, we will in fact only look at those stability conditions that fulfil a certain criterion called locally finiteness, which explains definition 2.5.17.

And hence we can define the stability space of a triangulated category as

Notation 2.6.1. By $\mathrm{Stab}(\mathcal{TR})$ we denote the set of locally finite stability conditions on the triangulated category $\mathcal{TR}$.

Remark 2.6.2. Note that the locally finiteness is a requirement for the proof of corollary 2.6.8 and will therefore give the stability space a somewhat nicer structure.

By defining a topology on the set of stability conditions on a triangulated category $\mathcal{TR}$ one obtains a topological space – this is what we call the stability space of $\mathcal{TR}$. This topology is induced by the generalised metric that will be defined next. Prior to this we will clarify what we mean by a generalised metric as there does not seem to be a common definition – the ways different authors define generalised metrics differ quite significantly in fact.

Definition 2.6.3. A "generalised metric" is a map $d(-,-) : S \to [0, \infty]$, defined on a set S, with the properties:

1. $d(\sigma_1, \sigma_2) = 0$ if and only if $\sigma_1 = \sigma_2$;
2. $d(\sigma_1, \sigma_2) = d(\sigma_2, \sigma_1)$;
3. $d(\sigma_1, \sigma_3) \leq d(\sigma_1, \sigma_2) + d(\sigma_2, \sigma_3)$

for any $\sigma_1, \sigma_2, \sigma_3 \in S$.

Remark 2.6.4. Note that other authors do not necessarily assume $d(\sigma_1, \sigma_2) = 0$ implies $\sigma_1 = \sigma_2$ and the second axiom of 2.6.3.

Definition 2.6.5. For stability conditions $\sigma_1 = (Z_1, \mathcal{P}_1)$ and $\sigma_2 = (Z_2, \mathcal{P}_2)$ we denote by $\phi^+_{\sigma_i}$ and $\phi^-_{\sigma_i}$ the ϕ^+ and ϕ^- defined in 2.5.2 of an $E \in \mathcal{TR}$ for the particular slicing $\mathcal{P}_i$ of σ_i. Similarly we denote by m_{σ_i} the m defined in 2.5.17 for the central charge Z_i of σ_i. Define

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{TR}} \{|\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|, |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|, |\ln(\frac{m_{\phi_{\sigma_2}}}{m_{\phi_{\sigma_1}}})|\}.$$

Lemma 2.6.6. *The mapping* $d(-,-) : \mathrm{Stab}(\mathcal{TR}) \times \mathrm{Stab}(\mathcal{TR}) \to \mathbb{R}_{\geq 0} \subset \mathbb{R}$ *defined in 2.6.5 defines a generalised metric on* $\mathrm{Stab}(\mathcal{TR})$.

Proof. To prove the first axiom of definition 2.6.3, on observes that

$$d(\sigma_1, \sigma_1) = \sup_{0 \neq E \in \mathcal{TR}} \{|\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_1}(E)|, |\ln(\frac{m_{\phi_{\sigma_1}}}{m_{\phi_{\sigma_1}}})|\} = 0.$$

for any $\sigma_1 \in \mathrm{Stab}(\mathcal{TR})$. To prove that $d(\sigma_1, \sigma_2) = 0$ implies $\sigma_1 = \sigma_2$ one notes that $d(\sigma_1, \sigma_2) = 0$ implies $\phi^-_{\sigma_1}(E) = \phi^-_{\sigma_2}(E)$ and $\phi^+_{\sigma_1}(E) = \phi^+_{\sigma_2}(E)$. Hence an object is semistable with regard to σ_1 if and only if it is semistable with regard to σ_2. This means that $\mathcal{P}_1 = \mathcal{P}_2$. Since $d(\sigma_1, \sigma_2)$ additionally implies that $m_{\sigma_1}(E) = m_{\sigma_2}(E)$ for any $E \in \mathcal{TR}$, one concludes that the central charges of σ_1 and of σ_2 agree on the semistable objects. The semistable objects, on the other hand, generate $\mathcal{N}(\mathcal{TR})$ and hence $Z_1 = Z_2$ due to the fact that the central charge is a homomorphism of groups. We conclude that $\sigma_1 = \sigma_2$.

Is is directly due to the properties of the absolute value, that the second axiom holds. The third axiom follows from the triangle inequality of $\mathbb{R}$.

□

There is more about the topological features of $\mathrm{Stab}(\mathcal{TR})$. Bridgeland proved a crucial fact, which allows us to formulate the following theorem and a direct implication. Besides Bridgeland's paper [18] from which this is taken, we refer to [5, Section 5.5] for a sketch of the proof.

Theorem 2.6.7. *Let $\mathcal{TR}$ be a triangulated category. For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{TR})$ there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}(\Lambda, \mathbb{C})$ with a well-defined linear topology and a local homeomorphism $\mathcal{Z}$, given by*

$$\mathcal{Z} : \Sigma \to V(\Sigma)$$
$$\mathcal{Z}(\sigma) = Z,$$

where $\sigma = (\mathcal{P}, Z)$.

Proof. See [18, Theorem 1.2]. □

Corollary 2.6.8. *Each connected component of the stability space* $\mathrm{Stab}(\mathcal{TR})$ *of a triangulated category $\mathcal{TR}$ is a complex manifold.*

Proof. This is an implication of 2.6.7 – see [18, Section 1]. □

The aim in dealing with stability conditions is, hence, to reach a full description of the stability space of the particular triangulated category one is interested in. This provides an interesting invariant and therefore bares information about the triangulated category itself. The particular example that we are interested in is to compute the stability space of the bounded derived category $\mathcal{D}^{\uparrow}$ of the arrow category $\mathcal{A}^{\uparrow}$ of an abelian category $\mathcal{A}$. More precisely are we interested in the stability space of the triangulated category $\mathcal{D}^b(\mathrm{Coh}(C)^{\uparrow})$, it will be our aim to research it throughout the following chapters.

3 CP-Gluing

This section aims at the introduction of a technique to compute stability conditions. At a pre-Serre functor level (see section 4) we can compute stability conditions using the technique of CP-gluing and hence obtain pre-stability conditions. The CP-gluing-technique, introduced by Collins and Polishchuk (CP) in [21], can be thought of as a special case of recollement which will be introduced in the subsequent section. There is, however, a fundamental difference – to be able to set up recollement in our case one requires the existence of a Serre functor, which is a crucial concept that simplifies many problems. However, this section develops the theory entirely without it. The section is joint work with Eva Martínez and Alejandra Rincón.

3.1 The CP-gluing-technique

This chapter provides the background theory regarding the technique of CP-gluing t-structures on a triangulated category from t-structures of certain triangulated subcategories that sit particularly nicely in the triangulated category one started with. Lemma 3.1.4 will clarify what this means.

Lemma 3.1.1. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$. The exact triangle*

$$i_2(E_2) \to E \to i_1(E_1) \overset{+}{\to}$$

of definition 2.1.6 is unique up to isomorphisms of exact triangles.

Proof. For $E \in \mathcal{TR}$ consider exact triangles

$$i_2(E_2) \overset{u}{\to} E \overset{v}{\to} i_1(E_1) \overset{+}{\to}$$

and

$$i_2(E_2') \overset{u'}{\to} E \overset{v'}{\to} i_1(E_1') \overset{+}{\to}$$

with $E_1, E_1' \in \mathcal{TR}^1$ and $E_2, E_2' \in \mathcal{TR}^2$. This gives

$$v' \circ \mathrm{id}_E \circ u \in \mathrm{Hom}(i_2(E_2), i_1(E_1')) \subset \mathrm{Hom}(i_2(\mathcal{TR}^2), i_1(\mathcal{TR}^1)) = 0$$

by definition 2.1.6. By [32, Lemma 1.6] there are now morphisms f and g such that the diagram

$$\begin{array}{ccccccc} i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \\ \downarrow f & & \downarrow \mathrm{id}_E & & \downarrow g & & \\ i_2(E_2') & \xrightarrow{u'} & E & \xrightarrow{v'} & i_1(E_1') & \xrightarrow{+} & \end{array}$$

is a morphism of triangles. Since we have

$$\mathrm{Hom}(i_2(E_2), i_1(E_1')[-1]) \subset \mathrm{Hom}(\mathcal{TR}^2, \mathcal{TR}^1) = 0, \tag{3.1}$$

again using definition 2.1.6, it is also implied in [32, Lemma 1.6], that f and g are unique. To prove that f and g are indeed isomorphisms, we extend the above diagram to

$$\begin{array}{ccccccc} i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \\ \downarrow f & & \downarrow \mathrm{id}_E & & \downarrow g & & \\ i_2(E_2') & \xrightarrow{u'} & E & \xrightarrow{v'} & i_1(E_1') & \xrightarrow{+} & \\ \downarrow f' & & \downarrow \mathrm{id}_E & & \downarrow g' & & \\ i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \end{array}$$

and obtain a morphism of triangles

$$\begin{array}{ccccccc} i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \\ \downarrow{\scriptstyle f'\circ f} & & \downarrow{\scriptstyle \mathrm{id}_E} & & \downarrow{\scriptstyle g'\circ g} & & \\ i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \end{array}$$

by composing the vertical arrows. On the other hand,

$$\begin{array}{ccccccc} i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \\ \downarrow{\scriptstyle \mathrm{id}_{i_2(E_2)}} & & \downarrow{\scriptstyle \mathrm{id}_E} & & \downarrow{\scriptstyle \mathrm{id}_{i_1(E_1)}} & & \\ i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \end{array}$$

is a morphism of triangles and since, once again combining definition 2.1.6 with [32, Lemma 1.6], the morphism id_E determines the left and right vertical arrow uniquely, we obtain $f' \circ f = \mathrm{id}_{i_2(E_2)}$ and $g' \circ g = \mathrm{id}_{i_1(E_1)}$. Similarly, now using the diagram

$$\begin{array}{ccccccc} i_2(E_2') & \xrightarrow{u'} & E & \xrightarrow{v'} & i_1(E_1') & \xrightarrow{+} & \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \mathrm{id}_E} & & \downarrow{\scriptstyle g'} & & \\ i_2(E_2) & \xrightarrow{u} & E & \xrightarrow{v} & i_1(E_1) & \xrightarrow{+} & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_E} & & \downarrow{\scriptstyle g} & & \\ i_2(E_2') & \xrightarrow{u'} & E & \xrightarrow{v'} & i_1(E_1') & \xrightarrow{+} & \end{array}$$

we see that $f \circ f' = \mathrm{id}_{i_2(E_2)}$ and $g \circ g' = \mathrm{id}_{i_1(E_1)}$. Hence f and g are isomorphisms. This implies that $i_1(E_1)$ and $i_2(E_2)$ are unique up to isomorphism which finishes the proof since i_1 and i_2 are embeddings. $\square$

Remark 3.1.2. We can in fact adapt the proof of lemma 3.1.1 to show that the exact triangle (2.7) is unique. This is obtained in the same way as its existence from the uniqueness of the exact triangle (2.6). The uniqueness of (2.6), however, can be seen in the same way as in the case of the semiorthogonal decomposition. A semiorthogonal decomposition is after all a special t-structure for which the condition to be closed under certain shifts is stricter. However, the proof of lemma 3.1.1 only uses the stability under shift to prove (3.1). Since for a t-structure $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$, we have that $\mathcal{TR}^{\geq 1}$ is – by definition – closed under negative shift however, (3.1) is still true for t-structures.

Lemma 3.1.3. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$. For λ_1 and ρ_2 as in definition 2.1.8, there are identities*

$$\lambda_1 \circ i_1 \cong \mathrm{id} \cong \rho_2 \circ i_2.$$

Proof. Let $E_2 \in \mathcal{TR}^2$. Then

$$i_2(E_2) \to i_2(E_2) \to 0 \xrightarrow{+}$$

is an exact triangle in $\mathcal{TR}$, equal to

$$i_2(E_2) \to i_2(E_2) \to i_1(0) \xrightarrow{+} .$$

Since $E_2 \in \mathcal{TR}^2$ and $0 \in \mathcal{TR}^1$ this is the, by lemma 3.1.1 unique, exact triangle that the semiorthogonal decomposition $\langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ provides for $i_2(E_2)$. Hence $\rho_2(i_2(E_2)) = E_2$ and in the same manner one sees that $\rho_2(i_2(f)) = f$ for a morphism f on E_2. The proof for $\lambda_1 \circ i_1 \cong \mathrm{id}$ is similar. □

Lemma 3.1.4. *Assume that on the triangulated category $\mathcal{TR}$ a semiorthogonal decomposition $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ is given. The functor ρ_2 is the right adjoint to the inclusion $i_2 : \mathcal{TR}^2 \to \mathcal{TR}$, and λ_1 is the left adjoint functor to the inclusion $i_1 : \mathcal{TR}^1 \to \mathcal{TR}$.*

Proof. Let $Y \in \mathcal{TR}^2$ and $X \in \mathcal{TR}$. The semiorthogonal decomposition $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ hence provides us with an exact triangle

$$i_2(X_2) \to X \to i_1(X_1) \xrightarrow{+}$$

where $X_1 \in \mathcal{TR}^1$ and $X_2 \in \mathcal{TR}^2$. Applying the functor $\mathrm{Hom}_{\mathcal{TR}}(i_2(Y), -)$ to this exact triangle provides us with the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}(i_2(Y), i_1(X_1)[-1]) &\to \mathrm{Hom}(i_2(Y), i_2(X_2)) \\ \to \mathrm{Hom}(i_2(Y), X) &\to \mathrm{Hom}(i_2(Y), i_1(X_1)) \to \dots \end{aligned}$$

which equals to

$$0 \to \mathrm{Hom}(i_2(Y), i_2(X_2)) \to \mathrm{Hom}(i_2(Y), X) \to 0$$

since $\langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ is a semiorthogonal decomposition, implying that $\mathrm{Hom}(i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)) = 0$. Therefore we obtain

$$\mathrm{Hom}(Y, \rho_2(X)) = \mathrm{Hom}(Y, X_2) \cong \mathrm{Hom}(i_2(Y), i_2(X_2)) \cong \mathrm{Hom}(i_2(Y), X),$$

where $Hom(Y, X_2) \cong \mathrm{Hom}(i_2(Y), i_2(X_2))$ is provided via the fact that i_2 is full by definition 2.1.6.

The functoriality, using lemma 3.1.3 is seen by

$$\mathrm{Hom}_{\mathcal{TR}}(i_2(Y), X) \xrightarrow{\rho_2} \mathrm{Hom}_{\mathcal{TR}^2}(\rho_2(i_2(Y)), \rho_2(X)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{TR}^2}(Y, \rho_2(X)).$$

The proof for λ_1 is similar. □

We can compute t-structures and hence in particular bounded t-structures – the type that we are interested in – on a triangulated category via the semiorthogonal decomposition by a method called "CP-gluing" which was introduced by Collins and Polishchuk in [21]. The idea of CP-gluing is to combine the data provided by the hearts of bounded t-structures on both components of the semiorthogonal decomposition and compute a heart of a bounded t-structure on the triangulated category $\mathcal{TR}$ in this manner.

Lemma 3.1.5. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ be a semiorthogonal decomposition and H_i the hearts of bounded t-structures on $\mathcal{TR}^i$ for $i = 1, 2$, such that $\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(i_1(H_1), i_2(H_2)) = 0$, then*

$$H = \{X \in \mathcal{TR} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$$

is the heart of a bounded t-structure on $\mathcal{TR}$. We will refer to hearts of this kind as "hearts obtained by CP-gluing".

Proof. For the proof of

$$H = \{X \in \mathcal{TR} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$$

being the heart of a t-structure on $\mathcal{TR}$ see [21, Lemma 2.1]. Note, that also [21, Lemma 2.1] claims that a t-structure obtained in this manner is bounded, a proof is not given, we therefore include a proof, provided in 3.1.7. □

Remark 3.1.6. Note that the t-structure $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ on $\mathcal{TR}$ that corresponds to the heart H of lemma 3.1.5 is then given by

$$\mathcal{TR}^{\leq 0} = \{X \in \mathcal{TR} \mid \lambda_1(X) \in (\mathcal{TR}^1)^{\leq 0}, \rho_2(X) \in (\mathcal{TR}^2)^{\leq 0}\}$$

and

$$\mathcal{TR}^{\geq 1} = \{X \in \mathcal{TR} \mid \lambda_1(X) \in (\mathcal{TR}^1)^{\geq 1}, \rho_2(X) \in (\mathcal{TR}^2)^{\geq 1}\}$$

where $((\mathcal{TR}^i)^{\leq 0}, (\mathcal{TR}^i)^{\geq 1})$ are the t-structures corresponding to the hearts H_i on $\mathcal{TR}^i$.

Lemma 3.1.7. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ be a semiorthogonal decomposition and H_i the hearts of bounded t-structures on $\mathcal{TR}^i$ for $i = 1, 2$. If*

$$H = \{X \in \mathcal{TR} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$$

is the heart of a t-structure on $\mathcal{TR}$, then that t-structure is bounded.

Proof. Recall the definition of boundedness of a t-structure in 2.5.26. Let $E \in \mathcal{TR}$ then the semiorthogonal decomposition provides us with the exact triangle $i_2(\rho_2(E)) \to E \to i_1(\lambda_1(E)) \xrightarrow{+}$. The definition of H implies that if for an object $E \in \mathcal{TR}$ we have $\lambda_1(E) \in (\mathcal{TR}^1)^{\leq 0}$ and $\rho_2(E) \in (\mathcal{TR}^2)^{\leq 0}$, then $E \in \mathcal{TR}^{\leq 0}$, where $\mathcal{TR}^{\leq 0}, (\mathcal{TR}^1)^{\leq 0}$ and $(\mathcal{TR}^2)^{\leq 0}$ are the categories corresponding to H, H_1 and H_2. For any $F \in \mathcal{TR}$ we have $\lambda_1(F) \in \cup_{n\in\mathbb{Z}}(\mathcal{TR}^1)^{\leq n}$ and $\rho_2(F) \in \cup_{n\in\mathbb{Z}}(\mathcal{TR}^2)^{\leq n}$ which implies that $F \in \cup_{n\in\mathbb{Z}}\mathcal{TR}^{\leq n}$ and hence $\cup_{n\in\mathbb{Z}}\mathcal{TR}^{\leq n}$. The proof is finished by repeating the argument for $\mathcal{TR}^{\geq n}$. □

Lemma 3.1.8. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$, be a semiorthogonal decomposition, $E_1 \in \mathcal{TR}^1$ and $\phi : \mathcal{TR}^1 \to \mathcal{TR}^2$ an equivalence of categories. Assume additionally, that ρ_2 has a right adjoint functor Δ and $\Delta\phi$ left adjoint to λ_1 and that Δ is fully faithful. There exist an exact triangle*

$$i_2(\phi(E_1)) \to \Delta(\phi(E_1)) \to i_1(E_1) \xrightarrow{+} .$$

Proof. Since $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ is a semiorthogonal decomposition, any $F \in \mathcal{TR}$ can be embedded into an exact triangle

$$i_2(\rho_2(F)) \to F \to i_1(\lambda_1(F)) \xrightarrow{+} .$$

If we, on the other hand, now let $F = \Delta(\phi(E_1)) \in \mathcal{TR}$, we obtain an exact triangle $i_2(\rho_2(\Delta(\phi(E_1)))) \to \Delta(\phi(E_1)) \to i_1(\lambda_1(\Delta(\phi(E_1)))) \xrightarrow{+}$. As Δ was assumed to be fully faithful, so is $\Delta\phi$ since ϕ is an equivalence of categories. As, additionally, Δ is the right adjoint functor of ρ_2 and $\Delta\phi$ the left adjoint to λ_1, we conclude with [44, Subsection 4.3, Theorem 1] that $\rho_2\Delta \cong \mathrm{id}_{\mathcal{TR}^2}$ and $\mathrm{id}_{\mathcal{TR}^1} \cong \lambda_1\Delta\phi$. Hence we obtain an exact triangle $i_2(\phi(E_1)) \to \Delta(\phi(E_1)) \to i_1(E_1) \xrightarrow{+}$. □

Remark 3.1.9. Subsection 3.2 will see the introduction of an example of such a Δ.

Lemma 3.1.10. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$. For $X \in \mathcal{TR}$ and $i \in \mathbb{Z}$,we have$(\lambda_1(X))[i] = \lambda_1((X)[i])$.*

Proof. For any $E_1 \in \mathcal{TR}^1$ we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{TR}^1}((\lambda_1(X))[i], E_1) = \mathrm{Hom}_{\mathcal{TR}^1}(\lambda_1(X), E_1[-i]) \\
&\cong \mathrm{Hom}_{\mathcal{TR}}(X, i_1(E_1[-i])) = \mathrm{Hom}_{\mathcal{TR}}(X, (i_1(E_1))[-i]) \\
&= \mathrm{Hom}_{\mathcal{TR}}(X[i], i_1(E_1)) = \mathrm{Hom}_{\mathcal{TR}^1}(\lambda_1((X)[i]), E_1)
\end{aligned}$$

since λ_1 is the left adjoint functor to i_1 and i_1 commutes with the shift functor by definition 2.1.6. This finishes the proof as E_1 was freely chosen from $\mathcal{TR}^1$. □

Lemma 3.1.11. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ be a semiorthogonal decomposition, then the composition $\lambda_1 i_2 = 0$.*

Proof. For any $E_1 \in \mathcal{TR}^1, E_2 \in \mathcal{TR}^2$ we obtain

$$\mathrm{Hom}_{\mathcal{TR}^1}(\lambda_1(i_2(E_2)), E_1) = \mathrm{Hom}_{\mathcal{TR}}(i_2(E_2), i_1(E_1)) = 0$$

since λ_1 is the left adjoint to i_1 and $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ is a semiorthogonal decomposition implying that $\mathrm{Hom}(i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)) = 0$. This provides us with the required statement, as E_1, E_2 were chosen freely from $\mathcal{TR}^1$ and $\mathcal{TR}^2$. □

Lemma 3.1.12. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$, then*

$$\mathrm{Hom}^i_{\mathcal{TR}}(\Delta(\phi(E_1)), i_2(E_2)) = 0$$

for any $i \in \mathbb{Z}, E_1 \in \mathcal{TR}^1$ and $E_2 \in \mathcal{TR}^2$.

Proof. Since λ_1 is the right adjoint functor of $\Delta\phi$ we obtain

$$\mathrm{Hom}(\Delta(\phi(E_1)), (i_2(E_2))[i]) = \mathrm{Hom}(E_1, \lambda_1((i_2(E_2))[i])).$$

But λ_1 commutes with the shift functor due to lemma 3.1.10 and hence

$$\begin{aligned}\mathrm{Hom}(\Delta(\phi(E_1)), (i_2(E_2))[i]) &= \mathrm{Hom}(E_1, \lambda_1((i_2(E_2))[i])) \\ &= \mathrm{Hom}(E_1, (\lambda_1(i_2(E_2))[i]) = \mathrm{Hom}(E_1, 0) = 0\end{aligned}$$

as $\lambda_1 i_2 = 0$ due to lemma 3.1.11. □

Theorem 3.1.13. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ be a semiorthogonal decomposition, $E_1 \in \mathcal{TR}^1, E_2 \in \mathcal{TR}^2$ and $\phi : \mathcal{TR}^1 \to \mathcal{TR}^2$ an equivalence of categories. Assume additionally, that there exists a Δ which is the right adjoint functor of ρ_2 and for which $\Delta\phi$ is the left adjoint functor to λ_1 and that Δ is fully faithful. Then*

$$\mathrm{Hom}^i_{\mathcal{TR}^2}(\phi(E_1), E_2) \cong \mathrm{Hom}^{i+1}_{\mathcal{TR}}(i_1(E_1), i_2(E_2))$$

for any $i \in \mathbb{Z}$.

Proof. Due to lemma 3.1.8 there is an exact triangle

$$i_2(\phi(E_1)) \to \Delta(\phi(E_1)) \to i_1(E_1) \stackrel{+}{\to} .$$

If we apply the functor $\mathrm{Hom}(-, i_2(E_2))$ to this triangle we obtain an exact sequence

$$\begin{aligned}\cdots \to \mathrm{Hom}^i_{\mathcal{TR}}(\Delta(\phi(E_1)), i_2(E_2)) &\to \mathrm{Hom}^i_{\mathcal{TR}}(i_2(\phi(E_1)), i_2(E_2)) \to \\ \mathrm{Hom}^{i+1}_{\mathcal{TR}}(i_1(E_1), i_2(E_2)) &\to \mathrm{Hom}^{i+1}_{\mathcal{TR}}(\Delta(\phi(E_1)), i_2(E_2)) \to \dots\end{aligned}$$

and $\mathrm{Hom}^i_{\mathcal{TR}}(\Delta(\phi(E_1)), i_2(E_2)) = \mathrm{Hom}^{i+1}_{\mathcal{TR}}(\Delta(\phi(E_1)), i_2(E_2)) = 0$ due to lemma 3.1.12. This provides us with the exact sequence

$$0 \to \mathrm{Hom}^i_{\mathcal{TR}}(i_2(\phi(E_1)), i_2(E_2)) \to \mathrm{Hom}^{i+1}_{\mathcal{TR}}(i_1(E_1), i_2(E_2)) \to 0.$$

Hence, as i_2 is fully faithful, we have

$$\begin{aligned}\mathrm{Hom}^i_{\mathcal{TR}^2}(\phi(E_1), E_2) &\cong \mathrm{Hom}^i_{\mathcal{TR}}(i_2(\phi(E_1)), i_2(E_2)) \\ &\cong \mathrm{Hom}^{i+1}_{\mathcal{TR}}(i_1(E_1), i_2(E_2))\end{aligned}$$

which finishes the proof. □

The following can be seen as an "almost-inverse" of lemma 3.1.5.

Lemma 3.1.14. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ be a semiorthogonal decomposition and H_i the hearts of bounded t-structures on $\mathcal{TR}^i$ for $i = 1, 2$, such that*

$$H = \{X \in \mathcal{TR} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$$

is the heart of a t-structure on $\mathcal{TR}$, then $\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(i_1(H_1), i_2(H_2)) = 0$.

Proof. Apply lemma 2.5.29. □

In order to obtain the data that we need to define stability conditions on $\mathcal{TR}$ we need the following observation (that has – without proof – been stated in [21, Equation (2.5)]):

Lemma 3.1.15. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ be a semiorthogonal decomposition and H_i the hearts of bounded t-structures on $\mathcal{TR}^i$ for $i = 1, 2$, such that* $\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(i_1(H_1), i_2(H_2)) = 0$, *then*

$$Z(X) = Z_1(\lambda_1(X)) + Z_2(\rho_2(X))$$

defines a stability function on the heart

$$H = \{X \in \mathcal{TR} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}.$$

Proof. The function Z is a homomorphism of groups on $\mathcal{K}(H)$ as every element $X \in \mathcal{K}(H)$ decomposes into a sum of elements of $\mathcal{K}(H_i)$ via the exact triangle $i_2(X_2) \to X \to i_1(X_1) \xrightarrow{+}$. By definition of H we have $X_1 \in H_1$ and $X_2 \in H_2$, since Z_1, Z_2 are stability functions $Z_1(X_1), Z_2(X_2)$ and hence their sum are in $\mathbb{H}$, therefore $Z(X) \in \mathbb{H}$. □

Lemma 3.1.16. *Let $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$ and σ be a stability condition on $\mathcal{TR}$ obtained by CP-gluing stability conditions σ_1 on $\mathcal{TR}_1$ and σ_2 on $\mathcal{TR}_2$ via the semiorthogonal decomposition $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$. Let $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, then for $\sigma g = (W, H)$, $\sigma_1 g = (W_1, H_1)$ and $\sigma_2 g = (W_2, H_2)$ and $d \in \{1, 2\}$ we have*

- $i_d(H_d) \subset H$,
- $W\big|_{i_d(\mathcal{TR}_d)} = W_d$ *and*
- *if* $\mathrm{Hom}^{\leq 0}(i_1(H), i_2(H)) = 0$ *then σg is a stability condition on $\mathcal{TR}$ obtained by CP-gluing stability conditions $\sigma_1 g$ on $\mathcal{TR}_1$ and $\sigma_2 g$ on $\mathcal{TR}_2$ via the semiorthogonal decomposition $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$.*

Proof. For $d \in \{1, 2\}$ let $\mathcal{P}_d$ be the slicing corresponding to σ_d and $\mathcal{P}$ the slicing that corresponds to σ. The action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ works in such a way that we obtain $H_d = \mathcal{P}_d(f(0), f(1)]$ as well as $H = \mathcal{P}_d(f(0), f(1)]$. Therefore we have $i_d(H_d) \subset H$ by [21, Proposition 2.2(3)].

Let Z and Z_d be the stability functions corresponding to σ and σ_d. We have $W = T^{-1} \circ Z$ such that $W(E) = T^{-1} \circ Z_d(E) = W_d(E)$.

Assume now $\mathrm{Hom}^{\leq 0}(i_1(H), i_2(H))$. By [21, Proposition 2.2(1)] σg is a stability condition on $\mathcal{TR}$ obtained by CP-gluing stability conditions $\sigma_1 g$ on $\mathcal{TR}_1$ and $\sigma_2 g$ on $\mathcal{TR}_2$ via the semiorthogonal decomposition $\mathcal{TR} = \langle i_1(\mathcal{TR}^1), i_2(\mathcal{TR}^2)\rangle$. □

3.2 Application of CP-gluing to $\mathcal{D}^\uparrow$

We will now use the theory outlined in this chapter to compute t-structures on $\mathcal{D}^\uparrow$ by means of extending our knowledge about t-structures on $\mathcal{D}$ to $\mathcal{D}^\uparrow$ using the technique of CP-gluing.

In order to find new bounded t-structures on $\mathcal{D}^\uparrow$ we want to make use of the fact that $\mathcal{D}^\uparrow$ consists of two copies of $\mathcal{D}$ and use the technique of CP-gluing to obtain bounded t-structures on $\mathcal{D}^\uparrow$ of the bounded t-structures on $\mathcal{D}$.

We start by defining the following functors in the context of $\mathcal{A}$ and $\mathcal{A}^\uparrow$ and subsequently in terms of $\mathcal{D}$ and $\mathcal{D}^\uparrow$.

Definition 3.2.1. For an object $(A \to B) \in \mathcal{A}^\uparrow$ and a morphism $f = (f_1, f_2) \in \mathcal{A}^\uparrow$, define

1.

$$\begin{aligned}\lambda_1^{\mathcal{A}} : \mathcal{A}^{\uparrow} &\to \mathcal{A}\\ \lambda_1^{\mathcal{A}}(A \to B) &= A\\ \lambda_1^{\mathcal{A}}(f) &= f_1,\end{aligned}$$

2.

$$\begin{aligned}\rho_2^{\mathcal{A}} : \mathcal{A}^{\uparrow} &\to \mathcal{A}\\ \rho_2^{\mathcal{A}}(A \to B) &= B\\ \rho_2^{\mathcal{A}}(f) &= f_2.\end{aligned}$$

Definition 3.2.2. For an object $A \in \mathcal{A}$ and a morphism $f_1 \in \mathcal{A}$ define

1.

$$\begin{aligned}i_1^{\mathcal{A}} : \mathcal{A} &\to \mathcal{A}^{\uparrow}\\ i_1^{\mathcal{A}}(A) &= A \to 0\\ i_1^{\mathcal{A}}(f_1) &= (f_1, 0),\end{aligned}$$

2.

$$\begin{aligned}i_2^{\mathcal{A}} : \mathcal{A} &\to \mathcal{A}^{\uparrow}\\ i_2^{\mathcal{A}}(A) &= 0 \to A\\ i_2^{\mathcal{A}}(f_1) &= (0, f_1),\end{aligned}$$

3.

$$\begin{aligned}\Delta^{\mathcal{A}} : \mathcal{A} &\to \mathcal{A}^{\uparrow}\\ \Delta^{\mathcal{A}}(A) &= A \xrightarrow{\mathrm{id}} A\\ \Delta^{\mathcal{A}}(f_1) &= (f_1, f_1).\end{aligned}$$

We obtain the following.

Lemma 3.2.3. *There are adjoint pairs $\lambda_1^{\mathcal{A}} \dashv i_1^{\mathcal{A}}$ and $i_2^{\mathcal{A}} \dashv \rho_2^{\mathcal{A}}$ for the functors defined in 3.2.1 and 3.2.2.*

Proof. To see that $(i_2^{\mathcal{A}}, \rho_2^{\mathcal{A}})$ is an adjoint pair, consider

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}^{\uparrow}}(i_2^{\mathcal{A}}(E), B) \cong \mathrm{Hom}_{\mathcal{A}}(0, B_1) \times \mathrm{Hom}_{\mathcal{A}}(E, B_2) &= 0 \times \mathrm{Hom}_{\mathcal{A}}(E, B_2)\\ &\cong \mathrm{Hom}_{\mathcal{A}}(E, B_2) = \mathrm{Hom}_{\mathcal{A}}(E, \rho_2^{\mathcal{A}}(B)).\end{aligned}$$

In order to demonstrate explicitly how this works, let us define a morphism $\mathrm{Hom}_{\mathcal{A}^\uparrow}(i_2^{\mathcal{A}}(E), B) \to \mathrm{Hom}_{\mathcal{A}}(E, \rho_2^{\mathcal{A}}(B))$ as $(f,g) \mapsto g$. Since $f : 0 \to \lambda_1^{\mathcal{A}}(B)$ implies $f = 0$ and therefore $(f,g) = (f,0)$, this is indeed an isomorphism. To prove that this isomorphism is bifunctorial, on the other hand, we regard the isomorphism

$$\mathrm{Hom}_{\mathcal{A}^\uparrow}(i_2^{\mathcal{A}}(E), B) \cong \mathrm{Hom}_{\mathcal{A}}(E, \rho_2^{\mathcal{A}}(B))$$

as an embedding of the set $\mathrm{Hom}_{\mathcal{A}}(E, B_2)$ into the product of sets that is $\mathrm{Hom}_{\mathcal{A}}(0, A) \times \mathrm{Hom}_{\mathcal{A}}(E, B_2) \supset \mathrm{Hom}_{\mathcal{A}^\uparrow}(i_2^{\mathcal{A}}(E), B)$. This embedding, however, is indeed bifunctorial – seen by the fact that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}_{\mathcal{A}}(E,B_2) & \xrightarrow{(0,-\circ t)} & \mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}_{\mathcal{A}}(E',B_2) \\ \uparrow{\scriptstyle (0,-)} & & \uparrow{\scriptstyle (0,-)} \\ \mathrm{Hom}_{\mathcal{A}}(E,B_2) & \xrightarrow{(-\circ t)} & \mathrm{Hom}_{\mathcal{A}}(E',B_2) \end{array}$$

commutes for $X, Y, E' \in \mathcal{A}$ and $t : E' \to E$ a morphism in $\mathcal{A}$. Therefore, the proof is finished. The proof that $(\lambda_1^{\mathcal{A}}, i_1^{\mathcal{A}})$ is an adjoint pair is similar. □

Lemma 3.2.4. *There are adjoint pairs* $\Delta^{\mathcal{A}} \dashv \lambda_1^{\mathcal{A}}$ *and adjoint pairs* $\rho_2^{\mathcal{A}} \dashv \Delta^{\mathcal{A}}$ *for* $\Delta^{\mathcal{A}}, \lambda_1^{\mathcal{A}}, \rho_2^{\mathcal{A}}$ *as in lemma 3.2.8.*

Proof. Let $A \in \mathcal{A}$ and $B = (B_1 \to B_2) \in \mathcal{A}^\uparrow$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}^\uparrow}(\Delta^{\mathcal{A}}(A), B) &= \mathrm{Hom}_{\mathcal{A}^\uparrow}((A \to A), (B_1 \to B_2)) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(A, B_1) = \mathrm{Hom}_{\mathcal{A}}(A, \lambda_1^{\mathcal{A}} B). \end{aligned}$$

To prove that this isomorphism is bifunctorial, on the other hand, we regard the isomorphism

$$\mathrm{Hom}_{\mathcal{A}^\uparrow}(\Delta^{\mathcal{A}}(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, \lambda_1^{\mathcal{A}} B)$$

as an embedding of the set $\mathrm{Hom}_{\mathcal{A}}(A, B_1)$ into the product of sets that is $\mathrm{Hom}_{\mathcal{A}}(A, B_1) \times \mathrm{Hom}_{\mathcal{A}}(A, B_2) \supset \mathrm{Hom}_{\mathcal{A}^\uparrow}(\Delta^{\mathcal{A}}(A), B)$. This embedding, however, is indeed bifunctorial – seen by the fact that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A,B_1) \times \mathrm{Hom}_{\mathcal{A}}(A,B_2) & \xrightarrow{(-\circ t,-\circ t)} & \mathrm{Hom}_{\mathcal{A}}(A',B_1) \times \mathrm{Hom}_{\mathcal{A}}(A',B_2) \\ \uparrow{\scriptstyle (-,f\circ -)} & & \uparrow{\scriptstyle (-,f\circ -)} \\ \mathrm{Hom}_{\mathcal{A}}(A,B_1) & \xrightarrow{(-\circ t)} & \mathrm{Hom}_{\mathcal{A}}(A',B_1) \end{array}$$

commutes where f equals to $(B_1 \to B_2)$ regarded as a morphism in $\mathcal{A}$. The proof for $\Delta^{\mathcal{A}} \dashv \lambda_1^{\mathcal{A}}$ is similar. □

We will now generalise the facts that we previously presented to the case of $\mathcal{D}$ by means of the next lemmas.

Lemma 3.2.5. *The functors* $\lambda_1^{\mathcal{A}}$ *and* $\rho_2^{\mathcal{A}}$ *are exact.*

Proof. This is obvious from corollary 2.2.7. □

Lemma 3.2.6. *The functors* $\lambda_1^{\mathcal{A}}, \rho_2^{\mathcal{A}}, i_1^{\mathcal{A}}, i_2^{\mathcal{A}}$ *and* $\Delta^{\mathcal{A}}$ *are exact.*

Proof. We obtain that $\lambda_1^{\mathcal{A}}$ and $\rho_2^{\mathcal{A}}$ are exact from lemma 3.2.5. The exactness of $\Delta^{\mathcal{A}}$ is an implication of lemma 3.2.4. We obtain the statement for $i_1^{\mathcal{A}}$ by realising that $\lambda_1^{\mathcal{A}} \circ i_1^{\mathcal{A}} = \mathrm{id}$ and $\rho_2^{\mathcal{A}} \circ i_1^{\mathcal{A}} = 0$ and applying corollary 2.2.7. The proof for $i_2^{\mathcal{A}}$ is similar. □

We need to combine this with a fact provided in [60, 10.5.2]. Since the details of the proof are left to the reader in the named literature, we will, for the convenience of the reader, prove the fact stated in proposition 3.2.7. It is subsequently our aim to obtain a well-defined functor $F_{\mathcal{D}(\mathcal{A})^b}$ on the derived category $\mathcal{D}(\mathcal{A})^b$ of an abelian category $\mathcal{A}$, out of an exact functor $F_{\mathcal{A}}$ defined on $\mathcal{A}$.

Proposition 3.2.7. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian categories and* $\mathcal{D}^b(\mathcal{A})$ *and* $\mathcal{D}^b(\mathcal{B})$ *their bounded derived categories. If a functor*

$$F_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$$

is exact, it extents to a functor

$$F_{\mathcal{D}^b(\mathcal{A})} : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B}).$$

by applying $F_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ *to the objects and differentials in a complex and to the components of localised homotopy classes of complex-morphisms – the morphisms in* $\mathcal{D}^b(\mathcal{A})$.

Proof. Consider the functor $F_{\mathcal{C}(\mathcal{A})}$ which is the functor on the chain-complex category $\mathcal{C}(\mathcal{A})$ that is induced by the functor $F_{\mathcal{A}}$ operating componentwise on the objects and morphisms in $\mathcal{A}$, that a complex in $\mathcal{C}(\mathcal{A})$ is composed of. This is consistent with chain-maps and boundary operators as well, since functors preserve commutative diagrams. Again for the same reason – the preservation of commutative diagrams – we can now define $F_{\mathcal{K}(\mathcal{A})}$ to be the functor on the homotopy category that is induced by $F_{\mathcal{C}(\mathcal{A})}$.

It is at this point, however, that we require the exactness of $F_{\mathcal{A}}$ in order to define $F_{\mathcal{D}^b(\mathcal{A})}$ as the functor on $\mathcal{D}^b(\mathcal{A})$ induced by $F_{\mathcal{K}(\mathcal{A})}$. To obtain that the functor $F_{\mathcal{D}^b(\mathcal{A})}$ is well-defined, we now need to verify that it preserves quasi-isomorphisms. Assume that $A^\bullet \xrightarrow{f} B^\bullet$ is a quasi-isomorphism. We need to prove that $F_{\mathcal{A}}(A^\bullet) \overset{F_{\mathcal{A}}(f)}{\to} F_{\mathcal{A}}(B^\bullet)$ is a quasi-isomorphism as well. The morphism $A^\bullet \xrightarrow{f} B^\bullet$ is a quasi-isomorphism if and only if the induced morphism $H^n(A^\bullet) \overset{H^n(f)}{\to} H^n(B^\bullet)$ is an isomorphism in $\mathcal{A}$ for any $n \in \mathbb{Z}$, where H^n is the usual complex-cohomology. In other words,

$$0 \to H^n(A^\bullet) \overset{H^n(f)}{\to} H^n(B^\bullet) \to 0$$

is exact. Hence, so is

$$F_{\mathcal{A}}(0) \to F_{\mathcal{A}}(H^n(A^\bullet)) \overset{F_{\mathcal{A}}(H^n(f))}{\to} F_{\mathcal{A}}(H^n(B^\bullet)) \to F_{\mathcal{A}}(0)$$

which means that

$$0 \to F_{\mathcal{A}}(H^n(A^\bullet)) \overset{F_{\mathcal{A}}(H^n(f))}{\to} F_{\mathcal{A}}(H^n(B^\bullet)) \to 0$$

is exact and hence $F_{\mathcal{A}}(H^n(A^\bullet)) \overset{F_{\mathcal{A}}(H^n(f))}{\to} F_{\mathcal{A}}(H^n(B^\bullet))$ is an isomorphism in $\mathcal{B}$.

Since $F_{\mathcal{A}}$ is exact, it preserves quotients – to see this, we let d^n be the boundary operator of the complex $A^\bullet$ and consider the exact sequence

$$0 \to \mathrm{im}(d^{n-1}) \to \ker(d^n). \tag{3.2}$$

On one hand, if we apply the exact functor $F_{\mathcal{A}}$ we obtain the exact sequence

$$F(0) \to F(\mathrm{im}(d^{n-1})) \to F(\ker(d^n))$$

which equals to

$$0 \to F(\mathrm{im}(d^{n-1})) \to F(\ker(d^n))$$

and can be completed to the short exact sequence

$$0 \to F(\mathrm{im}(d^{n-1})) \to F(\ker(d^n)) \to F_{\mathcal{A}}(\ker(d^n)) \big/ F_{\mathcal{A}}(\mathrm{im}(d^{n-1})) \to 0.$$

On the other hand, (3.2) can be completed to the exact sequence

$$0 \to \mathrm{im}(d^{n-1}) \to \ker(d^n) \to \ker(d^n) \big/ \mathrm{im}(d^{n-1}) \to 0$$

and applying the exact functor $F_{\mathcal{A}}$ this gives

$$0 \to F(\mathrm{im}(d^{n-1})) \to F(\ker(d^n)) \to F(\ker(d^n) \big/ \mathrm{im}(d^{n-1})) \to 0.$$

We obtain

$$F_{\mathcal{A}}(\ker(d^n)) \Big/ F_{\mathcal{A}}(\mathrm{im}(d^{n-1})) \cong F_{\mathcal{A}}\big(\ker(d^n) \Big/ \mathrm{im}(d^{n-1})\big).$$

Again by exactness, an exact functor also preserves kernels and images such that we obtain

$$\ker(F_{\mathcal{A}}(d^n)) \Big/ \mathrm{im}(F_{\mathcal{A}}(d^{n-1})) = F_{\mathcal{A}}(\ker(d^n)) \Big/ F_{\mathcal{A}}(\mathrm{im}(d^{n-1})),$$

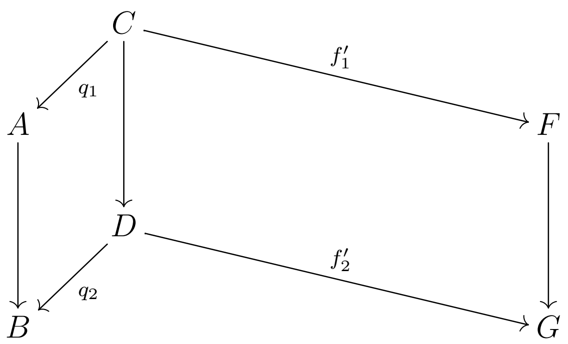
which hence gives

$$\begin{aligned} H^n(F_{\mathcal{D}^b(\mathcal{A})}(A^\bullet)) = \ker(F_{\mathcal{A}}(d^n)) \Big/ \mathrm{im}(F_{\mathcal{A}}(d^{n-1})) &= F_{\mathcal{A}}(\ker(d^n)) \Big/ F_{\mathcal{A}}(\mathrm{im}(d^{n-1})) \\ &= F_{\mathcal{A}}\big(\ker(d^n) \Big/ \mathrm{im}(d^{n-1})\big) = F_{\mathcal{A}}(H^n(A^\bullet)) \end{aligned}$$

and so we obtain that $H^n(F_{\mathcal{D}^b(\mathcal{A})}(A^\bullet)) \overset{H^n(F_{\mathcal{D}^b(\mathcal{A})}(f))}{\to} H^n(F_{\mathcal{D}^b(\mathcal{A})}(B^\bullet))$ is an isomorphism in $\mathcal{A}$, which finishes the proof. □

We can now introduce the two following lemmas.

Lemma 3.2.8. *For an object $(A \to B) \in \mathcal{D}^\uparrow$ and a morphism $f =$*



in $\mathcal{D}^\uparrow$, where we define $f_i = f'_i \circ q_i^{-1}, i \in \{1, 2\}$, we obtain

1. *The functor $\lambda_1^{\mathcal{A}}$ extends to a functor*

$$\begin{aligned} \lambda_1 : \mathcal{D}^\uparrow &\to \mathcal{D} \\ \lambda_1(A \to B) &= A \\ \lambda_1(f) &= f_1, \end{aligned}$$

2. *The functor $\rho_2^{\mathcal{A}}$ extends to a functor*

$$\begin{aligned} \rho_2 : \mathcal{D}^\uparrow &\to \mathcal{D} \\ \rho_2(A \to B) &= B \\ \rho_2(f) &= f_2. \end{aligned}$$

Proof. This combines lemma 3.2.6 with proposition 3.2.7. □

Lemma 3.2.9. *For an object $A \in \mathcal{D}$ and a morphism $f_1 \in \mathcal{D}$, we obtain the following.*

1. *The functor $i_1^{\mathcal{A}}$ extends to a functor*

$$\begin{aligned} i_1 &: \mathcal{D} \to \mathcal{D}^{\uparrow} \\ i_1(A) &= A \to 0 \\ i_1(f_1) &= (f_1, 0). \end{aligned}$$

2. *The functor $i_2^{\mathcal{A}}$ extends to a functor*

$$\begin{aligned} i_2 &: \mathcal{D} \to \mathcal{D}^{\uparrow} \\ i_2(A) &= 0 \to A \\ i_2(f_1) &= (0, f_1). \end{aligned}$$

3. *The functor $\Delta^{\mathcal{A}}$ extends to a functor*

$$\begin{aligned} \Delta &: \mathcal{D} \to \mathcal{D}^{\uparrow} \\ \Delta(A) &= A \xrightarrow{\mathrm{id}} A \\ \Delta(f_1) &= (f_1, f_1). \end{aligned}$$

Here we want $(f_1, 0), (0, f_1)$ and (f_1, f_1) to be morphisms in $\mathcal{D}^{\uparrow}$ in the sense of the diagram in lemma 3.2.8.

Proof. Again, combine lemma 3.2.6 with proposition 3.2.7. □

We can now use the fact that certain adjunctions hold in the abelian case with regard to the previously defined functors, in order to obtain these adjunctions for the case of $\mathcal{D}$ as well.

Lemma 3.2.10. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and*

$$F_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$$

and

$$G_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$$

be exact functors such that $F_{\mathcal{A}} \dashv G_{\mathcal{B}}$. These extend trivially to functors $F_{\mathcal{A}}, G_{\mathcal{B}}$

$$F_{\mathcal{D}^b(\mathcal{A})} : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B}).$$

and

$$G_{\mathcal{D}^b(\mathcal{B})} : \mathcal{D}^b(\mathcal{B}) \to \mathcal{D}^b(\mathcal{A}).$$

and $F_{\mathcal{D}^b(\mathcal{A})} \dashv G_{\mathcal{D}^b(\mathcal{B})}$.

Proof. This is a special case of [47, Corollary] because it was assumed that both involved functors extend trivially to the respective derived categories – in other words they each are their own right- and left derived functor. □

We obtain the following.

Lemma 3.2.11. *There are adjoint pairs $\lambda_1 \dashv i_1$ and $i_2 \dashv \rho_2$ for the functors obtained by lemma 3.2.8 and 3.2.9.*

Proof. Combine lemma 3.2.3 with lemma 3.2.10. □

Moreover we have the following lemmas, in order to prepare lemma 3.2.14.

Lemma 3.2.12. *A morphism (f, g) in $\mathcal{A}^{\uparrow}$ is an isomorphism if and only if f, g are isomorphisms in $\mathcal{A}$.*

Proof. Let $X_1, X_2, Y_1, Y_2 \in \mathcal{A}$ such that the morphism (f, g) is given by the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{g} & Y_2 \end{array}$$

and consider this as a complex of the form

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & X_1 & \xrightarrow{f} & Y_1 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & X_2 & \xrightarrow{g} & Y_2 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

to which we can now apply corollary 2.2.7 which finishes the proof. □

Lemma 3.2.13. *A morphism f in $\mathcal{D}^{\uparrow}$ is an isomorphism if and only if $\lambda_1(f), \rho_2(f)$ are isomorphisms in $\mathcal{D}$.*

Proof. A morphism f in a derived category - viewed as a roof $f_1 \circ q^{-1}$ for a quasi-isomorphism q is an isomorphism if and only if a chain-map ξ that can be chosen as a representative for f, is a quasi-isomorphism. We have that the chain maps $\lambda_1^{\mathcal{C}(\mathcal{A})}(\xi)$ and $\rho_2^{\mathcal{C}(\mathcal{A})}(\xi)$ that represent the morphisms $\lambda_1(f), \rho_2(f)$ are quasi-isomorphisms if and only if $H^n(\lambda_1^{\mathcal{A}}(\xi)), H^n(\rho_2^{\mathcal{A}}(\xi))$ are isomorphisms in $\mathcal{A}$ for all $n \in \mathbb{Z}$. By lemma 3.2.5 however, both $\lambda_1^{\mathcal{A}}$ and $\rho_2^{\mathcal{A}}$ are exact and hence commute with cohomology. This, on the other hand, implies that $H^n(\lambda_1^{\mathcal{A}}(\xi)), H^n(\rho_2^{\mathcal{A}}(\xi))$ are isomorphisms in $\mathcal{A}$ for all $n \in \mathbb{Z}$ if and only if $\lambda_1^{\mathcal{A}}(H^n(\xi)), \rho_2^{\mathcal{A}}(H^n(\xi))$ are isomorphisms in $\mathcal{A}$ for all $n \in \mathbb{Z}$. The latter happens – by definition – exactly if and only if ξ is a quasi-isomorphism. □

Lemma 3.2.14. *The functors obtained in lemma 3.2.8 and lemma 3.2.9 fulfil*

1. $\ker(\lambda_1) = \operatorname{im}(i_2)$ *and*
2. $\ker(\rho_2) = \operatorname{im}(i_1)$.

Proof. First assume that $(E \to F) \in \ker(\lambda_1) \subset \mathcal{D}^\uparrow$. This means $E \cong 0$ in $\mathcal{D}$. In other words, there is a quasi-isomorphism q in $\mathcal{K}(\mathcal{A})$ such that $E \xrightarrow{q} 0$ in $\mathcal{D}$ (which – of course – implies $q = 0$ in $\mathcal{D}^\uparrow$). The diagram

$$\begin{array}{ccc} 0 & \xrightarrow{0} & E \\ \downarrow & & \downarrow \\ F & \longrightarrow & F \end{array}$$

commutes and both id and $q = 0$ are quasi-isomorphisms. Hence, by lemma 3.2.13, (q, id) is an isomorphism in $\mathcal{D}^\uparrow$. Since $\operatorname{im}(i_2)$ consists of all objects $(A \to B)$ that fulfil $(A \to B) \cong (0 \to C)$, we obtain $(E \to F) \in \operatorname{im}(i_2)$.

If, on the other hand, we take $(A \to B) \in \operatorname{im}(i_2) \subset \mathcal{D}^\uparrow$, we have $(A \to B) \cong (0 \to C)$. By lemma 3.2.13, this implies that there is a quasi-isomorphism between A and 0. Hence, A is exact which means $A \cong 0$ in $\mathcal{D}$ and hence $(A \to B) \in \ker(\lambda_1)$.

The proof for $\ker(\rho_2) = \operatorname{im}(i_1)$ is similar. □

We do – however – not need the complete statement of this particular lemma, but only the following corollary.

Corollary 3.2.15. *The functors obtained in lemma 3.2.8 and 3.2.9 fulfil*

$$\rho_2 \circ i_1 = 0 = \lambda_1 \circ i_2.$$

Proof. This is an immediate implication of lemma 3.2.14. □

We also have

Lemma 3.2.16. *For any $E, F \in \mathcal{D}$, the functors defined in 3.2.9 fulfil*

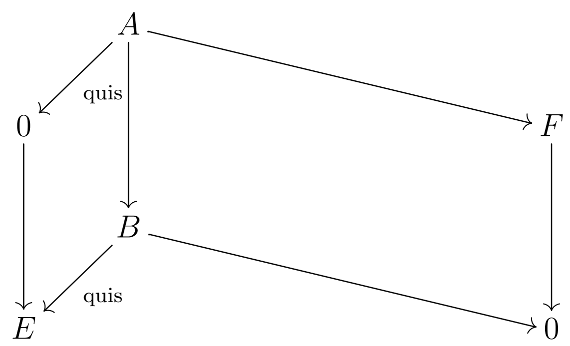
$$\operatorname{Hom}(i_2(E), i_1(F)) = 0.$$

Proof. Using lemma 3.2.11 and corollary 3.2.15 we obtain

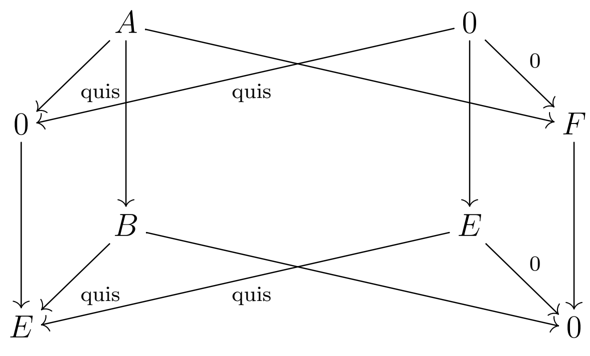
$$\operatorname{Hom}(i_2(E), i_1(F)) = \operatorname{Hom}(\lambda_1(i_2(E)), F) = \operatorname{Hom}(0, F) = 0.$$

□

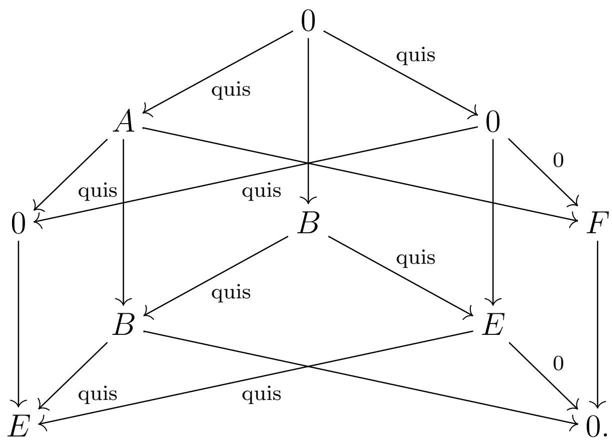
Remark 3.2.17. Another – more direct, yet more tedious – way to prove the crucial lemma 3.2.16 is to see that we have both $i_2(E) = 0 \to E$ and $i_1(F) = F \to 0$. A morphism in $\mathrm{Hom}(i_2(E), i_1(F))$ is given by the commutative diagram



where $(A \to B) \in \mathcal{D}^{\uparrow}$. We want to prove that this is in fact the zero-morphism, which follows by comparing it to a known representative of the zero-morphism in $\mathcal{D}^{\uparrow}$:



The next step is to now complete the above to the following commutative diagram:



This proves that every morphism from E_2 to E_1 in $\mathcal{D}^{\uparrow}$ is equivalent to the zero-morphism.

Furthermore we need this.

Lemma 3.2.18. *The functors* i_1 *and* i_2 *obtained by lemma 3.2.9 are fully faithful.*

Proof. Let $A, B \in \mathcal{D}$. Viewing i_1 as an induced map $i_1 : \mathrm{Hom}(A, B) \to \mathrm{Hom}(i_1(A), i_1(B))$ on the hom-sets, given by

$$i_1(f_1) = (f_1, 0)$$

for $f_1 \in \mathrm{Hom}(A, B)$ we obtain that i_1 is bijective. On one hand, if $f \in \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(A), i_1(B))$, then we have

$$f \in \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(A), i_1(B)) = \mathrm{Hom}_{\mathcal{D}}(\lambda_1(i_1(A)), B) \cong \mathrm{Hom}_{\mathcal{D}}(A, B).$$

In other words, f has a preimage under i_1 making i_1 a surjective map. On the other hand, if $g_1, g_2 \in \mathrm{Hom}(A, B)$ such that $i_1(g_1) = i_1(g_2)$, we have $(g_1, 0) = i_1(g_1) = i_1(g_2) = (g_2, 0)$ which implies $(g_1, 0) = (g_2, 0)$ and hence $g_1 = g_2$, proving that i_1 is also an injective map. Similarly one obtains the fully faithfulness of i_2. $\square$

This provides us with a corollary.

Corollary 3.2.19. *For the functors obtained by lemma 3.2.8 and by lemma 3.2.9 we obtain the identities*

$$\lambda_1 \circ i_1 \xrightarrow{\cong} \mathrm{id}_{\mathcal{D}}$$

and

$$\mathrm{id}_{\mathcal{D}} \xrightarrow{\cong} \rho_2 \circ i_2.$$

Proof. This is obtained by combining lemmas 3.2.18 and 3.2.11. □

Additionally we require this.

Lemma 3.2.20. *Assume $X = (E \to F) \in \mathcal{D}^{\uparrow}$. There is an exact triangle*

$$i_2(\rho_2(X)) \to X \to i_1(\lambda_1(E)) \xrightarrow{+} .$$

Proof. We obtain the exact triangle from the fact that the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & E & \longrightarrow & E & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

regarded as a diagram, commutes and has exact rows. In other words, it is an exact sequences of chain complexes. This implies that

$$(0 \to F) \to (E \to F) \to (E \to 0) \xrightarrow{+} \tag{3.3}$$

is an exact triangle. Because of the shape of the functors i_1, i_2, λ_1 and ρ_2 according to lemma 3.2.8 and lemma 3.2.9 the exact triangle (3.3) equals to

$$i_2(\rho_2(X)) \to X \to i_1(\lambda_1(E)) \xrightarrow{+} .$$

□

Definition 3.2.21. For i_j as in lemma 3.2.9, we define $\mathcal{D}_j$ for $j \in \{1, 2\}$ to be the smallest strictly full subcategory of $\mathcal{D}^{\uparrow}$ that contains all objects $i_j(E)$ for $E \in \mathcal{D}$.

Now we obtain a crucial theorem.

Proposition 3.2.22. *There is a semiorthogonal decomposition of $\mathcal{D}^{\uparrow}$ given by $\mathcal{D}^{\uparrow} = \langle i_1(\mathcal{D}), i_2(\mathcal{D}) \rangle$.*

Proof. Using lemmas 3.2.16 and 3.2.20 we see that the conditions of definition 2.1.6 are fulfilled. □

We can, however obtain more.

Lemma 3.2.23. *There is an adjoint pair $\Delta \dashv \lambda_1$ for Δ, λ_1 as in lemma 3.2.8.*

Proof. Similar to the proof of lemma 3.2.3, we deduce this result from lemma 3.2.4. □

Lemma 3.2.24. *There is an adjoint pair $\rho_2 \dashv \Delta$ for ρ_2, Δ as in lemma 3.2.8.*

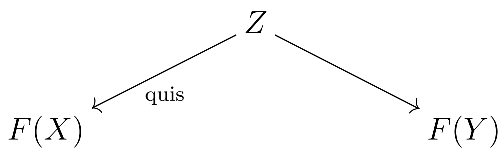
Proof. Similar to the proof of lemma 3.2.23. □

Remark 3.2.25. Note that the similarity of the proofs of lemmas 3.2.23 and 3.2.24 implies $\rho_2^{\mathcal{A}} \dashv \Delta^{\mathcal{A}}$.

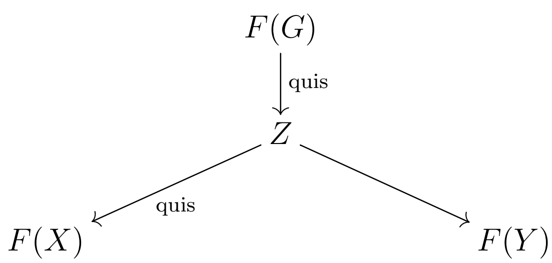
To be able to work with $\mathcal{D}$ in connection to $\mathcal{D}^{\uparrow}$ it will prove to be important to establish the connections between the corresponding Hom-sets. In the previous section we have done that in a more general framework. We will now use the results to obtain to necessary knowledge about $\mathcal{D}^{\uparrow}$.

Lemma 3.2.26. *Let $\mathcal{TR}^2$ and $\mathcal{TR}$ be derived categories of abelian categories $\mathcal{A}_2$ and $\mathcal{A}$ and $F : \mathcal{TR}^2 \to \mathcal{TR}$ a functor that is induced by an exact, fully-faithful functor $F' : \mathcal{A}_2 \to \mathcal{A}$, for which the following condition is fulfilled: For any $Z \in \mathcal{TR}$ such that $Z \overset{\text{quis}}{\to} F(X)$ for $X \in \mathcal{TR}^2$ there is a $G \in \mathcal{TR}^2$ such that $F(G) \overset{\text{quis}}{\to} Z$. Then F is a fully-faithful functor.*

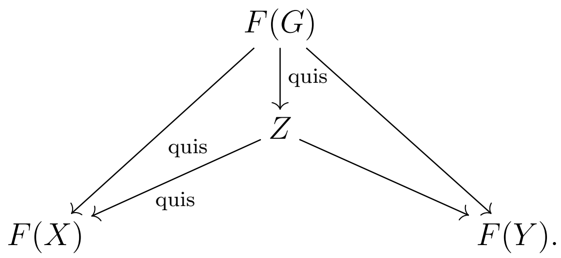
Proof. Assume all arrows to be morphisms of chain-complexes. Let $X, Y \in \mathcal{TR}^2$ and $\phi : F(X) \to F(Y)$ a morphism in $\mathcal{TR}$, hence ϕ is represented via a roof



for $Z \in \mathcal{TR}$.
For a $G \in \mathcal{TR}$ this can, due to the condition on F, be extended to:



for $G \in \mathcal{TR}$ and as we complete this to



We see that F is full if the functor of homotopy categories $F'' : \mathcal{K}(\mathcal{A}_2) \to \mathcal{K}(\mathcal{A})$ that is induced by F' is full. To see this, we prove, that the functor $F''' : \mathcal{C}(\mathcal{A}_2) \to \mathcal{C}(\mathcal{A})$ that is induced by F' and hence induces F'' is full. If f is a map of chain complexes in $\mathcal{C}(\mathcal{A})$, e.g. given by a collection f_i of morphisms $\mathcal{A}$ then by the assumption of F' being full, there are $g_i \in \mathcal{A}_2$ such that $F'(g_i) = f_i$. If we denote this collection by g, then g is indeed a map of chain complexes since the commutative squares:

$$\begin{array}{ccc} X_i & \xrightarrow{d_{i+1}} & X_{i+1} \\ {\scriptstyle f_i}\big\downarrow & & {\scriptstyle f_{i+1}}\big\downarrow \\ Y_i & \xrightarrow{d'_{i+1}} & Y_{i+1} \end{array}$$

that can now be rewritten as:

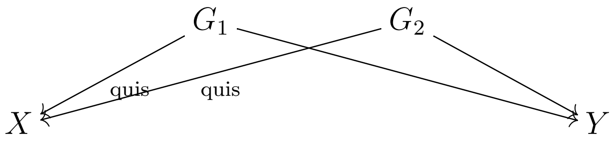
$$\begin{array}{ccc} X_i & \xrightarrow{F'(e_{i+1})} & X_{i+1} \\ {\scriptstyle F'(g_i)}\big\downarrow & & {\scriptstyle F'(g_{i+1})}\big\downarrow \\ Y_i & \xrightarrow{F'(e'_{i+1})} & Y_{i+1} \end{array}$$

as boundary operators $e, e' \in \mathcal{A}_2$, provide us with the equation $F'(g_{i+1}e_{i+1}) = F'(g_{i+1})F'(e_{i+1}) = F'(e'_{i+1})F'(g_i) = F'(e'_{i+1}g_i)$. As F' is faithful, by assumption, we obtain $g_{i+1}e_{i+1} = e'_{i+1}g_i$.
Now as $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y) \twoheadrightarrow \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ holds for any abelian category $\mathcal{A}$ the statement follows from the commutative diagram:

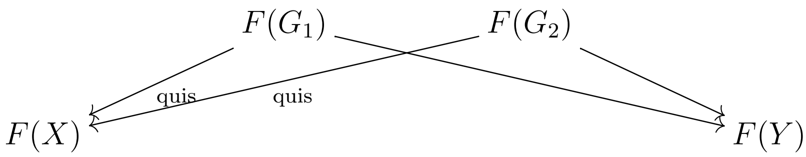
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}(\mathcal{A}_2)}(X, Y) & \xrightarrow{F'''} & \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(F(X), F(Y)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{\mathcal{K}(\mathcal{A}_2)}(X, Y) & \xrightarrow{F''} & \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(F(X), F(Y)). \end{array}$$

Since all other arrows are surjective maps, so is F''.

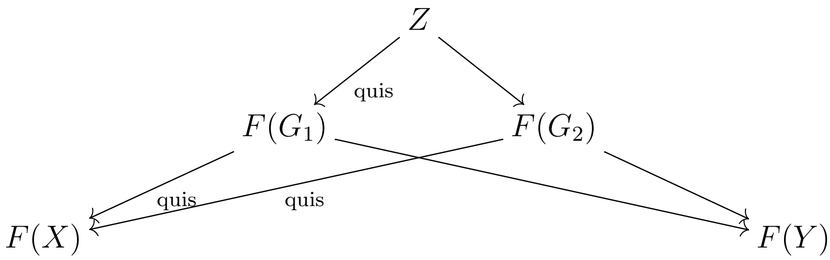
If we have, on the other hand, two morphisms $\phi : X \to Y$ and $\psi : X \to Y$, for $G_1, G_2 \in \mathcal{TR}^2$ presented by the diagram



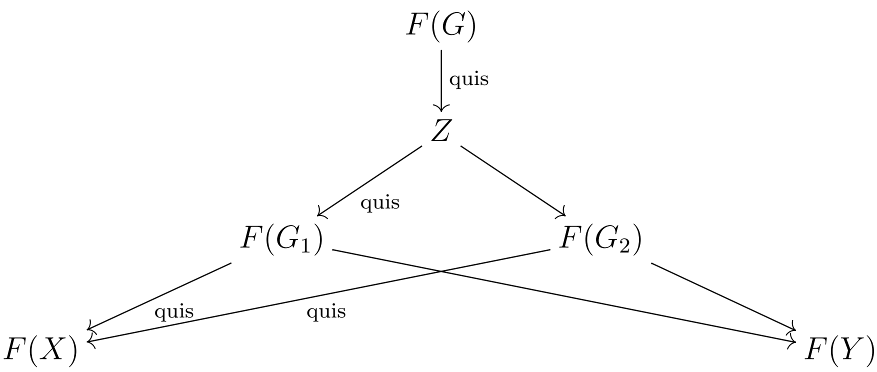
that map to the same morphism under F, which means that the diagram:



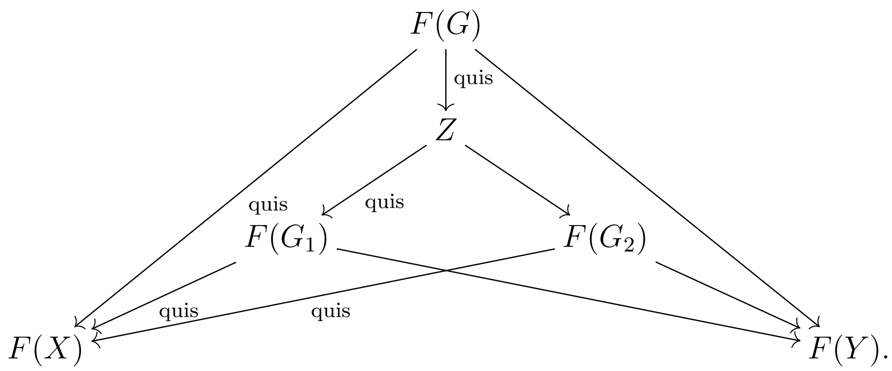
can be completed to:



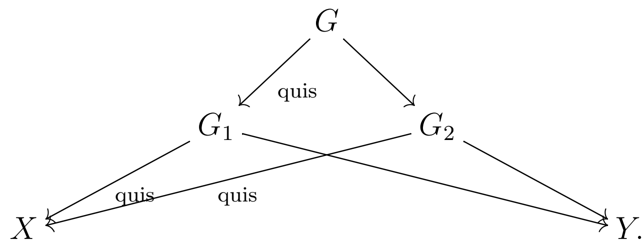
we can apply the condition on F once more to obtain:



and hence:



In order to prove, that ϕ and ψ come from the same preimage we now need F'' to be faithful. To see this, we must prove, that the preimage of any null-homotopic morphism in $\mathcal{C}(\mathcal{A})$ (hence a representative of the 0-morphism in $\mathcal{K}(\mathcal{A})$) is already null-homotopic. Let $f_i : X_i \to Y_i$ be a collection that represents a chain-map f in $\mathcal{C}(\mathcal{A})$ and $f_i = d'_i s_i + s_{i+1} d_{i+1}$ for $s_i : X_i \to Y_{i-1}$ and boundary operators d, d' (in other words f is null-homotopic). Similar to the proof of surjectivity we can now conclude, that a chain map g, that is mapped to f via F'' has to be null-homotopic as well, once again by the fully-faithfulness of F''. Since F'' is full, we can deduce from the previous diagram another diagram, that due to the faithfulness of F'' is also commutative:



This means that ϕ and ψ represent the same morphism in $\mathcal{TR}^2$, hence F is faithful and we conclude that it is fully-faithful.

□

Lemma 3.2.27. *The functor $\Delta^{\mathcal{A}} : \mathcal{A}_2 \to \mathcal{A}^{\uparrow}$ from definition 3.2.2 is fully faithful.*

Proof. This is obvious in the abelian case as Δ simply doubles everything. □

Lemma 3.2.28. *The functor $\Delta : \mathcal{D} \to \mathcal{D}^{\uparrow}$ is fully-faithful.*

Proof. Due to lemma 3.2.6 and lemma 3.2.27 exactness and fully-faithfulness of $\Delta : \mathcal{A}_2 \to \mathcal{A}$ is fulfilled. Consider a quasi-isomorphism

$$\begin{array}{ccc} A_1 & \xrightarrow{\text{quis}} & X_2 \\ \downarrow & & \downarrow{\scriptstyle =} \\ A_2 & \xrightarrow{\text{quis}} & X_2 \end{array}$$

where $A_1 \to A_2, X_2 \to X_2 \in \mathcal{D}^{\uparrow}$ and note that $\Delta(X_2) = X_2 \to X_2$. Then, as the diagram commutes, $A_1 \to A_2$ has to be a quasi-isomorphism and hence we get a commutative diagram:

$$\begin{array}{ccc} A_1 & \xrightarrow{=} & A_1 \\ {\scriptstyle =}\downarrow & & \downarrow \\ A_1 & \xrightarrow{\text{quis}} & A_2. \end{array}$$

As $A_1 \to A_2$ is G of lemma 3.2.26 we can now apply lemma 3.2.26 in the special case where $F = \Delta, \mathcal{TR} = \mathcal{D}^{\uparrow}$ and $\mathcal{TR}^2 = \mathcal{D}_2$. The statement follows. □

Corollary 3.2.29. *Let, $E_1, E_2 \in \mathcal{D}$, then*

$$\mathrm{Hom}^i_{\mathcal{D}}(E_1, E_2) \cong \mathrm{Hom}^{i+1}_{\mathcal{D}^{\uparrow}}(i_1(E_1), i_2(E_2))$$

for any $i \in \mathbb{Z}$.

Proof. Due to lemma 3.2.28, the functor Δ is fully faithful and hence we can apply theorem 3.1.13 letting $\mathcal{TR}^1 = \mathcal{D} = \mathcal{TR}^2$ and $\phi = \mathrm{id}_{\mathcal{D}}$. □

We can now make use of the connection between the Hom-sets of $\mathcal{D}$ and $\mathcal{D}^{\uparrow}$ provided by corollary 3.2.29 in order to prove that it is possible to glue hearts of bounded t-structures in this particular case.

Lemma 3.2.30. *For a standard slicing $\mathcal{P}$ (as it has been defined in 2.5.41) on $\mathcal{D}$ consider hearts of bounded t-structures on $\mathcal{D}$ that are given by $H_1 = \mathcal{P}(\alpha, \alpha + 1]$ and $H_2 = \mathcal{P}(\beta, \beta + 1]$, where $\alpha, \beta \in \mathbb{R}$. Then $\alpha \geq \beta$ if and only if $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$.*

Proof. In order to prove the "only if"-part it is sufficient to show that we obtain $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(i_1(E), i_2(F)) = 0$ for $E \in \mathcal{P}_1(\gamma), F \in \mathcal{P}_2(\delta)$, with $\gamma \in (\alpha, \alpha + 1], \delta \in (\beta, \beta + 1]$ as there are filtrations of any $G, H \in \mathcal{D}^{\uparrow}$ by objects in slices. Assume now for a contradiction that $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) \neq 0$. That means there are $E \in \mathcal{P}_1(\gamma) \subset H_1, F \in \mathcal{P}_1(\delta) \subset H_2$ such that $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(E), i_2(F)[m]) \neq 0$ for an $m \in \mathbb{Z}, m \leq 0$. Since $F \in \mathcal{P}_1(\delta)$, there is an $F_1 \in \mathcal{P}_1(\delta - n) \subset H_1$ such that $F_1[n] = F$ for $n \in \mathbb{Z}, n \leq 0$. Hence, making use of the Hom-connection provided by corollary 3.2.29 we obtain

$$\mathrm{Hom}^{m+n-1}_{\mathcal{D}}(E, F_1) = \mathrm{Hom}^{m+n}_{\mathcal{D}^{\uparrow}}(i_1(E), i_2(F_1)) = \mathrm{Hom}^{m}_{\mathcal{D}^{\uparrow}}(i_1(E), i_2(F)) \neq 0$$

and that provides the contradiction as $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(E, F_1) = 0$ (by lemma 2.5.29) and hence

$$\mathrm{Hom}^{m+n-1}_{\mathcal{D}}(E, F_1) \subset \mathrm{Hom}^{\leq -1}_{\mathcal{D}}(E, F_1) = \mathrm{Hom}^{\leq 0}_{\mathcal{D}}(E, F_1) = 0.$$

To prove the "if"-part assume that $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$. If we now have $\alpha < \beta$, this implies that there is a non-zero $E \in H_1$ such that $E \in \mathcal{P}(\alpha, \beta)$ and hence $i_2(E[n]) \in i_2(\mathcal{P}(\beta, \beta+1]) = i_2(H_2)$ for an $n \in \mathbb{N}_{>0}$. This gives

$$\mathrm{id}_E \in \mathrm{Hom}_{\mathcal{D}}(E, E) = \mathrm{Hom}^{-(n-1)}_{\mathcal{D}}(E, E[n-1]) = \mathrm{Hom}^{-(n-1)}_{\mathcal{D}^{\uparrow}}(i_1(E), i_2(E[n])).$$

This is a contradiction as $i_1(E) \in i_1(H_1)$ and $i_2(E[n]) \in i_2(H_2)$ and $-(n-1) \leq 0$ for $n > 0$. Hence we must have $\alpha \geq \beta$. □

Corollary 3.2.31. *For a standard slicing $\mathcal{P}$ on $\mathcal{D}$ consider hearts of bounded t-structures on $\mathcal{D}$ given by $H_1 = \mathcal{P}(\alpha, \alpha+1]$ and $H_2 = \mathcal{P}(\beta, \beta+1]$, where $\alpha, \beta \in \mathbb{R}$. If $\alpha \geq \beta$, then $H = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$ is the heart of a bounded t-structure on $\mathcal{D}^{\uparrow}$.*

Proof. Glue the heart using the technique described in lemma 3.1.5, as the necessary conditions are fulfilled by lemma 3.2.30. □

It is at this point a natural question if the requirements of corollary 3.2.31 are necessary for $H = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$ to be the heart of a bounded t-structure on $\mathcal{D}^{\uparrow}$. The "better part" of this question can be answered in general by lemma 3.1.14 which is simply a straightforward application of lemma 2.5.29.

However, the question that remains to be answered is, what happens in the case of $\mathrm{Hom}_{\mathcal{TR}}(H_1, H_2) \neq 0$. It is possible that this question cannot be answered in general. The following deals with this question in the special context of $\mathcal{D}^{\uparrow}$.

Lemma 3.2.32. *Let H_1, H_2 be hearts of bounded t-structures on $\mathcal{D}$. Assume $\mathrm{Hom}_{\mathcal{K}(A)}(H_1, H_2[-1]) = 0$, then $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$.*

Proof. Let $f \in \mathrm{Hom}_{\mathcal{D}}(H_1, H_2[-1])$. Hence f is of the form

$$\begin{array}{ccc} & U & \\ {}^{\text{quis}}\swarrow & & \searrow^{f_{\mathcal{K}(A)}} \\ A & & A' \end{array}$$

where $f_{\mathcal{K}(A)} \in \mathcal{K}(A), A \in H_1, A' \in H_2[-1]$ and $U \in \mathcal{D}$. By lemma 2.5.28 this implies $U \in H_1$. Hence $f_{\mathcal{K}(A)} \in \mathrm{Hom}_{\mathcal{K}(A)}(H_1, H_2[-1]) = 0$. Hence, f is a representative of the 0-morphism in $\mathcal{D}$ and therefore $\mathrm{Hom}_{\mathcal{D}}(H_1, H_2[-1]) = 0$. This implies, since we have

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) \cong \mathrm{Hom}_{\mathcal{D}}(H_1, H_2[-1]),$$

by corollary 3.2.29, that we obtain $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$. □

Lemma 3.2.33. *Let H_1, H_2 be the hearts of bounded t-structures on $\mathcal{D}$, such that*

$$H = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$$

is the heart of a t-structure on $\mathcal{D}^{\uparrow}$, then $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$.

Proof. Assume $\phi \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A}^{\uparrow})}(H_1, H_2[-1])$. This implies that there are objects $E \in H_1 \subset \mathcal{D}_1^{\leq 0}$ and $F \in H_2[-1] \subset \mathcal{D}_2^{\geq 1}$ and an object $\widetilde{\phi} = (E \to F) \in \mathcal{D}^{\uparrow}$ that corresponds to the morphism ϕ. We now denote the t-structures corresponding to H_1 and H_2 by $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ and $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$, then we have $E \in H_1 \subset \mathcal{D}_1^{\leq 0}$ and $F \in H_2[-1] \subset \mathcal{D}_2^{\geq 1}$. Denote the t-structure corresponding to H by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$. We can embed $E \xrightarrow{\widetilde{\phi}} F$ into the exact triangle

$$\begin{array}{ccccccc} X & \xrightarrow{e} & E & \xrightarrow{e'} & X' & \xrightarrow{+} & \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\widetilde{\phi}}\downarrow & & {\scriptstyle\psi'}\downarrow & & \\ Y & \xrightarrow{f} & F & \xrightarrow{f'} & Y' & \xrightarrow{+} & \end{array}$$

in $\mathcal{D}^{\uparrow}$, where $(X \to Y) \in \mathcal{D}^{\leq 0}$ and $(X' \to Y') \in \mathcal{D}^{\geq 1}$. Hence we obtain $X \cong \lambda_1(X \to Y) \in \mathcal{D}_1^{\leq 0}, X' \cong \lambda_1(X' \to Y') \in \mathcal{D}_1^{\geq 1}, Y \cong \rho_2(X \to Y) \in \mathcal{D}_2^{\leq 0}$ and $Y' \cong \rho_2(X' \to Y') \in \mathcal{D}_2^{\geq 1}$ (We refer to remark 3.1.6 regarding the definition of a t-structure that corresponds to a heart obtained by CP-gluing). This, however, implies that $e' \in \mathrm{Hom}(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1}) = 0$ and $f \in \mathrm{Hom}(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1}) = 0$. This means that $e' = f = 0$, which, by [32, Lemma 1.4] implies that e has a right inverse $\tilde{e}$ such that $e \circ \tilde{e} = \mathrm{id}$. Hence, viewing the object $\widetilde{\phi} \in \mathcal{D}^{\uparrow}$ as a morphism in $\mathcal{D}$, we obtain, using the commutativity of the exact triangle viewed as a diagram, that

$$\widetilde{\phi} = \widetilde{\phi} \circ \mathrm{id} = \widetilde{\phi} \circ e \circ \tilde{e} = f \circ \psi \circ \tilde{e} = 0 \circ \psi \circ \tilde{e} = 0.$$

Hence, we have $\mathrm{Hom}_{\mathcal{K}(\mathcal{A}^{\uparrow})}(H_1, H_2[-1]) = 0$ and therefore, by lemma 3.2.32, $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$. □

We can now conclude by formulating the following theorem.

Theorem 3.2.34. *Let H_1, H_2 be the hearts of bounded t-structures on $\mathcal{D}$,*

$$H = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$$

is the heart of a t-structure on $\mathcal{D}^{\uparrow}$ if and only if $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = 0$.

Proof. Apply lemma 3.1.5, lemma 3.1.14 and lemma 3.2.33. □

Remark 3.2.35. Note, that by lemma 3.1.7 the t-structure corresponding to H of theorem 3.2.34 is also bounded.

Corollary 3.2.36. *For a standard slicing $\mathcal{P}$ on $\mathcal{D}$ consider hearts of bounded t-structures on $\mathcal{D}$ given by $H_1 = \mathcal{P}(\alpha, \alpha + 1]$ and $H_2 = \mathcal{P}(\beta, \beta + 1]$, where $\alpha, \beta \in \mathbb{R}$. Then $H = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$ is the heart of a bounded t-structure on $\mathcal{D}^{\uparrow}$ if and only if $\alpha \geq \beta$.*

Proof. Combine corollary 3.2.31 with the "only if"-part of theorem 3.2.34. □

The question that remains, however, is whether these are the only t-structures on $\mathcal{D}^{\uparrow}$. In particular, it has been previously investigated how to obtain a t-structure on $\mathcal{D}^{\uparrow}$ coming from a pair of t-structures on $\mathcal{D}$, that will then have a heart H with the – moderately – straightforward description $H = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in H_1, \rho_2(X) \in H_2\}$. It is, however, possible to go further and combine pairs of t-structures that do not meet the requirement of corollary 3.2.36. This is done by the technique of recollement that we will introduce in the subsection 4.1.

Proposition 3.2.37. *Assume $\sigma = (Z, H)$ to be a pre-stability condition on $\mathcal{D}^{\uparrow}$, obtained by CP-gluing from stability conditions $\sigma_1 = (H_1, Z_1)$ and $\sigma_2 = (H_2, Z_2)$, such that for $i = 1, 2$, $H_i = \mathcal{P}_{\mu}(\theta_i, \theta_i + 1]$. If $\theta_1 \geq \theta_2 + 1$ then*

$$Z = Z_1 \circ i_1 + Z_2 \circ i_2$$

has the HN-property.

Proof. Since $\theta_1 \geq \theta_2 + 1$ we have

$$\mathrm{Hom}^{\leq 1}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)) = \mathrm{Hom}^{\leq -1}_{\mathcal{D}^{\uparrow}}(i_1(H_1), i_2(H_2)[1]) = 0$$

by lemma 3.2.30. Now we obtain from [21, Proposition 3.5] that Z has the HN-property on H. □

Remark 3.2.38. Note that it is in fact possible to complete the picture provided by proposition 3.2.37 under particularly favourable conditions. Under the right assumptions on $\mathcal{A}$ we obtain via proposition 4.10.16, that any heart obtained by CP-gluing via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ provides in fact a pre-stability condition – that is, the HN-property is fulfilled for Z as in proposition 3.2.37. Note that one can relax the condition $\theta_1 \geq \theta_2 + 1$ of proposition 3.2.37 slightly by using certain rationality assumptions (see [21, Proposition 3.5]).

We conclude with the following – preliminary – result. Its vague nature hints at the fact that one requires further conditions on $\mathcal{D}^\uparrow$ in order to obtain more satisfactory results.

Theorem 3.2.39. *If the space of pre-stability conditions of $\mathcal{D}$ is non-empty, then so is the space of pre-stability conditions of $\mathcal{D}^\uparrow$.*

Proof. Combine corollary 3.2.36 with proposition 3.2.37. □

The following is – in analogy to the standard example of a stability condition explained in example 2.5.37 – what could be seen as the standard example of a glued stability condition on a particular version of $\mathcal{A}^\uparrow$. It is the concept of alpha-stability for holomorphic triples introduced by Bradlow and O. García-Prada (see [14] and [15]).

Example 3.2.40. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve. Consider pre-stability conditions $(Z_1, \mathcal{A})$ and $(Z_2, \mathcal{A})$ on $\mathcal{D}$ where for $A_i, C_i \in \mathbb{R}_{>0}$ and $B_i \in \mathbb{R}$,*

$$Z_1(E_1) = -A_1 \deg(E_1) + B_1 \operatorname{rank}(E_1) + iC_1 \operatorname{rank}(E_1)$$

and

$$Z_2(E_2) = -A_2 \deg(E_2) + B_2 \operatorname{rank}(E_2) + iC_2 \operatorname{rank}(E_2)$$

with $E_1, E_2 \in \mathcal{A}$. The pre-stability condition obtained by CP-gluing via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from $(Z_1, \mathcal{A})$ and $(Z_2, \mathcal{A})$ is given by $(Z, \mathcal{A}^\uparrow)$ where

$$Z(F_1 \to F_2) = -A_1 \deg(F_1) - A_2 \deg(F_2) + B_1 \operatorname{rank}(F_1) + B_2 \operatorname{rank}(F_2) \\ + i(C_1 \operatorname{rank}(F_1) + C_2 \operatorname{rank}(F_2))$$

for $(F_1 \to F_2) \in \mathcal{A}^\uparrow$.

Letting $A_1 = A_2 = C_1 = C_2 = 1, B_1 = -\alpha \in \mathbb{R}$ and $B_2 = 0$, we obtain

$$Z(F_1 \to F_2) = \deg(F_1) - \deg(F_2) - \alpha \operatorname{rank}(F_1) + i(\operatorname{rank}(F_1) + \operatorname{rank}(F_2)).$$

This is the classical concept of α-stability translated into our language.

4 Recollement, tilting and the computation of $\mathrm{Stab}(\mathcal{D}^{\uparrow})$

In this section we introduce two additional techniques that can be used to compute stability conditions on a given triangulated category. The first is that of recollement which will prove to produce interesting results with regard to the computation of the data we are interested in – we will develop this throughout subsections 4.1 - 4.4. However, the main result (theorem 4.4.6) with regard to recollement will be, that given the particular situation that we are working in it will not produce stability conditions that we could not have obtained via gluing. However, the tilting-technique, that will be introduced – and subsequently applied – in subsection 4.7 completes the picture painted by the application of CP-gluing as will be demonstrated in the remaining subsections of this section. The section is joint work with Eva Martínez and Alejandra Rincón.

4.1 Recollement

The technique of "recollement" was introduced by Beilinson, Bernstein and Deligne in [10] and as a method of finding and classifying bounded t-structures it is crucial for the search of new stability conditions and goes – from some perspective – beyond the CP-gluing technique. The theory bases on the following definition. The motivation is the consideration of open ($j : \mathcal{X} \hookrightarrow \mathcal{Z}$) and closed ($i : \mathcal{Y} \hookrightarrow \mathcal{Z}$) embeddings among topological spaces and of sheaves that are being pushed and pulled along them. Note, that our own notation will be inspired by the notation of Liu and Vitoria from [43], instead of the original one of [10]:

Definition 4.1.1. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be triangulated categories. We say that $\mathcal{Z}$ admits recollement of $\mathcal{X}$ and $\mathcal{Y}$ if there are exact functors:

$$i^* : \mathcal{Z} \to \mathcal{Y}, i_* = i_! : \mathcal{Y} \to \mathcal{Z}, i^! : \mathcal{Z} \to \mathcal{Y},$$
$$j_! : \mathcal{X} \to \mathcal{Z}, j^* = j^! : \mathcal{Z} \to \mathcal{X} \text{ and } j_* : \mathcal{X} \to \mathcal{Z},$$

such that $i^* \dashv i_* = i_! \dashv i^!, j_! \dashv j^! = j^* \dashv j_*$, moreover $i_*, j_*, j_!$ are full embeddings, $i^! \circ j_* = 0$ and for any $Z \in \mathcal{Z}$ there are exact triangles:

$$i_! i^!(Z) \to Z \to j_* j^*(Z) \xrightarrow{+}$$

and

$$j_! j^!(Z) \to Z \to i_* i^*(Z) \xrightarrow{+} .$$

Notation 4.1.2. We will in the future refer to the data and conditions of definition 4.1.1 as "recollement-data".

The use of recollement in order to find t-structures is seen by the following lemma, that Beilinson, Bernstein and Deligne proved in [10].

Lemma 4.1.3. *Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be triangulated categories. Assume that $\mathcal{Z}$ admits recollement of $\mathcal{X}$ and $\mathcal{Y}$. Let $(\mathcal{X}^{\leq 0},\mathcal{X}^{\geq 1})$ and $(\mathcal{Y}^{\leq 0},\mathcal{Y}^{\geq 1})$ be t-structures on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Define*

$$\mathcal{Z}^{\leq 0} = \{Z \in \mathcal{Z} \mid j^*(Z) \in \mathcal{X}^{\leq 0}, i^*(Z) \in \mathcal{Y}^{\leq 0}\}$$
$$\mathcal{Z}^{\geq 1} = \{Z \in \mathcal{Z} \mid j^*(Z) \in \mathcal{X}^{\geq 1}, i^!(Z) \in \mathcal{Y}^{\geq 1}\}.$$

Then $(\mathcal{Z}^{\leq 0},\mathcal{Z}^{\geq 1})$ is a t-structure on $\mathcal{Z}$.

Proof. See [10, Theorem 1.4.10]. □

A straightforward – yet useful – implication of definition 4.1.1 is provided by the following lemma that will be required later.

Lemma 4.1.4. *Let $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ be triangulated categories. Assume that $\mathcal{Z}$ admits recollement of $\mathcal{X}$ and $\mathcal{Y}$ via the functors given in definition 4.1.1. Then*

$$\mathrm{Hom}(i_*(A), j_*(B)) = 0$$

Proof. By definition 4.1.1, the recollement-data fulfils the conditions $i^! \circ j_* = 0$ and $i_! \dashv i^!$. Hence

$$\mathrm{Hom}(i_!(A), j_*(B)) = \mathrm{Hom}(A, i^!(j_*(B)) = \mathrm{Hom}(A, 0) = 0$$

□

Remark 4.1.5. In subsection 5.1 we will provide an abundance of examples that will help the reader to understand the concept of recollement better. In particular will this help with regard to the difference of this concept compared to that of its related concept of CP-gluing which subsection 5.1 is build on and which subsection 4.4 investigates with regard to the issue of stability conditions in our particular situation.

The main difference between the similar techniques of gluing and recollement lies at the point where one needs to make restrictions in order to apply the theory. CP-Gluing works with two less adjoint functors then recollement and only requires one exact triangle. This means that it works in a more general context regarding the category that one works in – the findings of subsection 3.2 required no further restrictions on $\mathcal{A}$ (and hence on $\mathcal{D}$).

However, to use recollement, one may be forced to impose conditions, such as $\mathcal{D}$ having a Serre functor or $\mathcal{A}$ having enough injective objects in order to obtain the missing functors. On the other hand, in a situation where recollement is applicable, the orthogonality-assumption on the hearts is no longer required allowing to combine any two hearts which is not generally possible in the case of CP-gluing.

4.2 Application of the theory of Serre functors to $\mathcal{D}^{\uparrow}$

We can now use the theorems provided in the appendix (A.1.15 and A.1.16) to extend our knowledge on $\mathcal{D}^{\uparrow}$. This is accomplished by theorem 4.2.19, for which we will now provide some preparations. Note that from now on we also draw on notation and terminology that has been introduced in the appendix. The following lemma is taken from [11, Proposition 1.5].

Lemma 4.2.1. *If $\mathcal{B}$ is a strictly full triangulated subcategory of a triangulated category $\mathcal{A}$ the following statements are equivalent.*

1. *$\mathcal{B}$ is right- (respectively left-) admissible*
2. *The inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ has a right (respectively left) adjoint.*

Proof. At first we assume that $\mathcal{B}$ is right admissible. By definition, for any $Y \in \mathcal{A}$ there hence exists an exact triangle

$$B \to Y \to C \xrightarrow{+}$$

where $B \in \mathcal{B}$ and $C \in \mathcal{B}^{\perp}$. For any $B' \in \mathcal{B}$, applying the functor $\mathrm{Hom}(B', -)$, this provides us with the exact sequence

$$\mathrm{Hom}(B', C[-1]) \to \mathrm{Hom}(B', B) \to \mathrm{Hom}(B, Y) \to \mathrm{Hom}(B, C).$$

Since we have $\mathrm{Hom}(B', C[-1]) = 0 = \mathrm{Hom}(B, Y)$, we obtain the isomorphism $\mathrm{Hom}(B', B) \cong \mathrm{Hom}(B', Y)$ making the functor that maps Y onto B the right adjoint to the inclusion functor of $\mathcal{B}$ into $\mathcal{A}$.

If, on the other hand, the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ has a right adjoint, we obtain $\mathrm{Hom}(B', B) \cong \mathrm{Hom}(B', Y)$ for any $B' \in \mathcal{B}$ and $Y \in \mathcal{A}$, where $B \in \mathcal{B}$ is the image of Y under the functor that is right adjoint to the inclusion. Hence there is a $v \in \mathrm{Hom}(B, Y)$ such that for any $f \in \mathrm{Hom}(B', B)$ there is a $g \in \mathrm{Hom}(B', Y)$ for which we have $f = v \circ g$. Consider the canonical exact triangle

$$B \xrightarrow{v} Y \to \mathrm{Cone}(v) \xrightarrow{+} .$$

From $\mathrm{Hom}(B', Y) \cong \mathrm{Hom}(B', B)$ we obtain $\mathrm{Hom}(B', \mathrm{Cone}(v)) = 0$ and hence we have $\mathrm{Cone}(v) \in \mathcal{B}^{\perp}$, therefore, $\mathcal{B}$ is right admissible.

The proof for the equivalence with regard to the left admissibility and the existence of a left adjoint functor is similar. □

We require the following corollary.

Corollary 4.2.2. *Let $\mathcal{D}_3$ be the strictly full image of Δ, then $\mathcal{D}_3$ is admissible.*

Proof. Δ has a right adjoint functor by 3.2.23 and a left-adjoint by 3.2.24. Now apply lemma 4.2.1. □

In order to make use of corollary 4.2.2 we introduce another lemma provided in [12, Proposition 1.7] and – since it was omitted there – include the proof for the convenience of the reader.

Lemma 4.2.3. *If $\mathcal{B} \subset \mathcal{A}$ is a right-admissible subcategory of a triangulated category $\mathcal{A}$ then ${}^{\perp}(\mathcal{B}^{\perp}) = \mathcal{B}$.*

Proof. It is trivially always true that $\mathcal{B} \subset {}^{\perp}(\mathcal{B}^{\perp})$ and we have to prove ${}^{\perp}(\mathcal{B}^{\perp}) \subset \mathcal{B}$. Assume $E \in {}^{\perp}(\mathcal{B}^{\perp})$, since $\mathcal{B}$ is right-admissible we obtain an exact triangle

$$B \xrightarrow{v} E \to B' \xrightarrow{+}$$

where $B \in \mathcal{B}$ and $B' \in \mathcal{B}^{\perp}$. Since $E \in {}^{\perp}(\mathcal{B}^{\perp})$, we have $\mathrm{Hom}(E, B') = 0$. This gives

$$B'[-1] \xrightarrow{u} B \xrightarrow{v} E \xrightarrow{0}$$

and hence we obtain $\tilde{u} : B \to B'[-1]$ such that $\tilde{u} \circ u = \mathrm{id}_{B'[-1]}$. But we have

$$\tilde{u} \in \mathrm{Hom}(B, B'[-1]) \subset \mathrm{Hom}({}^{\perp}(\mathcal{B}^{\perp}), \mathcal{B}^{\perp}) = 0$$

and therefore obtain

$$\mathrm{id}_{B'[-1]} = \tilde{u} \circ u = 0 \circ u = 0$$

which implies $B'[-1] = 0$. Hence $E \cong B$ implying $E \in \mathcal{B}$, which finishes the proof. □

In analogy, we have the next lemma.

Lemma 4.2.4. *If $\mathcal{B} \subset \mathcal{A}$ is a left-admissible subcategory of a triangulated category $\mathcal{A}$ then $({}^{\perp}\mathcal{B})^{\perp} = \mathcal{B}$.*

Proof. It is trivially always true that $\mathcal{B} \subset ({}^{\perp}\mathcal{B})^{\perp}$. Assume $E \in ({}^{\perp}\mathcal{B})^{\perp}$, since $\mathcal{B}$ is left admissible we obtain an exact triangle

$$B' \to E \xrightarrow{v} B \xrightarrow{+}$$

where $B \in \mathcal{B}$ and $B' \in {}^{\perp}\mathcal{B}$. Since $E \in ({}^{\perp}\mathcal{B})^{\perp}$, we have $\mathrm{Hom}(B', E) = 0$. This gives

$$\xrightarrow{0} E \xrightarrow{v} B \xrightarrow{u} B'[1] \xrightarrow{+}$$

and hence we obtain $\tilde{u} : B'[1] \to B$ such that $u \circ \tilde{u} = \mathrm{id}_{B'[1]}$. But we have

$$\tilde{u} \in \mathrm{Hom}(B'[1], B) = \mathrm{Hom}(B', B[-1]) \subset \mathrm{Hom}({}^{\perp}\mathcal{B}, \mathcal{B}) = 0$$

and therefore we obtain

$$\mathrm{id}_{B'[1]} = u \circ \tilde{u} = u \circ 0 = 0$$

which implies $B'[1] = 0$. Hence $E \cong B$ implying $E \in \mathcal{B}$ which finishes the proof. □

Corollary 4.2.5. *We have* $({}^{\perp}(\mathcal{D}_3))^{\perp} = \mathcal{D}_3 = {}^{\perp}((\mathcal{D}_3)^{\perp})$.

Proof. By corollary 4.2.2, $\mathcal{D}_3$ is admissible, now use lemmas 4.2.3 and 4.2.4 to obtain the statement of the lemma. □

Additionally we require the following simple fact.

Lemma 4.2.6. *Let* $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ *be a t-structure on a triangulated category* $\mathcal{TR}$. *Then* $(\mathcal{TR}^{\leq n})^{\perp} = \mathcal{TR}^{\geq n+1}$ *and* ${}^{\perp}(\mathcal{TR}^{\geq n+1}) = \mathcal{TR}^{\leq n}$.

Proof. This is a simple redraft of lemma 2.5.24 in a language that will suite us better to work with in the following. □

In addition we need the – easy – observation that is next.

Lemma 4.2.7. *Let* $(\mathcal{TR}^1, \mathcal{TR}^2)$ *be a semiorthogonal decomposition on a triangulated category* $\mathcal{TR}$. *Then, if we define categories* $\mathcal{TR}^{\leq 0} = \mathcal{TR}^2$ *and* $\mathcal{TR}^{\geq 1} = \mathcal{TR}^1$, *we obtain that* $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ *is a t-structure on* $\mathcal{TR}$.

Proof. Definition 2.1.6 simply provides a special case of definition 2.5.20. □

Remark 4.2.8. Note that these t-structures are never bounded and hence useless as data of a stability condition. Therefore the fact that semiorthogonal decompositions are in fact t-structures is best disregarded in the context that we are working in.

We can, however, use the t-structure $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ to prove the following.

Lemma 4.2.9. *Let* $(\mathcal{TR}^1, \mathcal{TR}^2)$ *be a semiorthogonal decomposition. Then* $(\mathcal{TR}^2)^\perp = \mathcal{TR}^1$ *and* ${}^\perp(\mathcal{TR}^1) = \mathcal{TR}^2$.

Proof. Combine lemma 4.2.7 with lemma 4.2.6. □

Lemma 4.2.10. *For categories* $\mathcal{A}, \mathcal{B}$ *and* $f \dashv g$ *an adjoint pair of functors* $f : \mathcal{A} \to \mathcal{B}$ *and* $g : \mathcal{B} \to \mathcal{A}$*. Then*

$$\ker(f) = {}^\perp(\operatorname{im}(g)).$$

Proof. Assume $E \in \ker(f)$, for any $F \in \mathcal{B}$ we obtain

$$\operatorname{Hom}(E, g(F)) = \operatorname{Hom}(f(E), F) = \operatorname{Hom}(0, F) = 0$$

implying $E \in {}^\perp(\operatorname{im}(g))$ and hence $\ker(f) \subset {}^\perp(\operatorname{im}(g))$.

Assume now $E \in {}^\perp(\operatorname{im}(g))$ to obtain

$$\operatorname{Hom}(f(E), f(E)) = \operatorname{Hom}(E, gf(E)) = 0$$

which implies $f(E) = 0$ and therefore $E \in \ker(f)$. Hence ${}^\perp(\operatorname{im}(g)) \subset \ker(f)$ and the proof is finished. □

Lemma 4.2.11. *For categories* $\mathcal{A}, \mathcal{B}$ *and* $f \dashv g$ *an adjoint pair of functors* $f : \mathcal{A} \to \mathcal{B}$ *and* $g : \mathcal{B} \to \mathcal{A}$*. Then*

$$\ker(g) = (\operatorname{im}(f))^\perp.$$

Proof. Similar to 4.2.10. □

Lemma 4.2.12. *There is an equality* $\mathcal{D}_1 = {}^\perp\mathcal{D}_3$.

Proof. Since we have the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, we have $(\mathcal{D}_2)^\perp = \mathcal{D}_1$ and ${}^\perp(\mathcal{D}_1) = \mathcal{D}_2$ by lemma 4.2.9. Using the lemmas 4.2.10 and 4.2.11 together with the adjunctions $i_2, \dashv \rho_2$ and $\rho_2 \dashv \Delta$ we obtain

$$\mathcal{D}_1 = (\mathcal{D}_2)^\perp = (\operatorname{im}(i_2))^\perp = \ker(\rho_2) = {}^\perp(\operatorname{im} \Delta) = {}^\perp(\mathcal{D}_3).$$

□

In analogy to this we have the next lemma.

Lemma 4.2.13. *There is an equality* $\mathcal{D}_3^\perp = \mathcal{D}_2$.

Proof. Similar to the proof of lemma 4.2.12. □

We sum up with the following corollary.

Corollary 4.2.14. *The following equalities hold true.*

1. $\mathcal{D}_2^{\perp} = \mathcal{D}_1$
2. ${}^{\perp}\mathcal{D}_1 = \mathcal{D}_2$
3. $\mathcal{D}_1^{\perp} = \mathcal{D}_3$
4. ${}^{\perp}\mathcal{D}_2 = \mathcal{D}_3$
5. $\mathcal{D}_3^{\perp} = \mathcal{D}_2$
6. ${}^{\perp}\mathcal{D}_3 = \mathcal{D}_1$

Proof. Proving this in order of appearance we have

1. This was shown as part of the proof of lemma 4.2.12.
2. This was shown as part of the proof of lemma 4.2.12.
3. By lemma 4.2.12 we have $\mathcal{D}_1 = {}^{\perp}\mathcal{D}_3$. Hence, using corollary 4.2.5 we obtain $(\mathcal{D}_1)^{\perp} = ({}^{\perp}\mathcal{D}_3)^{\perp} = \mathcal{D}_3$.
4. Similar to the previous, this time use lemma 4.2.13 again combined with corollary 4.2.5.
5. By lemma 4.2.13.
6. By lemma 4.2.12.

□

It is at this point that we introduce a very important fact that can now be easily established.

Theorem 4.2.15. *We have* $\mathcal{D}^{\uparrow} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle, \mathcal{D}^{\uparrow} = \langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ *and* $\mathcal{D}^{\uparrow} = \langle \mathcal{D}_2, \mathcal{D}_3 \rangle$.

Proof. The equality $\mathcal{D}^{\uparrow} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is provided by proposition 3.2.22. For the others combine lemma A.1.6 with corollaries 4.2.2 and 4.2.14. □

Now returning to the development of the theory of Serre functors on $\mathcal{D}^{\uparrow}$, corollary 4.2.14 allows us to prove the following lemma.

Lemma 4.2.16. *The subcategory* $\mathcal{D}_1$ *of* $\mathcal{D}^{\uparrow}$ *is admissible.*

Proof. We obtain $\mathcal{D}_1$ left admissible from the fact that $\mathcal{D}^{\uparrow} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ (proposition 3.2.22). We must – therefore – prove that $\mathcal{D}_1$ is right admissible too. Since $\mathcal{D}_3$ is admissible by corollary 4.2.2, we obtain an exact triangle

$$A \to X \to B \xrightarrow{+} \tag{4.1}$$

where $A \in {}^{\perp}\mathcal{D}_3$ and $B \in \mathcal{D}_3$. By corollary 4.2.14, we obtain ${}^{\perp}\mathcal{D}_3 = \mathcal{D}_1$ and $\mathcal{D}_1^{\perp} = \mathcal{D}_3$. This means, the exact triangle (4.1) now becomes one where $A \in \mathcal{D}_1$ and $B \in \mathcal{D}_1^{\perp}$. In other words, $\mathcal{D}_1$ is right admissible and combined with the fact that it is also left admissible as we saw at the beginning, it is indeed admissible. □

And we also obtain the statement for $\mathcal{D}_2$.

Lemma 4.2.17. *The subcategory $\mathcal{D}_2$ of $\mathcal{D}^{\uparrow}$ is admissible.*

Proof. We obtain $\mathcal{D}_2$ right admissible from the fact that $\mathcal{D}^{\uparrow} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ (proposition 3.2.22). We must – therefore – prove that $\mathcal{D}_2$ is left admissible too. Since $\mathcal{D}_3$ is admissible by corollary 4.2.2, we obtain an exact triangle

$$A \to X \to B \xrightarrow{+} \tag{4.2}$$

where $A \in \mathcal{D}_3$ and $B \in (\mathcal{D}_3)^{\perp}$. By corollary 4.2.14, we obtain ${}^{\perp}\mathcal{D}_2 = \mathcal{D}_3$ and $(\mathcal{D}_3)^{\perp} = \mathcal{D}_2$. This means, the exact triangle (4.2) now becomes one where $A \in {}^{\perp}\mathcal{D}_2$ and $B \in \mathcal{D}_2$. In other words, $\mathcal{D}_2$ is left admissible and combined with the fact that it is also right admissible as we saw at the beginning, it is indeed admissible. □

Remark 4.2.18. After developing a lot of theory over section 3 and this one we are now able to prove theorem 4.2.19 without extra assumptions. In [49] the additional assumption of $\mathcal{D}$ being right- and left-saturated (see [12, Definition 2.5] for the meaning of "saturated") had to be made, which provides a short cut to the admissibility of $\mathcal{D}_2$ via [12, Proposition 2.6].

The admissibility shown before has the following nice consequence, that we will need later.

Theorem 4.2.19. *If $\mathcal{D}$ has a Serre functor then so has $\mathcal{D}^{\uparrow}$.*

Proof. For the categories $\mathcal{B}$ and $\mathcal{C}$ from theorem A.1.15 we let $\mathcal{B} = \mathcal{D}_3$, which, by corollary 4.2.2 is indeed admissible. By corollary 4.2.14, we have $(\mathcal{D}_3)^{\perp} = \mathcal{D}_2$ and therefore $\mathcal{C} = \mathcal{B}^{\perp} = (\mathcal{D}_3)^{\perp} = \mathcal{D}_2$. Since, by 4.2.17, $\mathcal{D}_2$ is admissible and, additionally both $\mathcal{D}_3$ and $\mathcal{D}_2$, as copies of $\mathcal{D}$ have a Serre functor, we can apply theorem A.1.15 to see that $\mathcal{D}^{\uparrow}$ has a Serre functor. □

We therefore introduce new language.

Notation 4.2.20. Let $S_{\mathcal{D}}$ denote the Serre functor on $\mathcal{D}$ and $S_{\mathcal{D}^\uparrow}$ the Serre functor on $\mathcal{D}^\uparrow$.

An immediate consequence of theorem 4.2.19, in combination with theorem 4.2.15, is that we can now compute additional hearts of t-structures.

We harvest from the previous work in order to provide recollement-data. At first we need to introduce a new functor.

Lemma 4.2.21. *The functor i_1 has a right adjoint.*

Proof. Combine lemma 4.2.16 with lemma 4.2.1. □

Definition 4.2.22. Define $\mathbb{K}$ to be the right adjoint functor of i_1.

Remark 4.2.23. The existence of $\mathbb{K}$ is granted by lemma 4.2.21.

Corollary 4.2.24. *Assume $\mathcal{D}$ has Serre functor. Let $\mathcal{P}$ be a standard slicing and $H_1 = \mathcal{P}(\alpha, \alpha+1], H_2 = \mathcal{P}(\beta, \beta+1]$ be hearts of t-structures on $\mathcal{D}$.*

- *There is a heart $H = \{E \in \mathcal{D}^\uparrow \mid \lambda_1(E) \in H_1, \mathbb{K}(E) \in H_2\}$ of a bounded t-structure on $\mathcal{D}^\uparrow$ obtained by CP-gluing via $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ from H_1, H_2 if $\alpha \geq \beta + 1$.*
- *There is a heart $H = \{E \in \mathcal{D}^\uparrow \mid \mathbb{K}(E)[1] \in H_1, \rho_2(E) \in H_2\}$ of a bounded t-structure on $\mathcal{D}^\uparrow$ obtained by CP-gluing via $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ from H_1, H_2 if $\alpha \geq \beta + 1$.*

Proof. To see that there is a heart H of a bounded t-structure on $\mathcal{D}^\uparrow$ obtained by CP-gluing via $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ (that theorem 4.2.15 provides) from H_1, H_2 if $\alpha \geq \beta + 1$ we use

$$\begin{aligned} \operatorname{Hom}^{\leq 0}(\Delta H_1, i_1 H_2) &= \operatorname{Hom}^{\leq 0}(\Delta H_1[-1], i_1 H_2[-1]) \\ = \operatorname{Hom}^{\leq 0}(S^{-1}_{\mathcal{D}^\uparrow} i_1 H_1, S^{-1}_{\mathcal{D}^\uparrow} i_2 H_2[1]) &= \operatorname{Hom}^{\leq 0}(i_1 H_1, i_2 H_2[1]) \end{aligned}$$

which we obtain from theorems 4.2.19 and A.1.16. The result now follows from lemmas 3.2.30 and 3.1.5 ([21, Lemma 2.1]). The proof that there is a heart H of a bounded t-structure on $\mathcal{D}^\uparrow$ obtained by CP-gluing via $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ from H_1, H_2 if $\alpha \geq \beta + 1$ is similar. □

We will use the admissibility of the categories $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ to prove the main theorem of this section (4.3.15). The proof is broken into a series of lemmas.

As a result of the adjunction of $\mathbb{K}$ and i_1 we can now introduce the vanishing of a composition of functors, that we also require.

Lemma 4.2.25. *For Δ as in lemma 3.2.8 we have*

$$\mathbb{K} \circ \Delta = 0.$$

Proof. By corollary 3.2.15, we have $\rho_2 \circ i_1 = 0$. Now, using the adjunctions provided by lemma 3.2.24 and lemma 4.2.21 in combination with definition 4.2.22, we obtain for any $F \in \mathcal{D}$,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(\mathbb{K}(\Delta(F)), \mathbb{K}(\Delta(F))) &\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(\mathbb{K}(\Delta(F))), \Delta(F)) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\rho_2(i_1(\mathbb{K}(\Delta(F)))), F) = \mathrm{Hom}_{\mathcal{D}}(0, F) = 0,\end{aligned}$$

which implies $\mathbb{K} \circ \Delta = 0$. □

Lemma 4.2.26. *For any $E \in \mathcal{D}$ there are exact triangles*

1. $i_1(\mathbb{K}(E)) \to E \to \Delta(\rho_2(E)) \xrightarrow{+}$
2. $\Delta(\lambda_1(E)) \to E \to i_2(\mathbb{C}(E)) \xrightarrow{+}$ *where $\mathbb{C}$ is defined to be the left adjoint functor of i_2 (which exists by the combined statements of lemma 4.2.17 and lemma 4.2.1).*

Proof. By theorem 4.2.15 we have semiorthogonal decompositions $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ and $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ of $\mathcal{D}^{\uparrow}$ where $\mathbb{K}, \mathbb{C}$ and Δ are the left and right adjoint functors to the respective inclusions. □

We proceed with the following technical lemmas.

Lemma 4.2.27. *We have $\mathbb{K} \circ i_2[1] \cong \mathrm{id}_{\mathcal{D}}$.*

Proof. For any $E \in \mathcal{D}^{\uparrow}$ we have an exact triangle

$$i_2(\rho_2(E)) \to E \to i_1(\lambda_1(E)) \xrightarrow{+} .$$

Hence, for any $F \in \mathcal{D}$ we obtain

$$i_2(\rho_2(\Delta(F))) \to \Delta(F) \to i_1(\lambda_1(\Delta(F))) \xrightarrow{+} .$$

Now, since $\rho_2 \circ \Delta = \mathrm{id}_{\mathcal{D}} = \lambda_1 \circ \Delta$ this becomes the exact triangle

$$i_2(F) \to \Delta(F) \to i_1(F) \xrightarrow{+} .$$

Using $\mathbb{K}$ on this we obtain the exact triangle

$$\mathbb{K}(i_2(F)) \to \mathbb{K}(\Delta(F)) \to \mathbb{K}(i_1(F)) \xrightarrow{+} .$$

But $\mathbb{K} \circ \Delta = 0$ by lemma 4.2.25 and hence we obtain $\mathbb{K}(i_1(F)) \cong \mathbb{K}(i_2(F))[1]$. Since i_1 is fully faithful and $i_1 \dashv \mathbb{K}$, we have $\mathrm{id}_{\mathcal{D}} \cong \mathbb{K} \circ i_1$. Hence $\mathrm{id}_{\mathcal{D}} \cong \mathbb{K} \circ i_2[1]$. □

Lemma 4.2.28. *Assume $\mathcal{D}$ has a Serre functor. We have $\mathbb{K} = \mathbb{C}[-1]$.*

Proof. Define $j_!$ to be the left adjoint functor of $\mathbb{C}$. Its existence is due to the combination of the statements of theorem 4.2.19 and theorem A.1.16. Since i_2 is fully faithful, so is $j_!$ and we define $\widetilde{\mathcal{D}}_! = \mathrm{im}(j_!)$. Hence, using lemma 4.2.11 and corollary 4.2.14 we obtain

$$(\widetilde{\mathcal{D}}_!)^\perp = (\mathrm{im}(j_!))^\perp = \ker(\mathbb{C}) = {}^\perp\mathcal{D}_2 = \mathcal{D}_3.$$

Moreover, $\widetilde{\mathcal{D}}_!$ is right admissible by lemma 4.2.1 which implies that for any $E \in \mathcal{D}^\uparrow$ there is an exact triangle

$$j_!(\mathbb{C}(E)) \to E \to \Delta(\rho_2(E)) \xrightarrow{+} .$$

Since, by lemma 4.2.26, there is also an exact triangle

$$i_1(\mathbb{K}(E)) \to E \to \Delta(\rho_2(E)) \xrightarrow{+},$$

we obtain $i_1(\mathbb{K}(E)) \cong j_!(\mathbb{C}(E))$. By lemma 4.2.27 we have $\mathrm{id}_{\mathcal{D}} \cong \mathbb{K} \circ i_2[1]$. Now, since i_2 is fully faithful and $\mathbb{C} \dashv i_2$, we obtain $\mathbb{C} \circ i_2 \cong \mathrm{id}_{\mathcal{D}}$, giving

$$i_1 = i_1 \circ \mathrm{id}_{\mathcal{D}} \cong i_1 \circ \mathbb{K} \circ i_2[1] \cong j_! \circ \mathbb{C} \circ i_2[1] \cong j_![1].$$

Since $i_1 \dashv \mathbb{K}$ is an adjoint pair. Hence we also have $j_![1] \dashv \mathbb{K}$ and therefore $\mathbb{K} \cong \mathbb{C}[-1]$.

□

4.3 Application of recollement to $\mathcal{D}^\uparrow$ with regard to the theory of Serre functors

We are now ready to make use of our previous preparations in order to introduce recollement data on $\mathcal{D}^\uparrow$.

Lemma 4.3.1. *For*

$$i_! = i_* = i_1, i^* = \lambda_1, j_! = i_2, j^* = j^! = \rho_2, j_* = \Delta, i^! = \mathbb{K},$$

the adjunctions of definition 4.1.1 are fulfilled.

Proof. This is provided by lemmas 3.2.11, 3.2.24 and the combination of lemma 4.2.21 with definition 4.2.22. □

Lemma 4.3.2. *Keep the assumptions of lemma 4.3.1. Then for $Z \in \mathcal{D}^{\uparrow}$,*

$$i_1\mathbb{K}(Z) \to Z \to \Delta\rho_2(Z) \xrightarrow{+} \tag{4.3}$$

and

$$i_2\rho_2(Z) \to Z \to i_1\lambda_1(Z) \xrightarrow{+} \tag{4.4}$$

are exact triangles.

Proof. The existence of (4.3) is due to lemma 4.2.26, the existence of (4.4) to proposition 3.2.22. □

It is at this point that we shall – prior to the continuation of the provision of the necessary facts that will lead to the usage of recollement – provide a small excursion on an implication that lemma 4.3.2 has.

Lemma 4.3.3. *For any $Z \in \mathcal{D}^{\uparrow}$, there is an exact triangle*

$$\mathbb{K}(Z) \to \lambda_1(Z) \to \rho_2(Z) \xrightarrow{+}$$

on $\mathcal{D}$.

Proof. By lemma 4.3.2,

$$i_1\mathbb{K}(Z) \to Z \to \Delta\rho_2(Z) \xrightarrow{+} \tag{4.5}$$

is an exact triangle. Since λ_1 is an exact functor, we obtain the exact triangle

$$\lambda_1(i_1\mathbb{K}(Z)) \to \lambda_1(Z) \to \lambda_1(\Delta(\rho_2(Z))) \xrightarrow{+}$$

on $\mathcal{D}$. By corollary 3.2.19 we have

$$\lambda_1(i_1(\mathbb{K}(Z))) \xrightarrow{\cong} \mathbb{K}(Z)$$

and, similarly to 3.2.19 we also obtain $\mathrm{id}_{\mathcal{D}} \xrightarrow{\cong} \lambda_1 \circ \Delta$, implying

$$\rho_2(Z) \xrightarrow{\cong} \lambda_1(\Delta(\rho_2(Z))),$$

which finishes the proof. □

This provides us with the following – quite useful – fact that will be presented next.

Corollary 4.3.4. *For $Z \in \mathcal{D}^{\uparrow}$ where $Z = (Z_1 \xrightarrow{f} Z_2)$, let*

$$\mathbb{K}(Z) \to \lambda_1(Z) \xrightarrow{\mu_Z} \rho_2(Z) \xrightarrow{+}$$

be the exact triangle of lemma 4.3.3. Then $\mu_Z = f \circ (\mathrm{quis})^{-1}$ where quis *is a quasi-isomorphism in $\mathcal{K}(\mathcal{A}^{\uparrow})$.*

Proof. Let $Z = (E_1 \xrightarrow{f} E_2)$. The three objects $(E_1 \xrightarrow{f} E_2), (E_2 \xrightarrow{\mathrm{id}_{E_2}} E_2)$ and $(\mathrm{Cone}(f) \to 0) \in \mathcal{D}^{\uparrow}$ fit into the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{f} & E_2 & \xrightarrow{i_{E_2}} & \mathrm{Cone}(f) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \mathrm{id}_{E_2} & & \downarrow & & \\ 0 & \longrightarrow & E_2 & \xrightarrow{i_{E_2}} & E_2 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

in $\mathcal{C}(\mathcal{A})$ and therefore into the corresponding exact triangle in $\mathcal{D}^{\uparrow}$. Hence, we obtain an exact triangle

$$(0 \to \mathrm{Cone}(f)[-1]) \to (E_1 \xrightarrow{f} E_2) \xrightarrow{\xi} (E_2 \xrightarrow{\mathrm{id}_{E_2}} E_2) \xrightarrow{+} \tag{4.6}$$

where $\xi =$

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow f & & \downarrow \mathrm{id}_{E_2} \\ E_2 & \xrightarrow{\mathrm{id}_{E_2}} & E_2. \end{array}$$

Since $(0 \to \mathrm{Cone}(f)[-1]) \in \mathcal{D}_1$ and $(E_2 \xrightarrow{\mathrm{id}_{E_2}} E_2) \in \mathcal{D}_3$, we see that (4.6) is a decomposition-triangle for the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_3 \rangle$ which exists by theorem 4.2.15. But – on the other hand – so is (4.3) and since decomposition-triangles are unique up to isomorphisms by lemma 3.1.1, we obtain (4.3) $\cong$ (4.6). Therefore we obtain $\lambda_1((4.3)) \cong \lambda_1((4.6))$ which implies $\mu_Z \cong \lambda_1(\xi) \cong f$. The proof is now finished. □

Remark 4.3.5. Note that we will form now on refrain from the – formally correct – referring to μ as $f \circ (\mathrm{quis})^{-1}$ and simply say that it equals to f.

Furthermore, we obtain the following nice result.

Corollary 4.3.6. *For any $Z = (Z_1 \xrightarrow{f} Z_2) \in \mathcal{D}^{\uparrow}$ we obtain the equality $\mathbb{K}(Z) \cong \mathrm{Cone}(f)[-1]$.*

Proof. Extend the morphism f to the exact triangle

$$Z_1 \xrightarrow{f} Z_2 \to \mathrm{Cone}(f) \xrightarrow{+} .$$

Since $\lambda_1(Z) = Z_1$ and $\rho_2(Z) = Z_2$, we obtain the result by considering the exact triangle

$$\mathbb{K}(Z) \to \lambda_1(Z) \xrightarrow{\mu_Z} \rho_2(Z) \xrightarrow{+}$$

from lemma 4.3.3 together with that statement of corollary 4.3.4. □

Remark 4.3.7. Note that we can too obtain the result $\mathbb{K}(Z) \cong \mathrm{Cone}(f)[-1]$ from the isomorphism of triangles (4.3) $\cong$ (4.6) that we used in corollary 4.3.4.

Lemma 4.3.8. *Keep the assumptions of lemma 4.3.1. Then $i_* = i_1, j_* = \Delta$ and $j_! = i_2$ are full embeddings.*

Proof. Let $A, B \in \mathcal{D}^\uparrow$ and $f \in \mathrm{Hom}_{\mathcal{D}^\uparrow}(i_1(A), i_1(B))$ then $f = (f_1, 0)$ where $f_1 \in \mathrm{Hom}_{\mathcal{D}}(A, B)$. Hence the map

$$\begin{aligned} i_1 : \mathrm{Hom}_{\mathcal{D}}(A, B) \to \mathrm{Hom}_{\mathcal{D}^\uparrow}(i_1(A), i_1(B)) \\ i_1(f) = (f, 0) \end{aligned}$$

is surjective – and therefore i_1 is full. In the same manner we obtain that i_2 and Δ are full. □

Remark 4.3.9. Note that the fact that Δ is full was already proven in the general situation where one does not have a Serre functor available – see lemma 3.2.28. However, the availability of the Serre functor makes the proofs a whole lot easier and less tedious.

Theorem 4.3.10. *Keep the assumptions of lemma 4.3.1. Then recollement data on $\mathcal{D}^\uparrow$ is provided by two copies of $\mathcal{D}$ via the functors given in lemma 4.3.1.*

Proof. The adjunction-condition of definition 4.1.1 is given by lemma 4.3.1. The vanishing condition on $\mathbb{K} \circ \Delta$ is provided by lemma 4.2.25, the fullness of the embeddings by lemma 4.3.8 and finally the two exact triangles by lemma 4.3.2. □

We can hence apply the theory developed in [10] to combine any two bounded t-structures on $\mathcal{D}$.

Corollary 4.3.11. *Let* $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ *and* $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ *be t-structures on* $\mathcal{D}$. *Then* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$*, defined by*

$$\mathcal{D}^{\leq 0} = \{Z \in \mathcal{D}^{\uparrow} \mid \rho_2(Z) \in \mathcal{D}_2^{\leq 0}, \lambda_1(Z) \in \mathcal{D}_1^{\leq 0}\}$$
$$\mathcal{D}^{\geq 1} = \{Z \in \mathcal{D}^{\uparrow} \mid \rho_2(Z) \in \mathcal{D}_2^{\geq 1}, \mathbb{K}(Z) \in \mathcal{D}_1^{\geq 1}\}$$

is a t-structure on $\mathcal{D}^{\uparrow}$.

Proof. We combine lemma 4.1.3 with theorem 4.3.10. □

We have seen that we can obtain a t-structure on $\mathcal{D}^{\uparrow}$ by combining any two t-structures on $\mathcal{D}$. In some of these cases the result of this process can be understood by applying corollary 3.2.36. It is hence our task to understand what the extra data gained by recollement means in terms of $\mathrm{Stab}(\mathcal{D}^{\uparrow})$.

Definition 4.3.12. Recollement-data on $\mathcal{D}^{\uparrow}$, given by

$$i_! = i_* = i_1, i^* = \lambda_1, j_! = i_2, j^* = j^! = \rho_2, j_* = \Delta, i^! = \mathbb{K}$$

will be called "type-1-recollement-data".

At this point it is not clear, whether this definition is of a mere theoretical nature, or if indeed other recollement-datas can be chosen. We will see now that the latter is – indeed – the case.

Lemma 4.3.13. *Assume* $\mathcal{D}$ *has a Serre functor. Recollement-data on* $\mathcal{D}^{\uparrow}$ *is given by*

$$i_! = i_* = \Delta, i^* = \rho_2, j_! = i_1, j^* = j^! = \mathbb{K}, j_* = i_2[1], i^! = \lambda_1$$

Proof. All adjunctions of definition 4.1.1 and the fullness are fulfilled – the key-point being $\mathbb{K} = \mathbb{C}[-1]$ by lemma 4.2.28, which implies $\mathbb{K} \dashv i_2[1]$. The existence of the required exact triangles is due to the lemmas 4.3.2 and 4.2.26. □

Lemma 4.3.14. *Assume* $\mathcal{D}$ *has a Serre functor. Recollement-data on* $\mathcal{D}^{\uparrow}$ *is given by*

$$i_! = i_* = i_2, i^* = \mathbb{C}, j_! = \Delta, j^* = j^! = \lambda_1, j_* = i_1, i^! = \rho_2.$$

Proof. Similar to the proof of 4.3.13 but without the implication of lemma 4.2.28. □

Theorem 4.3.15. *Three sets of recollement-data on* $\mathcal{D}^{\uparrow}$ *are given by the functors*

1. *equation* (4.3.12),

2. $i_! = i_* = \Delta, i^* = \rho_2, j_! = i_1, j^* = j^! = \mathbb{K}, j_* = i_2[1], i^! = \lambda_1,$

3. $i_! = i_* = i_2, i^* = \mathbb{C}, j_! = \Delta, j^* = j^! = \lambda_1, j_* = i_1, i^! = \rho_2.$

Proof. Use lemmas 4.3.13 and 4.3.14 for parts two and three. □

Definition 4.3.16. In extension of definition 4.3.12 we define the second and third set of recollement-data of theorem 4.3.15 to be "type-2-recollement-data" and "type-3-recollement-data", respectively.

We can hence use this to define more t-structures on $\mathcal{D}^{\uparrow}$, provided by the following.

Corollary 4.3.17. *There are t-structures on* $\mathcal{D}^{\uparrow}$ *given by*

1.

$$\begin{aligned}\mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow} \mid \rho_2(Z) \in \mathcal{D}_2^{\leq 0}, \lambda_1(Z) \in \mathcal{D}_1^{\leq 0}\}\\ \mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow} \mid \rho_2(Z) \in \mathcal{D}_2^{\geq 1}, \mathbb{K}(Z) \in \mathcal{D}_1^{\geq 1}\}\end{aligned}$$

that we refer to as t-structures obtained via type-1-recollement-data,

2.

$$\begin{aligned}\mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow} \mid \mathbb{K}(Z) \in \mathcal{D}_2^{\leq 0}, \rho_2(Z) \in \mathcal{D}_1^{\leq 0}\}\\ \mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow} \mid \mathbb{K}(Z) \in \mathcal{D}_2^{\geq 1}, \lambda_1(Z) \in \mathcal{D}_1^{\geq 1}\},\end{aligned}$$

that we refer to as t-structures obtained via type-2-recollement-data,

3.

$$\begin{aligned}\mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow} \mid \lambda_1(Z) \in \mathcal{D}_2^{\leq 0}, \mathbb{C}(Z) \in \mathcal{D}_1^{\leq 0}\}\\ \mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow} \mid \lambda_1(Z) \in \mathcal{D}_2^{\geq 1}, \rho_2(Z) \in \mathcal{D}_1^{\geq 1}\}\end{aligned}$$

that we refer to as t-structures obtained via type-3-recollement-data,

for t-structures $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ *and* $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ *on* $\mathcal{D}$.

Proof. The first is by 4.3.11, the second and the third are by 4.3.15 combined with lemma 4.1.3. □

Definition 4.3.18. Let t-structures on $\mathcal{D}$ be given by

$$(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1}) = (\mathcal{P}(\alpha, \infty), \mathcal{P}(-\infty, \alpha]), (\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1}) = (\mathcal{P}(\beta, \infty), \mathcal{P}(-\infty, \beta])$$

where $\mathcal{P}$ denotes the standard slicing. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ that is obtained by recollement from t-structures $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ and $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ via type-1-recollement-data, type-2-recollement-data or type-3-recollement-data in the sense of corollary 4.3.17, will be denoted $(\mathcal{D}_{1,\alpha,\beta}^{\leq 0}, \mathcal{D}_{1,\alpha,\beta}^{\geq 1})$, $(\mathcal{D}_{2,\alpha,\beta}^{\leq 0}, \mathcal{D}_{2,\alpha,\beta}^{\geq 1})$ or $(\mathcal{D}_{3,\alpha,\beta}^{\leq 0}, \mathcal{D}_{3,\alpha,\beta}^{\geq 1})$ respectively.

Lemma 4.3.19. *For a t-structure $(\mathcal{D}_{2,\beta,\alpha}^{\leq 0}, \mathcal{D}_{2,\alpha,\beta}^{\geq 1})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha \leq \beta - 1$, then $(\mathcal{D}_{2,\beta,\alpha}^{\leq 0}, \mathcal{D}_{2,\beta,\alpha}^{\geq 1}) = (\mathcal{D}_{1,\alpha,\beta}^{\leq 0}, \mathcal{D}_{1,\alpha,\beta}^{\geq 1})$.*

Proof. Assume $X \in \mathcal{D}_{2,\beta,\alpha}^{\leq 0} = \{X \mid \mathbb{K}(X) \in \mathcal{D}_1^{\leq 0}, \rho_2(X) \in \mathcal{D}_2^{\leq 0}\}$. Since $\alpha \leq \beta - 1$ was assumed, we have $\alpha \leq \beta$ and therefore $\mathcal{D}_2^{\leq 0} = \mathcal{P}(\beta, \infty) \subset \mathcal{P}(\alpha, \infty) = \mathcal{D}_1^{\leq 0}$. Hence $\rho_2(X) \in \mathcal{D}_1^{\leq 0}$. By lemma 4.3.3, there is an exact triangle

$$\mathbb{K}(X) \to \lambda_1(X) \to \rho_2(X) \xrightarrow{+} \tag{4.7}$$

and since $\mathbb{K}(X) \in \mathcal{D}_1^{\leq 0}$ and $\rho_2(X) \in \mathcal{D}_1^{\leq 0}$, we deduce from the fact that $\mathcal{D}_1^{\leq 0}$ is extension closed, that $\lambda_1(X) \in \mathcal{D}_1^{\leq 0}$. Hence, we now obtain that $X \in \{X \mid \lambda_1(X) \in \mathcal{D}_1^{\leq 0}, \rho_2(X) \in \mathcal{D}_2^{\leq 0}\} = \mathcal{D}_{1,\alpha,\beta}^{\leq 0}$. In other words, we have $\mathcal{D}_{2,\beta,\alpha}^{\leq 0} \subset \mathcal{D}_{1,\alpha,\beta}^{\leq 0}$.

On the other hand, if $X \in \mathcal{D}_{1,\alpha,\beta}^{\leq 0} = \{X \mid \lambda_1(X) \in \mathcal{D}_1^{\leq 0}, \rho_2(X) \in \mathcal{D}_2^{\leq 0}\}$, we obtain that $\rho_2(X)[-1] \in \mathcal{D}_2^{\leq 0}[-1] = \mathcal{P}(\beta, \infty)[-1] = \mathcal{P}(\beta - 1, \infty) \subset \mathcal{P}(\alpha, \infty) = \mathcal{D}_1^{\leq 0}$ since $\alpha \leq \beta - 1$ was assumed. From the exact triangle (4.7) we now obtain the exact triangle

$$\rho_2(X)[-1] \to \mathbb{K}(X) \to \lambda_1(X) \xrightarrow{+} \tag{4.8}$$

and deduce in the same way as we did before, that $X \in \mathcal{D}_{2,\beta,\alpha}^{\leq 0}$. In other words, we have $\mathcal{D}_{1,\alpha,\beta}^{\leq 0} \subset \mathcal{D}_{2,\beta,\alpha}^{\leq 0}$. This means that $\mathcal{D}_{1,\alpha,\beta}^{\leq 0} = \mathcal{D}_{2,\beta,\alpha}^{\leq 0}$. Now, applying lemma 4.2.6, we obtain

$$\mathcal{D}_{1,\alpha,\beta}^{\geq 1} = {}^{\perp}\mathcal{D}_{1,\alpha,\beta}^{\leq 0} = {}^{\perp}\mathcal{D}_{2,\beta,\alpha}^{\leq 0} = \mathcal{D}_{2,\beta,\alpha}^{\geq 1}$$

and the proof is finished. □

We obtain the alternative statement as well.

Lemma 4.3.20. *For a t-structure $(\mathcal{D}_{2,\beta,\alpha}^{\leq 0}, \mathcal{D}_{2,\beta,\alpha}^{\geq 1})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha \geq \beta$, then $(\mathcal{D}_{2,\alpha,\beta}^{\leq 0}, \mathcal{D}_{2,\alpha,\beta}^{\geq 1}) = (\mathcal{D}_{3,\alpha+1,\beta}^{\leq 0}, \mathcal{D}_{3,\alpha+1,\beta}^{\geq 1})$.*

Proof. Unlike the procedure of lemma 4.3.19, this time we will prove that $\mathcal{D}^{\geq 1}_{3,\alpha+1,\beta} = \mathcal{D}^{\geq 1}_{2,\beta,\alpha}$. Other then that, the proof is quite similar. To obtain the inclusion $\mathcal{D}^{\geq 1}_{2,\beta,\alpha} \subset \mathcal{D}^{\geq 1}_{3,\alpha+1,\beta}$ we use the exact triangle

$$\lambda_1(X) \to \rho_2(X) \to \mathbb{K}(X)[1] \stackrel{+}{\to}$$

that we obtain from the exact triangle (4.7). Then $\mathbb{K}(X)[1] \in \mathcal{P}(-\infty,\alpha][1] = \mathcal{P}(-\infty,\alpha+1]$ and $\lambda_1(X) \in \mathcal{P}(-\infty,\beta] \subset \mathcal{P}(-\infty,\alpha]$ (because of $\alpha \geq \beta$) and hence $\lambda_1(X) \in \mathcal{P}(-\infty,\alpha] \subset \mathcal{P}(-\infty,\alpha+1]$ provide us with $\rho_2(X) \in \mathcal{P}(-\infty,\alpha+1]$.

For the inclusion $\mathcal{D}^{\geq 1}_{3,\alpha+1,\beta} \subset \mathcal{D}^{\geq 1}_{2,\beta,\alpha}$, on the other hand, we see that if $\rho_2(X) \in \mathcal{P}(-\infty,\alpha+1]$, then $\rho_2(X)[-1] \in \mathcal{P}(-\infty,\alpha+1][-1] = \mathcal{P}(-\infty,\alpha]$ and since $\alpha \geq \beta$ implies $\mathcal{P}(-\infty,\beta] \subset \mathcal{P}(-\infty,\alpha]$ and we therefore obtain $\lambda_1(X) \in \mathcal{P}(-\infty,\beta] \subset \mathcal{P}(-\infty,\alpha]$, we obtain from (4.8) that $\mathbb{K}(X) \in \mathcal{P}(-\infty,\alpha]$. Again, applying lemma 4.2.6 finishes the proof. □

We combine lemmas 4.3.19 and 4.3.20 to the following proposition.

Proposition 4.3.21. *For a t-structure* $(\mathcal{D}^{\leq 0}_{2,\beta,\alpha}, \mathcal{D}^{\geq 1}_{2,\beta,\alpha})$ *on* $\mathcal{D}^{\uparrow}$ *we have*

1. $(\mathcal{D}^{\leq 0}_{2,\beta,\alpha}, \mathcal{D}^{\geq 1}_{2,\beta,\alpha}) = (\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ *if* $\alpha \leq \beta - 1$ *and*
2. $(\mathcal{D}^{\leq 0}_{2,\beta,\alpha}, \mathcal{D}^{\geq 1}_{2,\beta,\alpha}) = (\mathcal{D}^{\leq 0}_{3,\alpha+1,\beta}, \mathcal{D}^{\geq 1}_{3,\alpha+1,\beta})$ *if* $\alpha \geq \beta$.

Proof. Apply lemmas 4.3.19 and 4.3.20. □

Remark 4.3.22. Of course, as for the hearts of these t-structures, this means, we can say at this point, that we do not obtain anything new either as long as $\alpha \notin (\beta - 1, \beta)$.

4.4 The Jealousy Lemma

This subsection deals with the question of the necessity of extending CP-gluing to recollement with regard to the computation of $\mathrm{Stab}(\mathcal{D}^{\uparrow})$. With respect to this aim, theorem 4.4.6 deals with the problem hinted at in remark 4.3.22.

Definition 4.4.1. Denote the heart of a t-structure

- $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ by $H_{1,\alpha,\beta}$,
- $(\mathcal{D}^{\leq 0}_{2,\alpha,\beta}, \mathcal{D}^{\geq 1}_{2,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ by $H_{2,\alpha,\beta}$,

- $(\mathcal{D}^{\leq 0}_{3,\alpha,\beta}, \mathcal{D}^{\geq 1}_{3,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ by $H_{3,\alpha,\beta}$.

The following lemma investigates in which situation a heart of a t-structure on $\mathcal{D}^{\uparrow}$ that is obtained by recollement via type-1-recollement-data contains hearts of bounded t-structures on $\mathcal{D}$ embedded into it in three different ways.

Lemma 4.4.2. *Let $\mathcal{P}$ be a slicing on $\mathcal{D}$ and $\alpha \leq \beta \leq \alpha + 1$ for $\alpha, \beta \in \mathbb{R}$. There are $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that*

1.

$$i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1]) \subset H_{1,\alpha,\beta} \tag{4.9}$$

if and only if $\gamma_1 = \alpha$

2.

$$i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1]) \subset H_{1,\alpha,\beta} \tag{4.10}$$

if and only if $\gamma_2 = \beta$ and

3.

$$\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1]) \subset H_{1,\alpha,\beta} \tag{4.11}$$

if and only if $\gamma_3 = \beta$.

Proof. The inclusion from (4.9), $i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1]) \subset H_{1,\alpha,\beta}$ holds if and only if $\rho_2(i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1])) \subset \mathcal{P}(\beta, \beta + 1], \lambda_1(i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1])) \subset \mathcal{P}(\alpha, \infty]$ and $\mathbb{K}(i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1] \subset \mathcal{P}(-\infty, \alpha + 1]))$. Since $\rho_2 \circ i_1 = 0$, we automatically obtain $\rho_2(i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1])) = 0 \subset \mathcal{P}(\beta, \beta + 1]$. On the other hand, we have $\lambda_1(i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1])) = \mathcal{P}(\gamma_1, \gamma_1 + 1] \subset \mathcal{P}(\alpha, \infty)$ and $\mathbb{K}(i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1])) = \mathcal{P}(\gamma_1, \gamma_1 + 1] \subset \mathcal{P}(-\infty, \alpha + 1]$ if and only if $\alpha = \gamma_1$.

Next, we have the inclusion $i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1]) \subset H_{1,\alpha,\beta}$ from (4.10) which holds if and only if $\rho_2(i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1])) \subset \mathcal{P}(\beta, \beta + 1], \lambda_1(i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1])) \subset \mathcal{P}(\alpha, \infty)$ and $\mathbb{K}(i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1])) \subset \mathcal{P}(-\infty, \alpha + 1]$. Since $\lambda_1 \circ i_2 = 0$, the inclusion $\lambda_1(i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1])) \subset \mathcal{P}(\alpha, \infty)$ is automatic. Then, we also obtain that $\rho_2(i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1])) = \mathcal{P}(\gamma_2, \gamma_2 + 1] \subset \mathcal{P}(\beta, \beta + 1]$ holds if and only if $\gamma_2 = \beta$. And, since $= \mathbb{K}(i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1])) = \mathcal{P}(\gamma_2, \gamma_2 + 1][-1] = \mathcal{P}(\gamma_2 - 1, \gamma_2] \subset \mathcal{P}(-\infty, \alpha + 1]$ holds if and only if $\gamma_2 \leq \alpha + 1$, we now have $\beta = \gamma_2 \leq \alpha + 1$ if and only if $i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1]) \subset H_{1,\alpha,\beta}$.

Finally, $\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1]) \subset H_{1,\alpha,\beta}$, from (4.11) which hold if and only if we – at the same time – have $\rho_2(\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1])) \subset \mathcal{P}(\beta, \beta + 1], \lambda_1(\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1])) \subset \mathcal{P}(\alpha, \infty)$ and $\mathbb{K}(\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1])) \subset \mathcal{P}(-\infty, \alpha + 1]$. First, $\mathbb{K} \circ \Delta = 0$ implies that $\mathbb{K}(\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1])) \subset \mathcal{P}(-\infty, \alpha + 1]$. Then, additionally, we have that $\rho_2(\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1])) = \mathcal{P}(\gamma_3, \gamma_3 + 1] \subset \mathcal{P}(\beta, \beta + 1]$ holds if and only if $\gamma_3 = \beta$. Since $\lambda_1(\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1])) = \mathcal{P}(\gamma_3, \gamma_3 + 1] \subset \mathcal{P}(\alpha, \infty)$ holds if and only if $\alpha \leq \gamma_3$, we obtain $\alpha \leq \gamma_3 = \beta$ if and only if $\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1]) \subset H_{1,\alpha,\beta}$. Hence, the proof is finished. □

We obtain a similar statement in the cases of $H_{2,\alpha,\beta}$ and $H_{3,\alpha,\beta}$.

Lemma 4.4.3. *We have*

- *let $\mathcal{P}$ be a slicing on $\mathcal{D}$. There are $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that*

  1. $i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1]) \subset H_{2,\alpha,\beta}$,
  2. $i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1]) \subset H_{2,\alpha,\beta}$ *and*
  3. $\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1]) \subset H_{2,\alpha,\beta}$.

  if and only if $\gamma_2 = \alpha, \gamma_1 = \gamma_3 = \beta$ and $\alpha - 1 \leq \beta \leq \alpha$. And

- *let $\mathcal{P}$ be a slicing on $\mathcal{D}$. There are $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ such that*

  1. $i_1(\mathcal{P}(\gamma_1, \gamma_1 + 1]) \subset H_{3,\alpha,\beta}$,
  2. $i_2(\mathcal{P}(\gamma_2, \gamma_2 + 1]) \subset H_{3,\alpha,\beta}$ *and*
  3. $\Delta(\mathcal{P}(\gamma_3, \gamma_3 + 1]) \subset H_{3,\alpha,\beta}$.

  if and only if $\gamma_3 = \alpha, \gamma_2 = \gamma_1 = \beta$ and $\alpha - 1 \leq \beta \leq \alpha$.

Proof. Similar to the proof of lemma 4.4.2. □

Lemma 4.4.4. *If $\alpha < \beta \leq \alpha + 1$, there is no stability condition with heart $H_{1,\alpha,\beta}$.*

Proof. Assume there is a stability function Z such that a stability condition σ is given by $H_{1,\alpha,\beta}$ and Z. We consider the imaginary part of Z which is given by

$$\begin{aligned}\Im(Z(X)) = D_1(\deg(\lambda_1(X))) + D_2(\deg(\rho_2(X)))\\ +C_1(\operatorname{rank}(\lambda_1(X))) + C_2(\operatorname{rank}(\rho_2(X)))\end{aligned}$$

for $X \in H_{1,\alpha,\beta}$ and $C_1, C_2, D_1, D_2 \in \mathbb{R}$. By lemma 4.4.2 we have that $i_1(\mathcal{P}(\alpha, \alpha{+}1]) \subset H_{1,\alpha,\beta}, i_2(\mathcal{P}(\beta, \beta{+}1]) \subset H_{1,\alpha,\beta}$ and $\Delta(\mathcal{P}(\beta_3, \beta_3{+}1]) \subset H_{1,\alpha,\beta}$. Therefore we obtain the restrictions

$$\begin{aligned}&\Im(Z(X))|_{i_1(\mathcal{P}(\alpha,\alpha+1])}\\ &= \Im(Z(i_1(X_1))) = D_1(\deg(X_1)) + C_1(\operatorname{rank}(X_1)),\\ &\Im(Z(X))|_{i_2(\mathcal{P}(\beta,\beta+1])}\\ &= \Im(Z(i_2(X_1))) = D_2(\deg(X_1)) + C_2(\operatorname{rank}(X_1))\\ &\text{and } \Im(Z(X))|_{\Delta(\mathcal{P}(\beta,\beta+1])}\\ &= \Im(Z(\Delta(X_1))) = (D_1 + D_2)(\deg(X_1)) + (C_1 + C_2)(\operatorname{rank}(X_1)).\end{aligned}$$

Let $\widetilde{\beta}$ be given by the equation $\widetilde{\beta} = \beta - \lfloor \beta \rfloor$ (and $\widetilde{\alpha}$ analogously). The value $\widetilde{\beta}$ is hence determined by both the quotient $\frac{D_2}{C_2}$ and $\frac{D_1+D_2}{C_1+C_2}$ at the same time. This gives $\frac{D_1}{C_1} = \frac{D_2}{C_2}$. However, since the value of $\widetilde{\alpha} = \alpha - \lfloor \alpha \rfloor$ is determined by $\frac{D_1}{C_2} = \frac{D_2}{C_2}$, we obtain $\widetilde{\alpha} = \widetilde{\beta}$ and hence $\alpha = \beta$. This is a contradiction, which finishes the proof. □

Again, we obtain a similar statement in the cases of $\subset H_{2,\alpha,\beta}$ and $\subset H_{3,\alpha,\beta}$.

Lemma 4.4.5. *If $\alpha - 1 \leq \beta \leq \alpha$, there is no stability condition with heart $H_{2,\alpha,\beta}$ or $H_{3,\alpha,\beta}$.*

Proof. Similar to the proof of lemma 4.4.4, now making use of lemma 4.4.3. □

Summing up, we obtain the Jealousy Lemma, which gives name to this subsection - the name reflects its proof which was given throughout the subsection and which hints at the contradiction being constructed via the assumption of having three hearts of t-structures on $\mathcal{D}$ embedded into the same heart of a t-structure on $\mathcal{D}^{\uparrow}$.

Theorem 4.4.6 (Jealousy Lemma)**.** *If $H_{i,\alpha,\beta}$ for $i \in \{1, 2, 3\}$ is the heart of a t-structure that is not obtained by CP-gluing via either of the three semiorthogonal decompositions $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ or $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ then there is no stability condition with heart $H_{i,\alpha,\beta}$.*

Proof. We can apply lemmas 4.4.4 and 4.4.5 because $H_{i,\alpha,\beta}$ is not a heart obtained by CP-gluing if and only if the respective inequalities on α and β are fulfilled. □

Remark 4.4.7. The significance of the Jealousy Lemma is the following. It demonstrates, that the (older) technique of recollement, which was not designed to construct stability conditions but, in fact, to compute t-structures produces certain hearts that $\mathrm{Stab}(\mathcal{D}^{\uparrow})$ does not "see".

4.5 Stability of embeddings

This subsection provides an important result that characterises $\mathrm{Stab}(\mathcal{D}^{\uparrow})$. It will, moreover, subsequently be used to give a complete description of it.

First, we provide certain background and necessary notation by the next lemma.

Lemma 4.5.1. *For the units and counits ϵ, η, we have*

$$\begin{aligned} \eta : \mathrm{id}_{\mathcal{D}} &\xrightarrow{\cong} \mathbb{K} \circ i_1, & \epsilon : \lambda_1 \circ i_1 &\xrightarrow{\cong} \mathrm{id}_{\mathcal{D}}, \\ \eta : \mathrm{id}_{\mathcal{D}} &\xrightarrow{\cong} \rho_2 \circ i_2, & \epsilon : \mathbb{K} \circ i_2 &\xrightarrow{\cong} \mathrm{id}_{\mathcal{D}}, \\ \eta : \mathrm{id}_{\mathcal{D}} &\xrightarrow{\cong} \lambda_1 \circ \Delta, & \epsilon : \rho_2 \circ \Delta &\xrightarrow{\cong} \mathrm{id}_{\mathcal{D}} \end{aligned} \tag{4.12}$$

while there are remaining units and counits

$$\begin{aligned} \epsilon_1 : i_1 \circ \mathbb{K} \to \mathrm{id}_{\mathcal{D}^\uparrow}, &\qquad \eta_1 : \mathrm{id}_{\mathcal{D}^\uparrow} \to i_1 \circ \lambda_1, \\ \epsilon_2 : i_2 \circ \rho_2 \to \mathrm{id}_{\mathcal{D}^\uparrow}, &\qquad \eta_2 : \mathrm{id}_{\mathcal{D}^\uparrow} \to I_2 \circ \mathbb{K}[1], \\ \epsilon_3 : \Delta \circ \lambda_1 \to \mathrm{id}_{\mathcal{D}^\uparrow}, &\qquad \eta_3 : \mathrm{id}_{\mathcal{D}^\uparrow} \to \Delta \circ \rho_2 \end{aligned}$$

(note that these are not isomorphic to $\mathrm{id}_{\mathcal{D}^\uparrow}$*) such that we have*

$I \; i_2\rho_2(A) \overset{\epsilon_2}{\to} A \overset{\eta_1}{\to} i_1\lambda_1(A) \overset{+}{\to}$

$II \; \Delta\lambda_1(A) \overset{\epsilon_3}{\to} A \overset{\eta_2}{\to} i_2\mathbb{K}[1](A) \overset{+}{\to}$

$III \; i_1\mathbb{K}(A) \overset{\epsilon_1}{\to} A \overset{\eta_3}{\to} \Delta\rho_2(A) \overset{+}{\to}$

for the respective exact triangles for the three semiorthogonal decompositions that we are using on $\mathcal{D}^\uparrow$*.*

Proof. The functors i_1, i_2 and Δ are fully faithful. □

Lemma 4.5.2. *Let* $A = (A_1 \overset{\varphi}{\to} A_2)$ *be an object in* $\mathcal{D}^\uparrow$ *and denote by* $[\varphi] \in \mathrm{Hom}_{\mathcal{D}}(\lambda_1(A), \rho_2(A))$ *the morphism induced by the chain map* φ*.*

a) The exact triangle $\rho_2((II)_A)$ *is isomorphic to a triangle of the form*

$$\lambda_1(A) \overset{[\varphi]}{\to} \rho_2(A) \overset{t_2}{\to} \mathbb{K}(A)[1] \to \lambda_1(A)[1].$$

b) The exact triangle $\lambda_1((III)_A)$ *is isomorphic to a triangle of the form*

$$\mathbb{K}(A) \to \lambda_1(A) \overset{[\varphi]}{\to} \rho_2(A) \overset{t_2}{\to} \mathbb{K}(A)[1].$$

The morphism t_2 *is determined by* $[\varphi]$ *up to an automorphism of the object* $\mathbb{K}(A)[1]$*.*

Proof. The objects of $\rho_2((II)_A)$ and $\lambda_1((III)_A)$ are identified with those in the given triangles with the aid of the natural isomorphisms provided in (4.12). The proof that, after the identifications from (4.12), $\rho_2(\epsilon_3)$ and $\lambda_1(\eta_3)$ are both equal to $[\varphi]$ requires to use the explicit description of the adjunction isomorphisms provided in lemma 3.2.4. □

Remark 4.5.3. It is a crucial for the proof of theorem 4.6.4 to have $\rho_2(\epsilon_3) = [\varphi] = \lambda_1(\eta_3)$, after the identifications using (4.12). Lemma 4.5.2 allows us to access the "structural" map φ of an object in $\mathcal{D}^\uparrow$ as a morphism in $\mathcal{D}$. It is not clear if $\rho_2(\epsilon_3)] = \lambda_1(\eta_3)$ holds in general, when $\rho_2 \dashv \Delta \dashv \lambda_1$ gets replaced by two adjunctions $L \dashv M \dashv R$ with M fully faithful. The Problem that, at this point, remains open is under which conditions we can guarantee that the diagram

$$\begin{array}{ccc} LMR & \xrightarrow{L(\epsilon)} & L \\ {\scriptstyle \epsilon_R}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle \eta_L} \\ R & \xrightarrow{R(\eta)} & RLM \end{array}$$

is commutative.

Lemma 4.5.4. *If we have $h \in \mathrm{Hom}_{\mathcal{D}^\uparrow}(A, B)$ where $A = (A_1 \overset{\varphi}{\to} A_2)$ and $B = (B_1 \overset{\psi}{\to} B_2)$, then*

$$\rho_2(h) \circ [\varphi] = [\psi] \circ \lambda_1(h)$$

in $\mathcal{D}$.

Proof. Because $\Delta\lambda_1 \overset{\epsilon_3}{\to} \mathrm{id}_{\mathcal{D}^\uparrow}$ and $\epsilon : \rho_2\Delta \overset{\sim}{\to} \mathrm{id}_{\mathcal{D}}$ are natural transformations, we have a commutative diagram

$$\begin{array}{ccccc} \lambda_1(A) & \xleftarrow[\sim]{\epsilon_{\lambda_1(A)}} & \rho_2\Delta\lambda_1(A) & \xrightarrow{\rho_2(\epsilon_{3,A})} & \rho_2(A) \\ {\scriptstyle \lambda_1(h)}\downarrow & & \downarrow{\scriptstyle \rho_2\Delta\lambda_1(h)} & & \downarrow{\scriptstyle \rho_2(h)} \\ \lambda_1(B) & \xleftarrow[\sim]{\epsilon_{\lambda_1(B)}} & \rho_2\Delta\lambda_1(B) & \xrightarrow{\rho_2(\epsilon_{3,B})} & \rho_2(B). \end{array}$$

□

Theorem 4.5.5. *If $A = (A_1 \overset{\varphi}{\to} A_2)$ and $B = (B_1 \overset{\psi}{\to} B_2)$ are objects in $\mathcal{D}^\uparrow$ such that $[\varphi] \cong [\psi]$ in $\mathcal{D}$, then $A \cong B$ in $\mathcal{D}^\uparrow$.*

Proof. We are going to prove a slightly stronger statement. Let t_A denote the connecting morphism of the decomposition triangle I_A :

$$i_2\rho_2(A) \overset{\epsilon_3}{\to} A \overset{\eta_1}{\to} i_1\lambda_1(A) \overset{t_A}{\to} i_2\rho_2(A)[1].$$

Note that the object $A \in \mathcal{D}^\uparrow$ is determined up to non-unique isomorphisms by t_A, which is a morphism in $\mathcal{D}$. Suppose $f : \lambda_1(A) \to \lambda_1(B)$ and $g : \rho_2(A) \to \rho_2(B)$ are morphisms in $\mathcal{D}$ such that $g \circ [\varphi] = [\psi] \circ f$. Our aim is to show that then

$$i_2(g[1]) \circ t_A = t_B \circ i_1(f). \tag{4.13}$$

As a consequence, if f and g are both isomorphisms in $\mathcal{D}$ then we obtain a commutative diagram

$$\begin{array}{ccc}
i_1\lambda_1(A) & \xrightarrow{t_A} & i_2\rho_2(A)[1] \\
{\scriptstyle i_1(f)}\downarrow\wr & & \wr\downarrow{\scriptstyle i_2(g[1])} \\
i_1\lambda_1(B) & \xrightarrow{t_B} & i_2\rho_2(B)
\end{array}$$

that extends to an isomorphism between the exact triangles $(I)_A$ and (I_B). In particular we obtain an isomorphism $A \xrightarrow{h} B$ which fits into the commutative diagram

$$\begin{array}{ccccccc}
i_2\rho_2(A) & \xrightarrow{\epsilon_{2,A}} & A & \xrightarrow{\eta_{1,A}} & i_1\lambda_1(A) & \xrightarrow{t_A} & \\
\wr\downarrow{\scriptstyle \rho_2(g)} & & \wr\downarrow{\scriptstyle h} & & \wr\downarrow{\scriptstyle i_1(f)} & & \\
i_2\rho_2(B) & \xrightarrow{\epsilon_{2,B}} & B & \xrightarrow{\eta_{1,B}} & i_1\lambda_1(B) & \xrightarrow{t_B} & .
\end{array}$$

To prove (4.13) we first note that (I_A) is functorial in A. This means – in particular – that the morphism $\eta_{3,A} : A \to \Delta(\rho_2(A))$ induces a commutative diagram

$$\begin{array}{ccc}
i_1\lambda_1(A) & \xrightarrow{t_A} & i_2\rho_2(A)[1] \\
{\scriptstyle i_1\lambda_1(\eta_{3,A})}\downarrow\wr & & \wr\downarrow{\scriptstyle i_2\rho_2\Delta\rho_2(A[1])} \\
i_1\lambda_1\Delta\rho_2(A) & \xrightarrow{t_{\Delta\rho_2(A)}} & i_2\rho_2\Delta\rho_2(A).
\end{array}$$

The isomorphisms $\eta : \mathrm{id}_{\mathcal{D}} \overset{\cong}{\to} \lambda_1 \circ \Delta$ and $\epsilon : \rho_2 \circ \Delta \overset{\cong}{\to} \mathrm{id}_{\mathcal{D}}$ from (4.12) allow us to define a natural transformation $\widetilde{t}_\Delta$ by saying that $\widetilde{t}_{\Delta(X)} : i_1(X) \to i_2(X)[1]$ is the composition

$$i_1(X) \xrightarrow[i_1(\eta_X)]{\cong} i_1\lambda_1\Delta(X) \xrightarrow[t_{\Delta(X)}]{} i_2\rho_2\Delta(X)[1] \xrightarrow[i_2(\epsilon_{X[1]})]{\sim} i_2(X)[1].$$

Using $X = \rho_2(A)$, we obtain a commutative diagram

$$\begin{array}{ccc}
i_1\lambda_1(A) & \xrightarrow{t_A} & i_2\rho_2(A)[1] \\
{\scriptstyle i_1\lambda_1(\eta_{3,A})}\downarrow & & \downarrow{\scriptstyle i_1\rho_2(\eta_{3,A[1]})} \\
i_1\lambda_1(\Delta\rho_2 A) & \xrightarrow{t_{\Delta\rho_2 A}} & i_2\rho_2(\Delta\rho_2 A)[1] \\
{\scriptstyle i_1(\eta_{\rho_2 A})}\uparrow\wr & & \wr\downarrow{\scriptstyle i_2(\epsilon_{\rho_2 A_1})} \\
i_1\rho_2 A & \xrightarrow{\widetilde{t}_{\Delta\rho_2 A}} & i_2\rho_2 A[1]
\end{array}$$

in which the composition of the two vertical arrows on the right is the identity by [3, Proposition 10.1]. The composition of the two vertical arrows on the left is equal to $i_1[\varphi]$ by lemma 4.5.2 (b). Therefore, t_A is equal to the composition

$$i_1\lambda_1(A) \stackrel{i_1[\varphi]}{\to} i_1\rho_2(A) \stackrel{\widetilde{t}_{\Delta\rho_2 A}}{\to} i_2\rho_2(A)[1].$$

For any $f : \lambda_1 A \to \lambda_1 B$ and $g : \rho_2 A \to \rho_2 B$ that satisfy $g \circ [\varphi] = [\psi] \circ f$, we therefore obtain a commutative diagram

$$\begin{array}{ccccc} i_1\lambda_1(A) & \xrightarrow{i_1[\varphi]} & i_1\rho_2(A) & \xrightarrow{\widetilde{t}_{\Delta\rho_2 A}} & i_2\rho_2(A)[1] \\ {\scriptstyle i_1(f)}\downarrow & & {\scriptstyle i_1(g)}\downarrow & & \downarrow{\scriptstyle i_1(g[1])} \\ i_1\lambda_1(B) & \xrightarrow{i_1[\psi]} & i_1\rho_2(B) & \xrightarrow{\widetilde{t}_{\Delta\rho_2 B}} & i_2\rho_2(B)[1]. \end{array}$$

This establishes 4.13 and thereby finishes the proof. □

Corollary 4.5.6. *If $A = (A_1 \stackrel{\varphi}{\to} A_2) \in \mathcal{D}^{\uparrow}$ such that $[\varphi] = 0$, then we have $A = i_1\lambda_1 A \oplus i_2\rho_2 A$ in $\mathcal{D}^{\uparrow}$.*

Proof. Apply theorem 4.5.5 combined with $i_1\lambda_1 A \oplus i_2\rho_2 A \cong (A_1 \stackrel{0}{\to} A_2)$ in $\mathcal{D}^{\uparrow}$. □

Hence, we conclude with the next lemma.

Lemma 4.5.7. *Let $A, E \in \mathcal{D}^{\uparrow}$ such that $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, A) = 0$. Assume now that $A = (A_1 \stackrel{\varphi}{\to} A_2)$ such that $\varphi = 0$ in $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2)$, then $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_1(A_1)) = 0$.*

Proof. Using corollary 4.5.6 we obtain

$$\begin{aligned} \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_1(A_1)) &\subset \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_1(A_1)) \oplus \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_2(A_2)) \\ &\cong \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_1(A_1)) \oplus i_2(A_2) \cong \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, A) = 0. \end{aligned}$$

□

We are now able to turn our attention to the question of stability. We require the following definition.

Definition 4.5.8. A triangulated category $\mathcal{TR}$ is called n-Calabi-Yau if it has a Serre functor and this Serre functor equals to the shift $[n]$.

Lemma 4.5.9. *Assume that $\mathcal{D}$ is* 1*-Calabi-Yau and that the ordered pair* $\{M, N\}$ *equals to either of the ordered pairs* $\{i_1, \lambda_1\}, \{i_2, \mathbb{K}\}$ *or* $\{\Delta, \rho_2\}$*. Let*

$$X_1 \to M(X) \to X_2 \xrightarrow{+}$$

in $\mathcal{D}^{\uparrow}$ *where* $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(X_1, X_2) = 0$*. Then*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(N(X_1), N(X_2)) = 0.$$

Proof. We have to prove the following three statements.

1. Consider an exact triangle

$$E \to i_1(X) \to A \xrightarrow{+} . \tag{4.14}$$

   in $\mathcal{D}^{\uparrow}$ where $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, A) = 0$. Then

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\lambda_1(E), \lambda_1(A)) = 0.$$

2. Consider an exact triangle

$$F \to i_2(X) \to B \xrightarrow{+} \tag{4.15}$$

   in $\mathcal{D}^{\uparrow}$ where $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(F, B) = 0$. Then

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathbb{K}(F), \mathbb{K}(B)) = 0.$$

3. Consider an exact triangle

$$G \to \Delta(X) \to C \xrightarrow{+} \tag{4.16}$$

   in $\mathcal{D}^{\uparrow}$ where $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(G, C) = 0$. Then

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\rho_2(G), \rho_2(C)) = 0.$$

We will prove statement 1 first and will subsequently use it to prove the other two statements. Applying ρ_2 to (4.14) gives $\rho_2(A) \cong \rho_2(E[1]) = \rho_2(E)[1] = S_{\mathcal{D}}(\rho_2(E))$ via $\rho_2 \circ i_1 = 0$. Furthermore, using the definition of the Serre functor, combined with the fact that $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(E, A) \subset \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, A) = 0$, we obtain $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, S_{\mathcal{D}^{\uparrow}}(E)) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(E, A)^* = 0$. Moreover, the adjunctions $\rho_2 \dashv \Delta$ and $\Delta \dashv \lambda_1$ provided by lemmas 3.2.23 and 3.2.24 give us an isomorphism $\rho_2 \cong S_{\mathcal{D}}^{-1} \circ \lambda_1 \circ S_{\mathcal{D}^{\uparrow}}$ which implies $S_{\mathcal{D}}(\rho_2(E)) \cong \lambda_1(S_{\mathcal{D}^{\uparrow}}(E))$. Additionally, the morphism

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi_A} & A_2 \\ \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \mathrm{id}_{A_2}} \\ A_2 & \xrightarrow{\mathrm{id}_{A_2}} & A_2 \end{array}$$

in $\mathcal{C}(\mathcal{A}^{\uparrow})$ gives a (corresponding) morphism ν_A in $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, \Delta(\rho_2(A)))$. Hence, we obtain $\lambda_1(\nu_A) = \varphi_A$, where as of now we regard φ_A as the corresponding morphism in $\mathrm{mor}(\mathcal{D})$.

The adjunction $\Delta \dashv \lambda_1$, on the other hand, provides us with the morphism $\Delta \circ \lambda_1 \overset{\epsilon}{\to} \mathbb{1}$, the counit of the adjunction. Applying λ_1 to this now gives us the natural transformation $\lambda_1 \circ \Delta \circ \lambda_1 \overset{\lambda_1(\epsilon)}{\to} \lambda_1$. Consider now the unit of the adjunction, $\mathbb{1} \overset{\eta}{\to} \lambda_1 \circ \Delta$, that provides us with $\lambda_1 \overset{\eta_{\lambda_1}}{\to} \lambda_1 \circ \Delta \circ \lambda_1$. By the triangle equalities we have

$$\lambda_1(\epsilon) \circ \eta_{\lambda_1} = \mathrm{id}_{\lambda_1} \tag{4.17}$$

provided via the adjointness of the functors. However, Δ is fully-faithful and hence η is an isomorphism, making η_{λ_1} an isomorphism as well. Therefore, (4.17) implies, that $\lambda_1(\epsilon)$ is an isomorphism too. We are now ready to prove the first statement of the lemma using the previous considerations. We obtain a morphism from A to $S_{\mathcal{D}^{\uparrow}}(E)$ via the following chain of morphisms

$$A \overset{\nu_A}{\to} \Delta(\rho_2(A)) \overset{\cong}{\to} \Delta(S_{\mathcal{D}}(\rho_2(E))) \overset{\cong}{\to} \Delta(\lambda_1(S_{\mathcal{D}^{\uparrow}}(E))) \overset{\epsilon}{\to} S_{\mathcal{D}^{\uparrow}}(E). \tag{4.18}$$

Applying λ_1 to (4.18), it becomes

$$\begin{aligned} \lambda_1(A) \overset{\varphi_A}{\to} \lambda_1(\Delta(\rho_2(A))) \overset{\cong}{\to} \lambda_1(\Delta(S_{\mathcal{D}}(\rho_2(E)))) \\ \overset{\cong}{\to} \lambda_1(\Delta(\lambda_1(S_{\mathcal{D}^{\uparrow}}(E)))) \overset{\cong}{\to} \lambda_1(S_{\mathcal{D}^{\uparrow}}(E)), \end{aligned} \tag{4.19}$$

since $\lambda_1(\epsilon)$ is an isomorphism and moreover $\lambda_1(\nu_A) = \varphi_A$ as we have seen before. In other words, the crucial point is what is happening to the two morphisms on both ends of (4.18). Now, since the morphism constructed in (4.18) is in $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, S_{\mathcal{D}^{\uparrow}}(E))$ and $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, S_{\mathcal{D}^{\uparrow}}(E)) \subset \mathrm{Hom}^{\geq 0}_{\mathcal{D}^{\uparrow}}(A, S_{\mathcal{D}^{\uparrow}}(E)) = 0$ as previously seen, this morphism is 0. Since λ_1 is a functor, the morphism obtained in (4.19) is therefore also 0. Since, on the other hand, the morphism constructed in (4.19) is obtained by composing φ_A with isomorphisms only, this implies that we must have $\varphi_A = 0 \in \mathrm{mor}(\mathcal{D})$. By lemma 4.5.7, this implies $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_1(A_1)) = 0$. By adjunction we obtain $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\lambda_1(E), \lambda_1(A)) = \mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\lambda_1(E), A_1) \cong \mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, i_1(A_1)) = 0$ and the proof of 1 is finished.

This being established, we can now go on to proving statement 3. Applying $S_{\mathcal{D}^\uparrow}$ and then $[-1]$ to (4.16) gives

$$S_{\mathcal{D}^\uparrow}(G)[-1] \to i_1(X) \to S_{\mathcal{D}^\uparrow}(C)[-1] \xrightarrow{+}$$

via $S_{\mathcal{D}^\uparrow}(\Delta(X)) = i_1(S_{\mathcal{D}}(X)) = i_1(X)[1]$. By statement 1, this implies

$$\mathrm{Hom}_{\mathcal{D}}(\lambda_1(S_{\mathcal{D}^\uparrow}(G)[-1]), \lambda_1(S_{\mathcal{D}^\uparrow}(C)[-1])) = 0.$$

Hence we obtain

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}}\, \rho_2(S_{\mathcal{D}^\uparrow}(G), \rho_2(S_{\mathcal{D}^\uparrow}(C))) \\ = {} &\mathrm{Hom}_{\mathcal{D}}(\lambda_1(S_{\mathcal{D}^\uparrow}(G)[-1]), \lambda_1(S_{\mathcal{D}^\uparrow}(C)[-1])) = 0 \end{aligned}$$

via $\lambda_1 \circ S_{\mathcal{D}^\uparrow} = S_{\mathcal{D}} \circ \rho_2 = \rho_2[1]$.

Finally, in order to prove statement 2, we proceed similar to the previous case. Applying $S_{\mathcal{D}^\uparrow}$ to (4.15) gives

$$S_{\mathcal{D}^\uparrow}(F)[-1] \to \Delta(X) \to S_{\mathcal{D}^\uparrow}(B)[-1] \xrightarrow{+}$$

via $S_{\mathcal{D}^\uparrow} \circ i_2 = \Delta[1]$ such that we obtain

$$\begin{aligned} \mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathbb{K}(F), \mathbb{K}(B)) &= \mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\rho_2(S_{\mathcal{D}^\uparrow}(F[-2])), \rho_2(S_{\mathcal{D}^\uparrow}(B[-2]))) \\ &= \mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\rho_2(S_{\mathcal{D}^\uparrow}(F)), \rho_2(S_{\mathcal{D}^\uparrow}(B))) = 0 \end{aligned}$$

from $\rho_2 \circ S_{\mathcal{D}^\uparrow}[-1] = \mathbb{K}[1]$. With this, the proof of statement 2 and hence the entire proof is finished. □

In order to introduce lemma 4.5.11, we need the following.

Lemma 4.5.10. *Suppose $E \in H$, where H is the heart of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. If H has homological dimension 1, we obtain for any $X \in \mathcal{D}$ that $X \cong \oplus_{i\in\mathbb{Z}} H^i(X)[-i]$, where H^i is the cohomology given by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$.*

Proof. To see this, note at first, that an immediate implication of H having homological dimension 1 is that $\mathrm{Hom}_{\mathcal{D}}(H, \mathcal{D}^{\leq -2}) = 0$. Since $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ is bounded, there is an $m \in \mathbb{Z}$ for any non-zero $X \in \mathcal{D}$, such that $X \in \mathcal{D}^{\leq m}$. Consider the exact triangle

$$\tau^{\leq m-1}(X) \to X \to \tau^{\geq m}(X) \xrightarrow{+}$$

which gives the canonical connecting morphism $c : \tau^{\geq m}(X) \to \tau^{\leq m-1}(X)[1]$. Since $X \in \mathcal{D}^{\leq m}$ we have $X = \tau^{\leq m}(X)$ and hence $\tau^{\geq m}(X) = \tau^{\geq m}(\tau^{\leq m}(X)) =$

$H^m(X)[-m] \in H[-m]$. On the other hand, $\tau^{\leq m-1}(X)[1] \in \mathcal{D}^{m-2}$ and therefore $c \in \mathrm{Hom}(H[-m], \mathcal{D}^{m-2}) = \mathrm{Hom}_{\mathcal{D}}(H, \mathcal{D}^{\leq -2}) = 0$. We hence obtain $c = 0$ and by [59, Tag 05QT] this gives

$$X = \tau^{\leq m-1}(X) \oplus \tau^{\geq m}(X) = \tau^{\leq m-1}(X) \oplus H^m(X)[-m].$$

Repeating this with $\tau^{\leq m-1}(X)$ taking the place of X one obtains the statement by induction. □

The following lemma is essentially a version of [30, Lemma 7.2]. In our situation we are able to give a somewhat shorter and less tedious proof.

Lemma 4.5.11. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Suppose $E \in H$, where H is the heart of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. If there is an exact triangle*

$$Y \to E \to X \xrightarrow{+} \tag{4.20}$$

in $\mathcal{D}$ with $\mathrm{Hom}^{\leq 0}(Y, X) = 0$, then $X, Y \in H$.

Proof. At first, we see that $\mathcal{D}$ being 1-Calabi-Yau implies that H has homological dimension 1 as for any $A, B \in H$ we have

$$\begin{aligned} \mathrm{Hom}(A, B[k])^* &= \mathrm{Hom}(B[k], S_{\mathcal{D}}(A)) \\ = \mathrm{Hom}(B[k], A[1]) &= \mathrm{Hom}(B, A[1-k]) = 0 \end{aligned} \tag{4.21}$$

if $k \geq 2$, provided by the fact that the morphisms between objects in hearts vanish if a negative shift is applied to the second component. Hence, we also have $\mathrm{Ext}^n(A, B) = 0$ for $n \geq 2$. By lemma 4.5.10, we have

$$X \cong \oplus_{i \in \mathbb{Z}} H^i(X)[-i] \text{ and } Y \cong \oplus_{j \in \mathbb{Z}} H^j(Y)[-j]$$

where H^i is the cohomology given by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$. Now consider the long exact cohomology sequence provided by the exact triangle (4.20). Since $E \in H$, we have $H^n(E) = 0$ for any $n \neq 0$ and therefore the long exact cohomology sequence is of the form

$$0 \to H^{m-1}(X) \xrightarrow{\xi^m} H^m(Y) \to 0$$

wherever $m \notin \{0, 1\}$, completed by

$$0 \to H^{-1}(X) \xrightarrow{\epsilon} H^0(Y) \to H^0(E) \to H^0(X) \xrightarrow{\tau} H^1(Y) \to 0,$$

which provides that ϵ is a monomorphism, τ is an epimorphism and that all the ξ^m are isomorphisms. Hence, each of the morphisms ξ^m has an inverse $(\xi^m)^{-1} \in \mathrm{Hom}(H^m(Y), H^{m-1}(X))$. Since any non-zero morphism within the set $\mathrm{Hom}(H^m(Y), H^{m-1}(X))$ would induce a non-zero morphism in $\mathrm{Hom}(Y, X[-1])$ which – as a condition of the lemma – equals 0, we now obtain that $(\xi^m)^{-1} \in \mathrm{Hom}(H^m(Y), H^{m-1}(X)) = 0$ which implies $\xi^m = 0$ also. Since all ξ^m are isomorphisms this gives $H^{m-1}(X) = H^m(Y) = 0$ for $m \notin \{0, 1\}$. To see that ϵ is equal to 0 we consider $\epsilon \in \mathrm{Hom}(H^{-1}(X), H^0(Y))$. We have

$$\mathrm{Hom}(H^{-1}(X), H^0(Y))^* = \mathrm{Hom}(H^0(Y), S_{\mathcal{D}}(H^{-1}(X))) = \\ \mathrm{Hom}(H^0(Y), H^{-1}(X)[1]) = 0$$

since $H^0(Y)$ and $H^{-1}(X)$ are summands of Y and X respectively such that $\mathrm{Hom}(H^0(Y), H^{-1}(X)[1]) \subset \mathrm{Hom}(Y, X) \subset \mathrm{Hom}^{\leq 0}(Y, X) = 0$ which implies $\mathrm{Hom}(H^{-1}(X), H^0(Y)) = 0$ and therefore $\epsilon = 0$. Hence, $H^{-1}(X) = 0$. Similarly we proceed in the case of τ, now using

$$\mathrm{Hom}(H^0(X), H^1(Y))^* = \mathrm{Hom}(H^1(Y), H^0(X)[1]) = \\ \mathrm{Hom}(H^1(Y)[-1], H^0(X)) = 0.$$

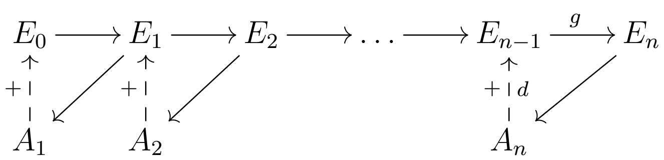
We obtain that $H^1(Y)$ too is 0. This leaves us with $X = H^0(X) \in H$ and $Y = H^0(Y) \in H$ which finishes the proof. □

Remark 4.5.12. Note that in the situation that we are in – which is that of $\mathcal{D}$ being 1-Calabi-Yau – we can in fact relax the hom-vanishing condition $\mathrm{Hom}^{\leq 0}(Y, X) = 0$ of lemma 4.5.11 to $\mathrm{Hom}(Y, X) = 0$ because within the proof of 4.5.11 one can alternatively simply argue the vanishing of the ξ^m in the same way as the vanishing of ϵ and τ. For that, one now considers the equation

$$\mathrm{Hom}(H^{m-1}(X), H^m(Y))^* = \mathrm{Hom}(H^m(Y), H^{m-1}(X)[1]) = \\ \mathrm{Hom}(H^m(Y)[-m], H^{m-1}(X)[-m+1]) = 0.$$

In the more general situation of [30, Lemma 7.2] however, $\mathrm{Hom}^{\leq 0}(Y, X) = 0$ is required.

Lemma 4.5.13. *For a slicing $\mathcal{P}$ on a triangulated category and a non-zero element $E \in \mathcal{TR}$, let*



with $E_0 = 0, E_n = E$ and $A_j \in \mathcal{P}(\phi_j)$ for all $j \in \{1, \dots, n\}$ be the Harder-Narashiman filtration of E with regard to $\mathcal{P}$. If $E \notin \mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$, then $g \neq 0$.

Proof. Consider the exact triangle

$$E_{n-1} \xrightarrow{g} E_n \to A_n \xrightarrow{d} \tag{4.22}$$

and assume $g = 0$. This provides us with the exact triangle

$$A_n \xrightarrow{d} E_{n-1}[1] \xrightarrow{0} E_n[1] \xrightarrow{+} .$$

By [32, Lemma 1.4], this implies that d is a retraction. This means, there is a morphism $f \in \mathrm{Hom}(E_{n-1}[1], A_n)$ such that $d \circ f = \mathrm{id}_{E_{n-1}[1]}$. We have

$$\phi^-_{\mathcal{P}}(E_{n-1}) = \phi_{\mathcal{P}}(A_{n-1}) > \phi_{\mathcal{P}}(A_n)$$

and hence

$$\phi^-_{\mathcal{P}}(E_{n-1}[1]) = \phi^-_{\mathcal{P}}(E_{n-1}) + 1 = \phi_{\mathcal{P}}(A_{n-1}) + 1 > \phi_{\mathcal{P}}(A_n) + 1 > \phi_{\mathcal{P}}(A_n).$$

Hence, $\mathrm{Hom}(E_{n-1}[1], A_n) = 0$, implying that $f = 0$. Therefore we obtain

$$\mathrm{id}_{E_{n-1}[1]} = d \circ f = d \circ 0 = 0,$$

implying that $E_{n-1}[1] = 0$. Hence (4.22) implies $E = E_n \cong A_n \in \mathcal{P}(\phi_n)$ which is a contradiction as $E \notin \mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ was assumed.

□

We will, with the help of the following lemma be able to produce the pieces of the – crucial – proposition 4.5.20.

Lemma 4.5.14. *Assume there are exact functors*

$$P : \mathcal{TR} \to \widetilde{\mathcal{TR}}; M : \mathcal{TR} \to \widetilde{\mathcal{TR}}; N : \mathcal{TR} \to \widetilde{\mathcal{TR}};$$
$$L : \widetilde{\mathcal{TR}} \to \mathcal{TR}; R : \widetilde{\mathcal{TR}} \to \mathcal{TR}$$

such that $P \dashv L \dashv M \dashv R \dashv N$. Assume furthermore that P, M and N are fully faithful, that $\mathrm{im}(N) = {}^{\perp}(\mathrm{im}(M)), \mathrm{im}(M)^{\perp} = \mathrm{im}(P)$ and that $\mathcal{TR}$ is 1-Calabi-Yau. If there is a bounded t-structure $(\mathcal{TR}^{\leq 0}, \mathcal{TR}^{\geq 1})$ on $\mathcal{TR}$ with heart H, X an indecomposable object in H and an exact triangle

$$E \xrightarrow{g} M(X) \to A \xrightarrow{+} \tag{4.23}$$

in $\widetilde{\mathcal{TR}}$ where $g \neq 0, A \neq 0$ and moreover we have that $\mathrm{Hom}^{\leq 0}_{\widetilde{\mathcal{TR}}}(E, A) = 0 = \mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(L(E), L(A))$ then we obtain $A \cong N(X)$ and $E \cong P(X)$.

Proof. Applying the functor L to (4.23) and letting $\epsilon_X : L(M(X)) \to X$ be the counit of the adjunction $L \dashv M$, provides us with the exact triangle

$$L(E) \overset{\epsilon_X \circ L(g)}{\to} X \to L(A) \overset{\gamma}{\to}$$

via $L \circ M \cong \mathrm{id}_{\mathcal{TR}}$ given by the fact that M is fully faithful. Since $\mathcal{TR}$ is 1-Calabi-Yau we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{TR}}(L(A), L(E)[1])^* &= \mathrm{Hom}_{\mathcal{TR}}(L(A), S_{\mathcal{TR}}(L(E)))^* \\ &= \mathrm{Hom}_{\mathcal{TR}}(L(E), L(A)) \subset \mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(L(E), L(A)) = 0 \end{aligned}$$

where $S_{\mathcal{TR}}$ is the Serre functor. Hence, $\mathrm{Hom}_{\mathcal{TR}}(L(A), L(E)[1]) = 0$ which implies $\gamma = 0$ from which we obtain $X \cong L(E) \oplus L(A)$. Next, we see that $L(E) \neq 0$, since otherwise we would get a contradiction via the fact that $g \in \mathrm{Hom}_{\widetilde{\mathcal{TR}}}(E, M(X)) = \mathrm{Hom}_{\mathcal{TR}}(L(E), X) = 0$. By lemma 4.5.11, we now obtain $L(E), L(A) \in H$ since $\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(L(E), L(A)) = 0$ by assumption. Because X is indecomposable in H we hence obtain $L(A) = 0$ and hence $L(E) \cong X$. By lemma 4.2.10 and by assumption we obtain $\ker(L) = {}^{\perp}(\mathrm{im}(M)) = \mathrm{im}(N)$ such that $L(A) = 0$ implies that there is an $A' \in \mathcal{TR}$ such that $N(A') = A$, turning (4.23) into the exact triangle

$$E \overset{g}{\to} M(X) \to N(A') \overset{+}{\to} .$$

Now applying R to this we obtain the exact triangle

$$R(E) \to X \to A' \overset{+}{\to} \tag{4.24}$$

via $R \circ M \cong \mathrm{id}_{\mathcal{TR}} \cong R \circ N$, using the fully-faithfulness of M and N. We have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(R(E), A') = \mathrm{Hom}^{\leq 0}_{\widetilde{\mathcal{TR}}}(E, N(A')) = \mathrm{Hom}^{\leq 0}_{\widetilde{\mathcal{TR}}}(E, A) = 0,$$

and therefore, by lemma 4.5.11 that $R(E), A' \in H$. On the other hand, we have $\mathrm{Hom}_{\mathcal{TR}}(R(E), A') \subset \mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(R(E), A') = 0$ and therefore – since, by definition 4.5.8, shift by one is the Serre functor – also

$$\mathrm{Hom}_{\mathcal{TR}}(A', R(E)[1])^* = \mathrm{Hom}_{\mathcal{TR}}(R(E), A') = 0,$$

giving $\mathrm{Hom}_{\mathcal{TR}}(A', R(E)[1]) = 0$. Hence, we obtain $f = 0$ in the exact triangle

$$X \to A' \overset{f}{\to} R(E)[1] \overset{+}{\to}$$

provided by (4.24). Therefore $X \cong R(E) \oplus A'$. Since $0 \neq A = N(A')$ implies $A' \neq 0$, we now have $R(E) = 0$ via the indecomposability of X in

H, combined with the fact that $R(E), A' \in H$. Therefore also $A' \cong X$, giving $A \cong N(X)$, the first statement of the lemma. By lemma 4.2.11 and by assumption we obtain $\ker(R) = \mathrm{im}(M)^{\perp} = \mathrm{im}(P)$ such that $R(E) = 0$ implies that there is an $E' \in \mathcal{TR}$ such that $P(E') = E$. We obtain

$$X = L(E) = L(P(E')) \cong E' \tag{4.25}$$

using the fully faithfulness of P. Applying P to (4.25) gives $E \cong P(X)$, which finishes the proof. □

We will use additional language for the following.

Definition 4.5.15. Let $\mathcal{TR}$ be a triangulated category and $\mathcal{P}$ be a slicing on $\mathcal{TR}$. We say that an object $A \in \mathcal{TR}$ is "slicy" if there is a $\phi \in \mathbb{R}$ such that $A \in \mathcal{P}(\phi)$. Furthermore denote this ϕ by ϕ_A.

Remark 4.5.16. In the following we will in situations were a slicing is given – but not necessarily a central charge – refer to the filtration of definition 2.5.1 as a Harder-Narashiman filtration (HNF for short).

Lemma 4.5.17. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let X be indecomposable in the heart H of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. Assume furthermore that $\mathcal{P}$ is a slicing on $\mathcal{D}^{\uparrow}$ and that $i_1(X)$ is not slicy, then $i_2(X)$ is slicy.*

Proof. Since $i_1(X) \notin \mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$, $i_1(X)$ has a non-trivial Harder-Narashiman filtration and – in particular – there is an exact triangle

$$E \xrightarrow{g} i_1(X) \to A \xrightarrow{+} \tag{4.26}$$

the "last" triangle in the HNF given by the slicing $\mathcal{P}$ (note, that this implies that there is a $\nu \in \mathbb{R}$ such that $A \in \mathcal{P}(\nu)$ and $A \neq 0$). Moreover, $\phi_-(E) > \phi_-(i_1(X)) = \phi(A)$ and hence $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(E, A) = 0$. By lemma 4.5.9, we obtain $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\lambda_1(E), \lambda_1(A)) = 0$ and by lemma 4.5.13 that $g \neq 0$. We can hence apply lemma 4.5.14. To do so let

$$P = \Delta, L = \lambda_1, M = i_1, R = \mathbb{K}, N = i_2[1].$$

We have $\Delta \dashv \lambda_1 \dashv i_1 \dashv \mathbb{K} \dashv i_2[1]$ by lemma 4.3.1, the fully faithfulness of i_1, i_2 and Δ by lemmas 3.2.18 and 3.2.28 and $\mathrm{im}(i_1)^{\perp} = \mathrm{im}(\Delta)$ by corollary 4.2.14, which at the same time provides us with $\mathrm{im}(i_2[1]) = {}^{\perp}(\mathrm{im}(i_1))$ if combined with the fact that $\mathrm{im}(i_2)$ is closed under shift and hence $\mathrm{im}(i_2[1]) = \mathrm{im}(i_2)$. We obtain that $A = i_2(X)$ which finishes the proof. □

Lemma 4.5.18. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let X be indecomposable in the heart H of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. Assume furthermore that $\mathcal{P}$ is a slicing on $\mathcal{D}^{\uparrow}$ and that $i_2(X)$ is not slicy, then $\Delta(X)$ is slicy.*

Proof. Since $\mathcal{D}$ has a Serre functor $S_{\mathcal{D}}$, by theorem 4.2.19, $\mathcal{D}^{\uparrow}$ also has a Serre functor $S_{\mathcal{D}^{\uparrow}}$. Since $i_2(X) \notin \mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$, $i_2(X)$ has a non-trivial Harder-Narashiman filtration and – in particular – there is an exact triangle

$$F \xrightarrow{g} i_2(X) \to B \xrightarrow{+}, \tag{4.27}$$

the "last" triangle in the HNF (note, that this implies that there is a $\nu \in \mathbb{R}$ such that $B \in \mathcal{P}(\nu)$ and $B \neq 0$). Moreover, $\phi_-(F) > \phi_-(i_2(X)) = \phi(B)$ and hence $\mathrm{Hom}^{\leq 0}_{\mathcal{D}^{\uparrow}}(F, B) = 0$. By lemma 4.5.9, we obtain $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathbb{K}(F), \mathbb{K}(B)) = 0$ and by lemma 4.5.13 that $g \neq 0$. We can hence apply lemma 4.5.14. To do so let

$$P = i_1[-1], L = \mathbb{K}[1], M = i_2, R = \rho_2, N = \Delta$$

and use lemmas 3.2.18 and 3.2.28 and corollary 4.2.14 in a similar way as it was done in lemma 4.5.17. We see that $B = \Delta(X)$ which finishes the proof. □

Lemma 4.5.19. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let X be indecomposable in the heart H of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. Assume furthermore that $\mathcal{P}$ is a slicing on $\mathcal{D}^{\uparrow}$ and that $\Delta(X)$ is not slicy, then $i_1(X)$ is slicy.*

Proof. We proceed similar to the proves of lemmas 4.5.17 and 4.5.18. We are now working with the exact triangle

$$G \xrightarrow{g} \Delta(X) \to C \xrightarrow{+}, \tag{4.28}$$

using lemmas 4.5.9, 4.5.13 and 4.5.14. □

Proposition 4.5.20. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let X be indecomposable in the heart H of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. Assume furthermore that $\mathcal{P}$ is a slicing on $\mathcal{D}^{\uparrow}$. Let the ordered triple (F, G, H) be equal to either of the ordered triples $(i_1[-1], i_2, \Delta), (i_2, \Delta, i_1)$ or $(\Delta, i_1, i_2[1])$. If $G(X)$ is not slicy, then $F(X)$ and $H(X)$ are slicy and $\phi_{F(X)} > \phi_{H(X)} + 1$.*

Proof. Note that the condition of being slicy is stable under shift. Hence, making use of the previous lemmas, we get the following.

1. Assume that (F, G, H) equals to $(i_1[-1], i_2, \Delta)$. By lemma 4.5.17 we directly obtain $i_2(X) \in \mathcal{P}(\bar{\phi})$ and moreover, since by lemma 4.5.19, $\Delta(X) \notin \mathcal{P}(\widetilde{\phi})$ for all $\widetilde{\phi} \in \mathbb{R}$ would imply the existence of a $\phi_{i_1(X)} \in \mathbb{R}$

such that $i_1(X) \in \mathcal{P}(\phi_{i_1(X)})$ which contradicts the assumption, we also have the existence of a $\phi_{\Delta(X)} \in \mathbb{R}$ such that $\Delta(X) \in \mathcal{P}(\phi_{\Delta(X)})$. To see that $\phi_{\Delta(X)} > \phi_{i_2(X)} + 1$, consider the exact triangle

$$\Delta(X) \to i_1(X) \to i_2(X)[1] \xrightarrow{+}$$

that we obtain from (4.26) since, by lemma 4.5.14, we have $E \cong \Delta(X)$ and $A \cong i_2(X)[1]$. Since (4.26) is the last triangle in the HNF of $i_1(X)$ we therefore obtain that $\phi_{\Delta(X)} > \phi_{i_2(X)[1]}$ and hence

$$\phi_{\Delta(X)} > \phi_{i_2(X)[1]} = \phi_{i_2(X)} + 1.$$

2. Assume that (F, G, H) equals to (i_2, Δ, i_1). This case is similar to 1, using lemmas 4.5.18 and 4.5.17. Then use the exact triangle

$$i_1(X)[-1] \to i_2(X) \to \Delta(X) \xrightarrow{+}$$

provided by (4.27) and – as above – lemma 4.5.14 to see that $\phi_{i_1(X)} - 1 > \phi_{\Delta(X)}$.

3. Assume that (F, G, H) equals to $(\Delta, i_1, i_2[1])$. This case is similar to 1, using lemmas 4.5.19 and 4.5.18. Then use the exact triangle

$$i_2(X) \to \Delta(X) \to i_1(X) \xrightarrow{+}$$

provided by (4.28) and – as above – lemma 4.5.14 to see that $\phi_{i_2(X)} > \phi_{i_1(X)}$.

□

In other words, at this point we can also give an exact description of the HNFs of the three embeddings.

Corollary 4.5.21. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let X be indecomposable in the heart H of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$. Assume furthermore that $\mathcal{P}$ is a slicing on $\mathcal{D}^{\uparrow}$,*

1. *If $i_1(X)$ is not slicy, then it's HNF is given by*

$$0 \to \Delta(X) \to i_1(X)$$

with slicy quotients $\Delta(X)$ and $i_2(X)[1]$.

2. *If $i_2(X)$ is not slicy, then its HNF is given by*

$$0 \to i_1(X)[1] \to i_2(X)$$

with slicy quotients $i_1(X)[1]$ and $\Delta(X)$.

3. *If $\Delta(X)$ is not slicy, then its HNF is given by*

$$0 \to i_2(X) \to \Delta(X)$$

with slicy quotients $i_2(X)$ and $i_1(X)$.

Proof. This corollary simply sums up the proof of proposition 4.5.20. □

We can also use proposition 4.5.20 to prove the following.

Lemma 4.5.22. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let $\mathcal{P}$ be a slicing on $\mathcal{D}^{\uparrow}$. Let X_1 be indecomposable in the heart H_1 of a bounded t-structure $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ on $\mathcal{D}$ and X_2 be indecomposable in the heart H_2 of a bounded t-structure $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ on $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(X_1, X_2) \neq 0$. Let the ordered triple (F, G, H) be equal to either of the ordered triples $(i_1[-1], i_2, \Delta), (i_2, \Delta, i_1)$ or $(\Delta, i_1, i_2[1])$. Assume that $G(X_1)$ is not slicy. Then $F(X_2)$ and $H(X_2)$ are slicy.*

Proof. There are six cases for which we must prove a contradiction – that is assuming that

1. $i_1(X_1), \Delta(X_2)$ or
2. $i_2(X_1), i_1(X_2)$ or
3. $\Delta(X_1), i_2(X_2)$ or
4. $i_1(X_2), \Delta(X_1)$ or
5. $i_2(X_2), i_1(X_1)$ or
6. $\Delta(X_2), i_2(X_1)$

both aren't in a slice. Let's assume, that this is the case for $i_1(X_1), \Delta(X_2)$. By proposition 4.5.20, $i_2(X_1), \Delta(X_1), i_1(X_2)$ and $i_2(X_2)$ are each in a slice. We also obtain by proposition 4.5.20 that $\phi_{i_2(X_1)}+1 < \phi_{\Delta(X_1)}$. Next, consider

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\Delta(X_1), i_1(X_2)) &\cong \mathrm{Hom}_{\mathcal{D}}(X_1, \lambda_1(i_1(X_2))) \\ &= \mathrm{Hom}_{\mathcal{D}}(X_1, \mathrm{id}(X_2)) = \mathrm{Hom}_{\mathcal{D}}(X_1, X_2) \neq 0\end{aligned}$$

which forces $\phi_{\Delta(X_1)} \leq \phi_{i_1(X_2)}$. Again by proposition 4.5.20 we obtain $\phi_{i_1(X_2)} < \phi_{i_2(X_2)}$. Finally, we have $S_{\mathcal{D}} = [1]$ which gives

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_2(X_2), i_2(X_1)[1]) = \mathrm{Hom}_{\mathcal{D}}(X_2, X_1[1]) \\ &= \mathrm{Hom}_{\mathcal{D}}(X_2, S_{\mathcal{D}}(X_1)) = \mathrm{Hom}_{\mathcal{D}}(X_1, X_2) \neq 0. \end{aligned}$$

and therefore forces $\phi_{i_2(X_2)} \leq \phi_{i_2(X_1)} + 1$. In other words, via

$$\phi_{i_2(X_1)} + 1 < \phi_{\Delta(X_1)} \leq \phi_{i_1(X_2)} < \phi_{i_2(X_2)} \leq \phi_{i_2(X_1)} + 1 \tag{4.29}$$

we obtain the contradiction $\phi_{i_2(X_1)} < \phi_{i_2(X_1)}$.

The statement for the situation where $i_2(X_1)$ and $i_1(X_2)$ aren't assumed to be in a slice and the situation where $\Delta(X_1)$ and $i_2(X_2)$ are not in a slice are now similar. In fact they are dealt with by using the fact that the proof is "stable" under the application of the Serre functor $S_{\mathcal{D}^{\uparrow}}$. In other words, applying $S_{\mathcal{D}^{\uparrow}}$ to everything will provide the analogous inequalities – and hence a similar contradiction – applying once will deal with the situation where $i_2(X_1)$ and $i_1(X_2)$ are not assumed to be in a slice and the situation where $\Delta(X_1)$ and $i_2(X_2)$ are not in a slice is dealt with by applying $S_{\mathcal{D}^{\uparrow}}$ again. Finally, we can prove the contradiction in the case of the situation where $i_1(X_2)$ and $\Delta(X_1)$ are not assumed to be in a slice by the inequality

$$\phi_{i_2(X_2)} + 1 < \phi_{\Delta(X_2)} < \phi_{i_1(X_1)} + 1 \leq \phi_{i_2(X_1)} + 1 \leq \phi_{i_2(X_2)} + 1$$

where the first inequality is provided by proposition 4.5.20, the second by applying $S_{\mathcal{D}^{\uparrow}}$ to $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\Delta(X_1), \Delta(X_2)) \neq 0$, the third one again by proposition 4.5.20 and the last one by $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_2(X_1), i_2(X_2)) = \mathrm{Hom}_{\mathcal{D}}(X_1, X_2) \neq 0$. The last two situations now – again – can be lead to a contradiction by applying $S_{\mathcal{D}^{\uparrow}}$.

□

We can extend this by putting more conditions on $\mathcal{D}$.

Lemma 4.5.23. *Assume that $\mathcal{D}$ is 1-Calabi-Yau. Let $\mathcal{P}$ be a slicing on $\mathcal{D}$ for which we have $\mathrm{Hom}_{\mathcal{D}}(A, B) \neq 0$ if $A \in \mathcal{P}_{\mathcal{D}}(\varphi), B \in \mathcal{P}_{\mathcal{D}}(\psi)$ with $\varphi < \psi < \varphi + 1, \varphi, \psi \in \mathbb{R}$ and A, B non-zero. Let $\mathcal{P}_{\mathcal{D}^{\uparrow}}$ be a slicing on $\mathcal{D}^{\uparrow}$ and let the ordered triple (F, G, H) be equal to either of the ordered triples $(i_1[-1], i_2, \Delta), (i_2, \Delta, i_1)$ or $(\Delta, i_1, i_2[1])$. Furthermore let X be indecomposable in the heart H_1 of a bounded t-structure $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ on $\mathcal{D}$ and Y be indecomposable in the heart H_2 of a bounded t-structure $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ on $\mathcal{D}$. Let X, Y be slicy such that $\phi_Y - \phi_X \notin \mathbb{Z}$. If $G(X)$ is not slicy, then $F(Y)$ and $H(Y)$ are slicy.*

Proof. We have $\phi_Y - \phi_X = m + \alpha$ where $m \in \mathbb{Z}$ and $0 < \alpha < 1$. Hence, by assumption, $\mathrm{Hom}_{\mathcal{D}}(X[m], Y) \neq 0$ and since, again by assumption $X[m], Y$ are indecomposable in their respective hearts (shifted by $[m]$ in the case of $X[m]$), we can apply proposition 4.5.20 combined with lemma 4.5.22, where $X_1 = X[m]$ and $X_2 = Y$. We obtain $\iota, \zeta \in \mathbb{R}$ such that $F(Y) \in \mathcal{P}_{\mathcal{D}^\uparrow}(\iota)$ and $H(Y) \in \mathcal{P}_{\mathcal{D}^\uparrow}(\zeta)$. □

We will give an improved version of our result in the special case of an elliptic curve C. We need the following technical lemma. We include a proof of this – well known – result which is based on [2].

Lemma 4.5.24. *Let X be a μ-stable object in $\mathcal{D}^b(\mathrm{Coh}(C))$ where C is an elliptic curve. Then* $\mathrm{rank}(X)$ *and* $\deg(X)$ *are coprime.*

Proof. Let E be a stable vector bundle, then E is simple by [20, Section 2.4]. This implies that E is indecomposable. Assume that $h = \gcd(r, d) > 1$ where $r = \mathrm{rank}(E)$ and $d = \deg(E)$. By [2, Theorem 10], since E is indecomposable we have $E \cong E_A(r, d) \otimes L$, where L is a line bundle. Since a vector bundle F is stable if and only if $F \otimes L'$ is stable for a line bundle L', we obtain $E_A(r, d)$ stable. By [2, Lemma 24] however, $E_A(r, d) \cong E_A(r', d') \otimes F_h$ where F_h is the Atiyah bundle from [2, Theorem 5] and $r' = \frac{r}{h}$ and $d' = \frac{d}{h}$. By [2, Theorem 5] we obtain an exact sequence

$$0 \to \mathcal{O}_C \to F_h \to F_{h-1} \to 0, \tag{4.30}$$

which implies that the structure sheaf $\mathcal{O}_C$ is a subbundle of F_h. In other words, we obtain an embedding $\mathcal{O}_C \overset{i}{\hookrightarrow} F_h$. Since tensoring vector bundles with vector bundles is an exact functor, we obtain from applying $\otimes E_A(r', d')$ to (4.30) that $E_A(r', d') \otimes \mathcal{O}_C \overset{E_A(r',d')\otimes i}{\hookrightarrow} E_A(r', d') \otimes F_h$ is still an embedding. Since $E_A(r', d') \otimes F_h \cong E_A(r, d)$ (and since $\mathcal{O}_C$ is the neutral element in the Picard-group), we obtain $E_A(r', d') \subset E_A(r, d)$. This gives

$$\mu(E_A(r, d)) = \frac{d}{r} = \frac{d'}{r'} = \mu(E_A(r', d'))$$

which is a contradiction because $E_A(r, d)$ is stable. Hence, $h = 1$ and the proof is finished. □

Lemma 4.5.25. *Assume that $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ where C is an elliptic curve. Let $\sigma = (\mathcal{P}_{\mathcal{D}^\uparrow}, Z) \in \mathrm{Stab}(\mathcal{D}^\uparrow)$. There are $F, H \in \{i_1, i_2, \Delta\}$, $F \neq H$, such that $F(X), H(X)$ is σ-semistable for any stable $X \in \mathcal{D}$.*

Proof. Let the ordered triple (F, G, H) be equal to either of the ordered triples $(i_1[-1], i_2, \Delta), (i_2, \Delta, i_1)$ or $(\Delta, i_1, i_2[1])$. For $X \in \mathcal{P}_\mu(\phi_X)$ given (note that on $\mathcal{D}^b(\mathrm{Coh}(C))$ we only need to consider $\mathcal{P}_\mu$), there are, by proposition 4.5.20, $F, H \in \{i_1, i_2, \Delta\}, F \neq H$ such that $F(X), H(X)$ are σ-semistable and $\phi^\sigma_{H(X)} + 1 < \phi^\sigma_{F(X)}$. We must prove that for any stable object $D \in \mathcal{D}$, $F(D)$ $H(D)$ are σ-semistable. Let $Y' \in (\mathcal{P}_\mu)_{\mathcal{D}}(\phi'_Y)$, we must prove that $F(Y'), H(Y')$ are σ-semistable. If $\phi_{Y'} \neq \phi_X + n$ for any $n \in \mathbb{Z}$, we obtain the result by lemma 4.5.23, since for a suitable shift of Y', the Hom vanishing condition is fulfilled by [20, Corollary 4.4]. If, on the other hand, we have $\phi_{Y'} = \phi_X + n$ for $n \in \mathbb{Z}$, we obtain $Y \in (\mathcal{P}_\mu)_{\mathcal{D}}(\phi_X)$ for $Y = Y'[-n]$. There is a $\xi \in \mathbb{R}$ with $\xi \neq \phi_X + m$ for any $m \in \mathbb{Z}$ such that $(\mathcal{P}_\mu)_{\mathcal{D}}(\xi) \neq \emptyset$. Hence there is a non-zero $Z' \in (\mathcal{P}_\mu)_{\mathcal{D}}(\xi)$ and we set $Z = Z'[d] \in (\mathcal{P}_\mu)(\xi + d)$ such that $\xi + d \in (\phi_X, \phi_X + 1)$. We obtain that $F(Z), H(Z)$ are σ-semistable by lemma 4.5.23. Assume now that $F(Y')$ is not σ-semistable, then – since this condition is invariant under shift – so is $F(Y)$ and proposition 4.5.20 provides that $H(Z)$ and, in particular $G(Z)$ are σ-semistable where $\phi^\sigma_{G(Y)} + 1 < \phi^\sigma_{H(Y)}$. From the exact sequence

$$F(Z) \to G(Z) \to H(Z) \xrightarrow{+}$$

we obtain the inequality

$$\phi^\sigma_{F(Z)} \leq \phi^\sigma_{G(Z)} \leq \phi^\sigma_{H(Z)} \leq \phi^\sigma_{F(Z)} + 1.$$

From $\mathrm{Hom}_{\mathcal{D}}(X, Z) \neq 0$ we obtain $\mathrm{Hom}_{\mathcal{D}^\uparrow}(F(X), F(Z)) \neq 0$ since F is a fully-faithful, which provides us with $\phi^\sigma_{F(X)} \leq \phi^\sigma_{F(Z)}$. Since, again by assumption, $\mathrm{Hom}_{\mathcal{D}}(Y, Z) \neq 0$, we obtain $\mathrm{Hom}_{\mathcal{D}}(Y, Z)^* \neq 0$. Since G is fully-faithful, this gives

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^\uparrow}(G(Z), G(Y[-n+1])) &= \mathrm{Hom}_{\mathcal{D}}(Z, Y[-n+1]) \\ &= \mathrm{Hom}_{\mathcal{D}}(Z, S_{\mathcal{D}}(Y)) = \mathrm{Hom}_{\mathcal{D}}(Y, Z)^* \neq 0\end{aligned}$$

and hence $\phi^\sigma_{G(Z)} \leq \phi^\sigma_{G(Y)} + 1$. Since C is an elliptic curve, we have that $\mathrm{Coh}(C) = \mathcal{P}_\mu(0, 1]$ and there is an $a \in \mathbb{Z}$ such that $\mathcal{P}_\mu(\phi + a) \subset \mathcal{P}_\mu(0, 1]$ which implies $X[a], Y[a] \in \mathrm{Coh}(C)$. Additionally, we have the equation

$$\frac{\deg(Y[a])}{\mathrm{rank}(Y[a])} = \mu(Y[a]) = \tan((\phi_X + a)\pi) = \mu(X[a]) = \frac{\deg(X[a])}{\mathrm{rank}(X[a])}$$

using the concept of μ-stability. Since X and Y are simple objects, they are stable by [20, 2.4] and hence so are $X[a]$ and $Y[a]$. By lemma 4.5.24 this implies $\deg(Y[a]) = \deg(X[a])$ and $\mathrm{rank}(Y[a]) = \mathrm{rank}(X[a])$. Since σ is a numerical stability condition, the central charge Z of the stability condition σ is given by $Z(A) = Z(\text{rank}(\lambda_1(A)), \deg(\lambda_1(A)), \text{rank}(\rho_2(A)), \deg(\rho_2(A)))$ for $A \in \text{Coh}(C)$. Therefore $Z(J(Y[a])) = Z(J(X[a]))$ for any $J \in \{i_1, i_2, \Delta\}$. This implies the equality $\exp(i\pi\phi^\sigma_{H(X[a])}) = \exp(i\pi\phi^\sigma_{H(Y[a])})$, in other words $\cos(\phi^\sigma_{H(X[a])}) + i\sin(\phi^\sigma_{H(X[a])}) = \cos(\phi^\sigma_{H(Y[a])}) + i\sin(\phi^\sigma_{H(Y[a])})$ and from this we now obtain $\phi^\sigma_{H(X[a])} = \phi^\sigma_{H(Y[a])} + 2f$ where $f \in \mathbb{Z}$ and therefore $\phi^\sigma_{H(X)} = \phi^\sigma_{H(Y[-n])} + 2f$. However, for $B \in \{X, Y[-m]\}$, we have $\text{Hom}_{\mathcal{D}}(B, Z) \neq 0$ which provides

$$\text{Hom}_{\mathcal{D}^\uparrow}(H(B), H(Z)) \neq 0 \tag{4.31}$$

via the fully-faithfulness of H. This implies that $\text{Hom}_{\mathcal{D}}(B, Z)^* \neq 0$ as well. We hence have $\text{Hom}_{\mathcal{D}}(Z, B[1]) = \text{Hom}_{\mathcal{D}}(Z, S_{\mathcal{D}}(B)) = \text{Hom}_{\mathcal{D}}(B, Z)^* \neq 0$, that is, again using the fully-faithfulness of H,

$$\text{Hom}_{\mathcal{D}^\uparrow}(H(Z), H(B[1])) \neq 0. \tag{4.32}$$

While (4.31) provides us with $\phi^\sigma_{H(B)} \leq \phi^\sigma_{H(Z)}$, we obtain $\phi^\sigma_{H(Z)} \leq \phi^\sigma_{H(B)} + 1$. But since we have that Z, and therefore $H(Z)$ is non-zero, we obtain from this, that $[\phi^\sigma_{H(X)}, \phi^\sigma_{H(X)} + 1] \cap [\phi^\sigma_{H(Y[-m])}, \phi^\sigma_{H(Y[-m])} + 1] \neq \emptyset$, which can be rephrased as $H(X) \in [\phi^\sigma_{H(Y[-m])} - 1, \phi^\sigma_{H(Y[-m])} + 1]$. Hence, $f = 0$ and we get $\phi^\sigma_{H(Y)} = \phi^\sigma_{H(X)}$. Summing up, we obtain

$$\phi^\sigma_{G(Z)} \leq \phi^\sigma_{G(Y)} + 1 < \phi^\sigma_{H(Y)} = \phi^\sigma_{H(X)} < \phi^\sigma_{F(X)} \leq \phi^\sigma_{F(Z)} \leq \phi^\sigma_{G(Z)}$$

which is a contradiction. Hence, $F(Y)$ is σ-semistable. To see that $H(Y)$ too is semistable, repeat the argument swapping X and Y and replacing (F, G) by (G, H). □

Expressing this in a more suitable – topological – language we define

Notation 4.5.26. Let $\text{pre}\,\text{Stab}(\mathcal{D}^\uparrow)$ denote the space of pre-stability conditions on $\mathcal{D}^\uparrow$.

Definition 4.5.27. Define

$$\widetilde{\Theta}_{12} = \{\sigma \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) \mid i_1(X), i_2(X)\sigma\text{-semistable for all } X \in \mathcal{D} \text{ with } X \text{ stable}\}$$
$$\Theta_{12} = \{\sigma \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) \mid i_1(X), i_2(X)\sigma\text{-stable for all } X \in \mathcal{D} \text{ with } X \text{ stable}\}$$
$$\widetilde{\Theta}_{31} = \{\sigma \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) \mid i_1(X), \Delta(X)\sigma\text{-semistable for all } X \in \mathcal{D} \text{ with } X \text{ stable}\}$$
$$\Theta_{31} = \{\sigma \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) \mid i_1(X), \Delta(X)\sigma\text{-stable for all } X \in \mathcal{D} \text{ with } X \text{ stable}\}$$
$$\widetilde{\Theta}_{23} = \{\sigma \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) \mid i_2(X), \Delta(X)\sigma\text{-semistable for all } X \in \mathcal{D} \text{ with } X \text{ stable}\}$$
$$\Theta_{23} = \{\sigma \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) \mid i_2(X), \Delta(X)\sigma\text{-stable for all } X \in \mathcal{D} \text{ with } X \text{ stable}\}$$

Corollary 4.5.28. *Assume that $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ where C is an elliptic curve, then*

$$\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) = \widetilde{\Theta}_{12} \cup \widetilde{\Theta}_{13} \cup \widetilde{\Theta}_{23}.$$

Proof. This is simply a version of lemma 4.5.25 using the language of definition 4.5.27. □

However, it is possible to say more.

Theorem 4.5.29. *Assume that $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ where C is an elliptic curve, then*

$$\operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}) = \Theta_{12} \cup \Theta_{31} \cup \Theta_{23}.$$

Proof. We provide the exemplary proof that $i_1(X)$ not σ-semistable gives $\Delta(X)$ σ-stable. Everything follows the same way as we have previously seen throughout this section. We assume $\Delta(X)$ is not σ-stable and hence consider its JHF, that is, all its σ-stable factors A_i have the same phase ϕ. We assume that $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A_{i_0}, \Delta(X)) \neq 0$ for a σ-stable factor A_{i_0}. Therefore by [38, Exercise 1.6] and the fact that $i_1(X)$ was assumed to be non-semistable, arguing as before, we have that all the stable factors of $\Delta(X)$ are isomorphic to A_{i_0}. Hence, $[\Delta(X)] = n[A_{i_0}]$, where n is the number of stable factors. Since C is an elliptic curve, $\mathrm{rank}(X)$ and $\deg(X)$ are coprime which implies that the vector $[\Delta(X)] = (a, b, c, d)$ is non-divisible. Therefore we must have $n = 1$, in other words, $\Delta(X)$ is isomorphic to a stable object and therefore stable.

To obtain the equivalent of lemma 4.5.25 we now obtain the analogous statement of lemma 4.5.22 (that was the key to proving lemma 4.5.25) because the inequality (4.29) is still strict despite the fact that if $i_1(X_1)$ is semistable, the two strict inequalities that were then provided by the HNF are now given by the JHF and are therefore equalities. To correct for this we can now prove that $\phi_{\Delta(X_1)} < \phi_{i_1(X_2)}$. To see this consider the fact that $\phi_{i_1(X_1)} < \phi_{i_1(X_2)}$ given via the fact that equality would imply both objects in the same slice, therefore violating the locally finiteness of the slicing. Since $\mathrm{Hom}_{\mathcal{D}^\uparrow}(\Delta(X_1), i_1(X_1)) \neq 0$ we have $\phi_{\Delta(X_1)} \leq \phi_{i_1(X_1)}$ such that $\phi_{\Delta(X_1)} \leq \phi_{i_1(X_1)} < \phi_{i_1(X_2)}$. If, on the other hand, $i_1(X_1)$ is not semistable, we obtain $\Delta(X_2)$ semistable by lemma 4.5.25, then providing us with $\phi_{\Delta(X_1)} < \phi_{\Delta(X_2)} \leq \phi_{i_1(X_2)}$ by the same reasoning as before. □

For the sake of completeness we shall add the following.

Corollary 4.5.30. *Let Θ'_{ij} be like Θ_{ij} where now we assume $\sigma \in \mathrm{Stab}(\mathcal{D}^\uparrow)$ instead of* $\mathrm{pre\,Stab}(\mathcal{D}^\uparrow)$*, then we have*

$$\mathrm{Stab}(\mathcal{D}^\uparrow) = \Theta'_{12} \cup \Theta'_{31} \cup \Theta'_{23}.$$

Proof. This is an obvious consequence of theorem 4.5.29. □

Before we continue our discussion we will provide the following useful lemma.

Lemma 4.5.31. *If $\mathcal{A} = \mathrm{Coh}(C)$ for an elliptic curve $C, \sigma \in \mathrm{pre\,Stab}(\mathcal{D}^\uparrow)$ and X stable, then*

- *$i_1(X)$ strictly σ-semistable implies that it has a Jordan-Hölder filtration given by*
$$0 \to \Delta(X) \to i_1(X),$$
- *$i_2(X)$ strictly σ-semistable implies that it has a Jordan-Hölder filtration given by*
$$0 \to i_1(X) \to i_2(X)$$
and
- *$\Delta(X)$ strictly σ-semistable implies that it has a Jordan-Hölder filtration given by*
$$0 \to i_2(X) \to \Delta(X).$$

Proof. This is a a by-product of the proof of theorem 4.5.29. □

We will combine the fundamental lemma 4.5.25 with a fact provided by the following lemmas.

Lemma 4.5.32. *Assume that* $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ *where* C *is an elliptic curve. Let* $X, Y \in \mathcal{D}$ *such that* $X \in \mathcal{P}_\mu(\phi_X), Y \in \mathcal{P}_\mu(\phi_Y)$*. Let* F *be a faithful triangle functor* $F : \mathcal{D} \hookrightarrow \mathcal{D}^\uparrow$ *for which for any* $t \in \mathbb{R}$ *there is a* $t' \in \mathbb{R}$ *and a slicing* $\mathcal{P}$ *on* $\mathcal{D}^\uparrow$ *such that* $F(\mathcal{P}_\mu(t)) \subset \mathcal{P}(t')$ *. Let* $r, s \in \mathbb{R}$ *such that* $F(X) \in \mathcal{P}(r)$ *and* $F(Y) \in \mathcal{P}(s)$*. If* $\phi_Y - 1 < \phi_X < \phi_Y$*, then* $s - 1 \leq r \leq s$*.*

Proof. From $\phi_X - 1 < \phi_Y - 1 < \phi_X < \phi_Y$, we obtain

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \neq 0 \neq \mathrm{Hom}_{\mathcal{D}}(Y[-1], X)$$

by [20, Corollary 4.4]. Since F is faithful, this implies

$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(F(X), F(Y)) \neq 0 \tag{4.33}$$

and

$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(F(Y[-1]), F(X)) \neq 0. \tag{4.34}$$

Now, (4.33) gives $r \leq s$ and (4.34) gives $s - 1 \leq r$. □

Lemma 4.5.33. *Assume that* $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ *where* C *is an elliptic curve. Let non-zero* $X, Y \in \mathcal{D}$ *such that* $X \in \mathcal{P}_\mu(\phi_X), Y \in \mathcal{P}_\mu(\phi_Y)$*. Let* F *be a faithful triangle functor* $F : \mathcal{D} \hookrightarrow \mathcal{D}^\uparrow$ *for which for any* $t \in \mathbb{R}$ *there is a* $t' \in \mathbb{R}$ *and a locally finite slicing* $\mathcal{P}$ *on* $\mathcal{D}^\uparrow$ *such that* $F(\mathcal{P}_\mu(t)) \subset \mathcal{P}(t')$ *. Let* $r, s \in \mathbb{R}$ *such that* $F(X) \in \mathcal{P}(r)$ *and* $F(Y) \in \mathcal{P}(s)$*. If* $\phi_Y - 1 < \phi_X < \phi_Y$ *and there is a* $\sigma \in \mathrm{pre\,Stab}(\mathcal{D}^\uparrow)$ *such that* $\sigma = (\mathcal{P}, Z)$*, then* $s - 1 < r < s$*.*

Proof. We have $s - 1 \leq r \leq s$ by lemma 4.5.32. If we assume $r = s$, we obtain $Z(F(\mathcal{P}_\mu(\phi_X))) = Z(F(\mathcal{P}_\mu(\phi_Y))) \subset \mathcal{P}(r)$. This implies both $Z(F(\mathcal{P}_\mu(\phi_X))) \subset \mathbb{R}_+ \exp(i\pi r)$ and $Z(F(\mathcal{P}_\mu(\phi_X))) \subset \mathbb{R}_+ \exp(i\pi r)$. However, $\phi_Y - 1 < \phi_X < \phi_Y$ implies that $\mathcal{K}(\mathcal{D}) = \langle [X], [Y] \rangle$. This gives $F(\mathcal{K}(\mathcal{D})) \subset \mathbb{R}_+ \exp(i\pi r)$, which, on the other hand, implies that for any $\varphi \in \mathbb{R}$ there is an $n \in \mathbb{Z}$ such that $F(\mathcal{P}_\mu(\varphi)) \subset \mathcal{P}(r)[n]$. Since it is part of the definition of a slicing $\mathcal{P}'$ that $\mathcal{P}'(a + m) = \mathcal{P}'(a)[m]$ for $a \in \mathbb{R}, m \in \mathbb{Z}$ and moreover F is a triangle functor (e.g. commutes with shift), we can conclude that there is an $\alpha \in \mathbb{R}$ such that $F(\mathcal{P}_\mu(\alpha, \alpha + 1)) \subset \mathcal{P}(r)$. If α is irrational, then $\mathcal{P}_\mu(\alpha - \lfloor \alpha \rfloor, \alpha - \lfloor \alpha \rfloor + 1)$ is not noetherian as a direct implication of [54, Proposition 3.1]. Hence $\mathcal{P}_\mu(\alpha, \alpha + 1)$ is not noetherian as this property does not change under shift. If, on the other hand, $\alpha \in \mathbb{Q}$, we first consider the case where $\alpha = 0$. Then $\mathcal{P}_\mu(\alpha, \alpha + 1) = \mathrm{Coh}(C)$ which is not an artinian category seen from the non-stabilising descending sequence of subobjects $\mathcal{O} \supset \mathcal{O}(-1) \supset \mathcal{O}(-2) \supset \ldots$. For a non-zero α consider the

$\widetilde{\mathrm{SL}}(2,\mathbb{Z})$-action under which $\mathcal{P}_\mu(\alpha,\alpha+1)$ can be deformed into $\mathrm{Coh}(C)$. Since taking subobjects is stable under this action, $\mathcal{P}_\mu(\alpha,\alpha+1)$ is not artinian.

But F is a triangle functor and hence preserves both monomorphisms and epimorphisms in the abelian category $F(\mathcal{P}_\mu(\alpha,\alpha+1))$. Hence there either is a non-stabilising descending or a non-stabilising ascending sequence S of subobjects such that $S \in F(\mathcal{P}_\mu(\alpha,\alpha+1)) \subset \mathcal{P}(r) \subset \mathcal{P}(r-\eta, r+\eta)$ for any $\eta \in \mathbb{R}_+$. Therefore, the quasi-abelian category $\mathcal{P}(r-\eta, r+\eta)$ is not of finite length for any $\eta \in \mathbb{R}_+$, since it has an abelian subcategory that is not of finite length. This provides a contradiction to the assumption of $\mathcal{P}$ being locally finite. The proof for $s-1<r$ is similar. □

Lemma 4.5.34. *Assume that $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ where C is an elliptic curve and let $\sigma = (\mathcal{P}, Z) \in \operatorname{pre}\mathrm{Stab}(\mathcal{D}^\uparrow)$.*

1. *If $i_1(X)$ and $i_2(X)$ are σ-semistable for all stable $X \in \mathcal{D}$ then there are $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq \beta - 1$ such that we have $i_1(\mathcal{P}_\mu(\alpha,\alpha+1]) \subset \mathcal{P}(0,1]$ and $i_2(\mathcal{P}_\mu(\beta,\beta+1]) \subset \mathcal{P}(0,1]$.*

2. *If $i_1(X)$ and $\Delta(X)$ are σ-semistable for all stable $X \in \mathcal{D}$ then there are $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq \beta$ such that we have $\Delta(\mathcal{P}_\mu(\alpha,\alpha+1]) \subset \mathcal{P}(0,1]$ and $i_1(\mathcal{P}_\mu(\beta,\beta+1]) \subset \mathcal{P}(0,1]$.*

3. *If $i_2(X)$ and $\Delta(X)$ are σ-semistable for all stable $X \in \mathcal{D}$ then there are $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq \beta$ such that we have $i_2(\mathcal{P}_\mu(\alpha,\alpha+1]) \subset \mathcal{P}(0,1]$ and $\Delta(\mathcal{P}_\mu(\beta,\beta+1]) \subset \mathcal{P}(0,1]$.*

Proof. To see that we have $i_1(\mathcal{P}_\mu(\alpha,\alpha+1]) \subset \mathcal{A}$ and $i_2(\mathcal{P}_\mu(\beta,\beta+1]) \subset \mathcal{A}$ with $\alpha \geq \beta - 1$ whenever $i_1(X)$ and $i_2(X)$ are σ-semistable for all stable $X \in \mathcal{D}$, it is enough to proof this for stable X. We obtain α and β from lemma 4.5.33. To see $\alpha \geq \beta - 1$ assume $\alpha < \beta - 1$. Then there is a stable $Y \in \mathcal{D}$ such that $i_1(Y) \in \mathcal{P}(0,1]$ and $i_2(Y)[n] = i_2(Y[n]) \in \mathcal{P}(0,1]$ for an $n \in \mathbb{Z}_{\geq 2}$ This gives $\mathrm{Hom}^{-n+1}_{\mathcal{D}^\uparrow}(i_1(Y), i_2(Y)[n]) = \mathrm{Hom}^{-n}_{\mathcal{D}}(Y, Y[n]) = \mathrm{Hom}_{\mathcal{D}}(i_1(Y), i_2(Y)) \neq 0$ because Y stable implies $Y \neq 0$. This is a contradiction to 2.5.29 since $-n+1 < 0$. The proof for the other cases is similar. □

Corollary 4.5.35. *Assume that $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ where C is an elliptic curve and let $\sigma = (\mathcal{P}, Z) \in \operatorname{pre}\mathrm{Stab}(\mathcal{D}^\uparrow)$. Either*

1. *there are $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq \beta - 1$ such that we have $i_1(\mathcal{P}_\mu(\alpha,\alpha+1]) \subset \mathcal{P}(0,1]$ and $i_2(\mathcal{P}_\mu(\beta,\beta+1]) \subset \mathcal{P}(0,1]$*

2. *or there are $\alpha,\beta \in \mathbb{R}$ with $\alpha \geq \beta$ such that we have $\Delta(\mathcal{P}_\mu(\alpha,\alpha+1]) \subset \mathcal{P}(0,1]$ and $i_1(\mathcal{P}_\mu(\beta,\beta+1]) \subset \mathcal{P}(0,1]$*

3. *or there are* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta$ *such that we have* $i_2(\mathcal{P}_\mu(\alpha, \alpha+1]) \subset \mathcal{P}(0,1]$ *and* $\Delta(\mathcal{P}_\mu(\beta, \beta+1]) \subset \mathcal{P}(0,1]$.

Proof. We combine lemma 4.5.25 with lemma 4.5.34. □

This gives the following preliminary result.

Proposition 4.5.36. *Assume that* $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(C))$ *where* C *is an elliptic curve and let* $\sigma = (\mathcal{P}, Z) \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^\uparrow)$. *Either* σ *is obtained by CP-gluing via the semiorthogonal decompositions* $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle, \langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ *or* $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ *or it fulfils at least one of the following conditions.*

1. *There are* $\beta > \alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta - 1$ *such that* $i_1(\mathcal{P}_\mu(\alpha, \alpha+1]) \subset \mathcal{P}(0,1]$ *and* $i_2(\mathcal{P}_\mu(\beta, \beta+1]) \subset \mathcal{P}(0,1]$

2. *or there are* $\beta + 1 > \alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta$ *such that* $\Delta(\mathcal{P}_\mu(\alpha, \alpha+1]) \subset \mathcal{P}(0,1]$ *and* $i_1(\mathcal{P}_\mu(\beta, \beta+1]) \subset \mathcal{P}(0,1]$

3. *or there are* $\beta + 1 > \alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq \beta$ *such that* $i_2(\mathcal{P}_\mu(\alpha, \alpha+1]) \subset \mathcal{P}(0,1]$ *and* $\Delta(\mathcal{P}_\mu(\beta, \beta+1]) \subset \mathcal{P}(0,1]$.

Proof. Using corollary 4.5.35 we have to investigate the situations where $\alpha \geq \beta$ (or $\alpha \geq \beta + 1$ respectively). If we have both $i_1(\mathcal{P}_\mu(\alpha, \alpha+1]) \subset \mathcal{P}(0,1]$ and $i_2(\mathcal{P}_\mu(\beta, \beta+1]) \subset \mathcal{P}(0,1]$ with $\alpha \geq \beta$ then this does indeed imply that the set $H = \{X \in \mathcal{D}^\uparrow \mid \lambda_1(X) \in \mathcal{P}_\mu(\alpha, \alpha+1], \rho_2(X) \in \mathcal{P}_\mu(\beta, \beta+1]\}$ is a subset of $\mathcal{P}(0,1]$. But by theorem 3.2.36, H is the heart of a bounded t-structure obtained by CP-gluing via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Since a heart of a bounded t-structure cannot contain another heart of a bounded t-structure as a proper subset, we obtain $H = \mathcal{P}(0,1]$, which gives the required statement. The proof for $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ and $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ is similar. □

4.6 Shape of the Serre functor on $\mathcal{D}^\uparrow$

A particularly nice implication of theorem 4.5.5 is, that we are now able to give a complete description of the Serre functor on $\mathcal{D}^\uparrow$, as well as being able to conjecture that $\mathcal{D}^\uparrow$ is fractional Calabi-Yau where we can also conjecture on the associated fraction (see [39] for more insights on Calabi-Yau and fractional Calabi-Yau categories and their construction). However, in addition to being interesting with regard to the understanding of $\mathcal{D}^\uparrow$, this chapter also provides an important finding with regard to the further study of $\operatorname{Stab}(\mathcal{D}^\uparrow)$, given by lemma 4.6.7.

Lemma 4.6.1. *If $M : \mathcal{TR} \to \mathcal{TR}$ is an equivalence of triangulated categories and $\mathcal{TR} = \langle \mathcal{TR}^a, \mathcal{TR}^b \rangle$ with decomposition triangle*

$$A_b \to A \to A_a \xrightarrow{+},$$

then the distinguished triangle

$$M(A_b) \to M(A) \to M(A_a) \xrightarrow{+},$$

is the decomposition triangle of $M(A)$ for the semiorthogonal decomposition $\mathcal{TR} = \langle M(\mathcal{TR}^a), M(\mathcal{TR}^b) \rangle$.

Proof. Since $\mathrm{Hom}(A_b, A_a) = 0$ for any $A_a \in \mathcal{TR}^a$ and $A_b \in \mathcal{TR}^b$, this follows from [32, Lemma 6]. □

Lemma 4.6.2. *The decomposition triangles I, II and III defined in lemma 4.5.1 are related to each other via the Serre functor by the formulas*

$$S_{\mathcal{D}^\uparrow}(I)_A = (II)_{S_{\mathcal{D}^\uparrow}(A)}, S_{\mathcal{D}^\uparrow}(II)_A = (III)_{S_{\mathcal{D}^\uparrow}(A)} \text{ and } S_{\mathcal{D}^\uparrow}(III)_A = (I)_{S_{\}DT}(A)}.$$

Proof. Apply lemma 4.6.1 and the equations

$$\begin{aligned} S_{\mathcal{D}^\uparrow} \circ i_1 = i_2[1] \circ S_{\mathcal{D}}, \quad & S_{\mathcal{D}} \circ \lambda_1 = \mathbb{K} \circ S_{\mathcal{D}^\uparrow}, \\ S_{\mathcal{D}^\uparrow} \circ i_2 = \Delta \circ S_{\mathcal{D}}, \quad & S_{\mathcal{D}} \circ \rho_2 = \lambda_1 \circ S_{\mathcal{D}^\uparrow}, \\ S_{\mathcal{D}^\uparrow} \circ \Delta = i_1 \circ S_{\mathcal{D}} \text{ and } & S_{\mathcal{D}} \circ \mathbb{K} = \rho_2 \circ S_{\mathcal{D}^\uparrow} \end{aligned} \tag{4.35}$$

related to the Serre functor. □

Lemma 4.6.3. *If $A = (A_1 \xrightarrow{\varphi} A_2) \in \mathcal{D}^\uparrow$ and $S_{\mathcal{D}^\uparrow}(A) = (B_1 \xrightarrow{\psi} B_2)$, then*

$$B_1 = \lambda_1(S_{\mathcal{D}^\uparrow}(A)) = S_{\mathcal{D}}\rho_2(A) \text{ and } B_2 = \rho_2(S_{\mathcal{D}^\uparrow}(A)) = S_{\mathcal{D}}\mathbb{K}(A)[1].$$

Proof. This is a consequence of (4.35). □

Theorem 4.6.4. *If $A = (A_1 \xrightarrow{\varphi} A_2) \in \mathcal{D}^\uparrow, S_{\mathcal{D}^\uparrow}(A) = (B_1 \xrightarrow{\psi} B_2)$ and t_2 as in 4.5.2 (b), then $[\psi] \cong S_{\mathcal{D}}(t_2) : S_{\mathcal{D}}\rho_2(A) \to S_{\mathcal{D}}\mathbb{K}(A)[1]$ as a morphism in $\mathcal{D}$.*

Proof. From lemma 4.6.2 and one of the equations provided in (4.35), we obtain

$$\lambda_1(III)_{S_{\mathcal{D}^\uparrow}(A)} \cong \lambda_1 S_{\mathcal{D}^\uparrow}(II)_A \cong S_{\mathcal{D}}\rho_2(II)_A.$$

Using lemma 4.5.2 this translates into the statement that the two exact triangles

$$\mathbb{K}S_{\mathcal{D}^\uparrow}(A) \to \lambda_1 S_{\mathcal{D}^\uparrow}(A) \xrightarrow{[\varphi]} \rho_2 S_{\mathcal{D}^\uparrow}(A) \xrightarrow{+}$$
$$S_{\mathcal{D}}\lambda_1(A) \xrightarrow{S_{\mathcal{D}}([\varphi])} S_{\mathcal{D}}\rho_2(A) \xrightarrow{S_{\mathcal{D}}(t_2)} S_{\mathcal{D}}\mathbb{K}(A)[1] \xrightarrow{+}$$

are isomorphic to each other in $\mathcal{D}$. Hence $[\psi] \cong S_{\mathcal{D}}(t_2)$ as a morphism in $\mathcal{D}$. □

Corollary 4.6.5. *We have* $S_{\mathcal{D}^\uparrow}(A) \cong (S_{\mathcal{D}}(A_2) \overset{S_{\mathcal{D}}(\tau)}{\to} S_{\mathcal{D}}(\mathrm{Cone}(\varphi)))$ *where* $\tau : A_2 \to \mathrm{Cone}(\varphi)$ *assumes the role of the canonical mapping given by the mapping cone (see the diagram of definition 5.2.2 in which* τ *assumes the role of the morphism* i_F*).*

Proof. This is a consequence of theorem 4.6.4 since $[\tau] \cong [t_2]$. □

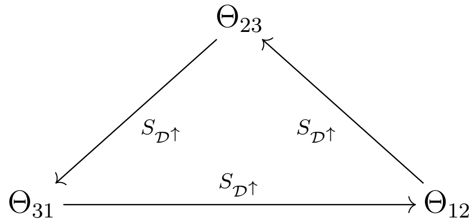
Corollary 4.6.6. *If* $\mathcal{D}$ *is* 1*-Calabi-Yau and* $X \in \mathcal{D}$*, we obtain*

- $S_{\mathcal{D}^\uparrow}(i_1(X)) = S_{\mathcal{D}^\uparrow}(X \to 0) = 0[1] \to X[1][1] = i_2(X)[2]$,
- $S_{\mathcal{D}^\uparrow}(i_2(X)) = S_{\mathcal{D}^\uparrow}(0 \to X) = X[1] \xrightarrow{\mathrm{id}_X[1]} X[1] = \Delta(X)[1]$ *and*
- $S_{\mathcal{D}^\uparrow}(\Delta(X)) = S_{\mathcal{D}^\uparrow}(X \xrightarrow{\mathrm{id}_X} X) = X[1] \to 0[1] = i_1(X)[1]$.

Proof. Let A equal to either $i_1(X), i_2(X)$ or $\Delta(X)$ in the formula provided in corollary 4.6.5 using that $S_{\mathcal{D}} = [1]$ since $\mathcal{D}$ is 1-Calabi-Yau. □

The following result allows us to generalise statements about one of the Θ_{ij} to the others.

Lemma 4.6.7. *The Serre functor* $S_{\mathcal{D}^\uparrow}$ *acts by circularly mapping the* Θ_{ij} *into one another as the following diagram demonstrates:*



Proof. This follows from corollary 4.6.6, combined with the fact that, for any Serre functor, we have $\mathrm{Hom}_{\mathcal{D}^\uparrow}(A, B) = \mathrm{Hom}_{\mathcal{D}^\uparrow}(S_{\mathcal{D}^\uparrow}(A), S_{\mathcal{D}^\uparrow}(B))$ and therefore, using lemma 4.5.31 the stability of the respective embedded objects is preserved. □

The following – if proven – would be a nice result that could add to the understanding of $\mathcal{D}^\uparrow$ further.

Definition 4.6.8. A triangulated category $\mathcal{TR}$ is called "$\frac{a}{b}$ fractional Calabi-Yau" for $a, b \in \mathbb{Z}$ if $\mathcal{TR}$ has a Serre functor $S_{\mathcal{TR}}$ and

$$S_{\mathcal{TR}}^b = [a].$$

Conjecture 4.6.9. *If* $\mathcal{D}$ *is* 1*-Calabi-Yau,* $\mathcal{D}^\uparrow$ *is fractional Calabi-Yau where the associated fraction is* $\frac{3}{4}$*.*

Remark 4.6.10. Conjecture 4.6.9 is based on corollary 4.6.5, which proves the statements as far as objects in $\mathcal{D}^\uparrow$ are concerned.

4.7 Application of tilting to $\mathcal{D}^{\uparrow}$

Tilting traces back to two articles by Bernstein, Gelfand and Ponomarev (see [1] for further reading) and operates on the heart of a bounded t-structure using a torsion pair to obtain a new heart – in other words, an essential ingredient of a stability condition. To perform tilting, one needs to make use of torsion pairs, originally introduced in [24]. Following [33] we provide

Definition 4.7.1. A "torsion pair" $(\mathcal{T}, \mathcal{F})$ on an abelian category $\mathcal{A}$ consists of full subcategories $\mathcal{T}$ and $\mathcal{F}$ such that

1. $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. For any $E \in \mathcal{A}$ there are $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that
$$0 \to T \to E \to F \to 0$$
is an exact sequence in $\mathcal{A}$.

Remark 4.7.2. Recalling definition 2.1.6 one can think of a torsion pair as the abelian equivalent of a semiorthogonal decomposition.

Lemma 4.7.3. *The category $\mathcal{F}$ is closed under subobjects and the category $\mathcal{T}$ closed under quotients.*

Proof. Let $E \in \mathcal{F}$ and $U \subset E$ Let
$$0 \to U_{\mathcal{T}} \xrightarrow{i_{U_{\mathcal{T}}}} U \to U_{\mathcal{F}} \to 0$$
be an exact triangle with $U_{\mathcal{T}} \in \mathcal{T}$ and $U_{\mathcal{F}} \in \mathcal{F}$. Let $i : U \to E$ be the embedding of U into E. We have that $i \circ i_{U_{\mathcal{T}}} \in \mathrm{Hom}(U_{\mathcal{T}}, E) = 0$ since $U_{\mathcal{T}} \in \mathcal{T}$ and $E \in \mathcal{F}$. Since $i_{U_{\mathcal{T}}}$ is an embedding as well as i, so is $i \circ i_{U_{\mathcal{T}}}$ and hence $U_{\mathcal{T}} = 0$. This provides $U \cong U_{\mathcal{F}}$.

Analogously we obtain the dual statement for $\mathcal{T}$. □

Lemma 4.7.4. *Let $\mathcal{P}$ be a slicing on $\mathcal{D}$ with $\mathcal{A} = \mathcal{P}(0, 1]$. Let $\phi \in (0, 1]$, then $(\mathcal{T}_1, \mathcal{F}_1) = (\mathcal{P}(\phi, 1], \mathcal{P}(0, \phi])$ and $(\mathcal{T}_2, \mathcal{F}_2) = (\mathcal{P}[\phi, 1], \mathcal{P}(0, \phi))$ are torsion pairs on $\mathcal{A}$.*

Proof. This follows from definition 4.7.1. □

The motivating example to be explained next ties in with the theory outlined in lemma 4.7.4.

Example 4.7.5. *Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is a smooth projective curve and consider the stability condition $(Z_\mu, \mathcal{A})$. Using the torsion pair described in lemma 4.7.4 gives*

$$\mathcal{T} = \mathcal{P}(1) = \{\textit{torsion sheaves on } X\}$$
$$\textit{and } \mathcal{F} = \mathcal{P}(0,1) = \{\textit{torsion-free sheaves on } X\}$$

if we let $\phi = 1$.

Remark 4.7.6. The torsion pair provided in lemma 4.7.4 is the most prevalent construction of a torsion pair in a situation where a slicing is available.

We now have the following lemma that gives this subsection its meaning.

Lemma 4.7.7. *Let $\mathcal{H}$ be the heart of a bounded t-structure on a derived category $\mathcal{D}$. Assume there is a torsion pair $(\mathcal{T}, \mathcal{F}\rangle)$ on $\mathcal{H}$. Then the full subcategory*

$$\mathcal{H}^\sharp = \{E \in \mathcal{D} \mid H^i_{\mathcal{H}}(E) = 0 \textit{ if } i \notin \{-1, 0\}, H^{-1}_{\mathcal{H}}(E) \in \mathcal{F} \textit{ and } H^0_{\mathcal{H}}(E) \in \mathcal{T}\}$$

is the heart of a bounded t-structure on $\mathcal{D}$.

Proof. See [33, Proposition 2.1]. □

Remark 4.7.8. The technique of obtaining new hearts provided by lemma 4.7.7 is often in the literature – and will be here – referred to as "tilting". Note that in some articles it is referred to as "left tilting" whereas left tilting combined with $[-1]$ is then referred to as "right tilting".

Lemma 4.7.9. *The pair $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair on $\mathcal{H}^\sharp$.*

Proof. This obvious fact is, for example, mentioned in [16, Section 5.2]. □

The theory we just introduced allows us to compute hearts of t-structures. Equipping a heart like this with a suitable stability function will subsequently provide a stability condition. For the construction that we are planning we have to radically restrict our category $\mathcal{A}$ to a smooth projective curve as otherwise we do not have the necessary concepts available.

Remark 4.7.10. Note that for $\mathcal{A} = \mathrm{Coh}(C)$, where C is a smooth projective curve, the category $\mathcal{D}$ has a Serre functor and – hence – so does $\mathcal{D}^\uparrow$.

Definition 4.7.11. Let $Y \in \mathcal{A}^\uparrow$, where $\mathcal{A} = \mathrm{Coh}(C)$, C a smooth projective curve, $C_1 \in \mathbb{R}$ and $D_1 \in \mathbb{R}_{<0}$. We define a group homomorphism $Z_\lambda : \mathcal{N}(\mathcal{A}^\uparrow) \xrightarrow{\cong} \mathbb{Z}^4 \to \mathbb{C}$ by

$$Z_\lambda(Y) = D_1 \deg(\lambda_1^{\mathcal{A}}(Y)) + (C_1 - 1)\,\mathrm{rank}(\lambda_1^{\mathcal{A}}(Y)) + i(\mathrm{rank}(\lambda_1^{\mathcal{A}}(Y)) + \mathrm{rank}(\rho_2^{\mathcal{A}}(Y)))$$

and

$$\lambda(Y) = (1/\pi)\arg(Z_\lambda([Y])) \in (0,1]$$

if $Y \neq (0 \to T)$ where $T \in \mathcal{A}$ is a torsion sheaf.

Lemma 4.7.12. *The group homomorphism Z_λ is a weak stability function on $\mathcal{A}^\uparrow$.*

Proof. Since $\Im(Z_\lambda) = \operatorname{rank}(\lambda_1^{\mathcal{A}}(Y)) + \operatorname{rank}(\rho_2^{\mathcal{A}}(Y)) \geq 0$ this follows from definition 2.4.3. □

Definition 4.7.13. Let $\mathcal{A} = \operatorname{Coh}(C)$, where C is a smooth projective curve. An object $Y \neq (0 \to T)$ where $T \in \mathcal{A}$ is a torsion sheaf, is λ-semistable if $\lambda(Y) \geq \lambda(U)$ for any subobject $U \subset Y$ with $U \neq (0 \to T')$ where $T' \in \mathcal{A}$ is a torsion sheaf.

Lemma 4.7.14. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence where $A, B, C \in \mathcal{A}^\uparrow$ and $A, B, C \neq 0 \to T$, where T is a torsion sheaf, then*

$$\lambda(A) < \lambda(B) \text{ if and only if } \lambda(B) < \lambda(C),$$
$$\lambda(A) > \lambda(B) \text{ if and only if } \lambda(B) > \lambda(C) \text{ and}$$
$$\lambda(A) = \lambda(B) \text{ if and only if } \lambda(B) = \lambda(C)$$

Proof. Since the Z used to define λ is a weak stability function by 2.4.3 and $Z_\lambda(A) \neq 0, Z_\lambda(B) \neq 0$ and $Z_\lambda(C) \neq 0$ we obtain that $\lambda(A), \lambda(B)$ and $\lambda(C)$ are defined and finally the result by lemma 2.4.22. □

Definition 4.7.15. An object $X \in \mathcal{A}^\uparrow$, $\mathcal{A} = \operatorname{Coh}(C)$, C a smooth projective curve, is "torsion-free" if $\lambda_1^{\mathcal{A}}(X)$ and $\rho_2^{\mathcal{A}}(X)$ are torsion-free in $\mathcal{A}$.

Remark 4.7.16. Note that this implies, that λ of definition 4.7.11 is defined for all torsion-free $X \in \mathcal{A}^\uparrow$.

Lemma 4.7.17. *Let $Y \in \mathcal{A}^\uparrow$, where $\mathcal{A} = \operatorname{Coh}(C), C$ a smooth projective curve. If Y is λ-semistable and there is non-zero torsion-free object $Q \in \mathcal{A}^\uparrow$ such that there is an epimorphism $Y \overset{q}{\twoheadrightarrow} Q$, then $\lambda(Y) \leq \lambda(Q)$.*

Proof. The surjectivity of q provides the short exact sequence

$$0 \to \ker(q) \hookrightarrow Y \overset{q}{\twoheadrightarrow} Q \to 0$$

where we have $\lambda(\ker(q)) \leq \lambda(Y)$ provided by the λ-semistability of Y, such that we obtain $\lambda(Y) \leq \lambda(Q)$ by lemma 4.7.14. □

In order to prove the existence of a HNF for torsion-free objects with regard to λ, provided by lemma 4.7.20, we adapt [42, Proposition 5.4.2] and as to do so give an enhanced version of [42, Lemma 5.4.1], provided by lemma 4.7.18. In classical textbooks, such as [42] or [34], one finds the Riemann-Roch formula

$$\chi(L) = \deg(L) + \operatorname{rank}(L) \cdot (1 - g)$$

for vector bundles L on a smooth projective curve C of genus g. On an elliptic curve this – hence – simplifies to

$$\chi(L) = \deg(L).$$

However, we define $\deg(F) = \deg(L) + \chi(T)$ for a coherent sheaf F where T is the torsion-subsheaf of F and L the quotient $L = F/T$ which is torsion-free – and on a smooth projective curve therefore also locally free. Since $\chi(F) = \chi(T) + \chi(L)$ provided by the additivity of χ on exact sequences we obtain $\deg(F) = \chi(F)$.

Lemma 4.7.18. *Let E be a coherent sheaf on an elliptic curve C. Then the degree of its subsheaves $F \subset E$ is bounded above.*

Proof. We generalise [52, Corollary 10.9] with regard to non-locally-free sheaves. Let $F \subset E$ be a coherent subsheaf. This means there is an exact sequence

$$0 \to F \to E \to F/E \to 0$$

and applying the (left exact) functor H^0 to it hence provides us with $H^0(F) \hookrightarrow H^0(E)$. This implies $h^0(F) \leq h^0(E)$. Which gives

$$\deg(F) = h^0(F) - h^1(F) \leq h^0(F) \leq h^0(E)$$

and the proof is finished. □

Corollary 4.7.19. *Let $A \in \mathcal{A}^{\uparrow}$ where $\mathcal{A} = \operatorname{Coh} C$ and C is an elliptic curve. Assume that non-zero A is torsion-free. There is a torsion-free $X \subset A$ such that $\lambda(X) \geq \lambda(Y)$ for any torsion-free $Y \subset A$.*

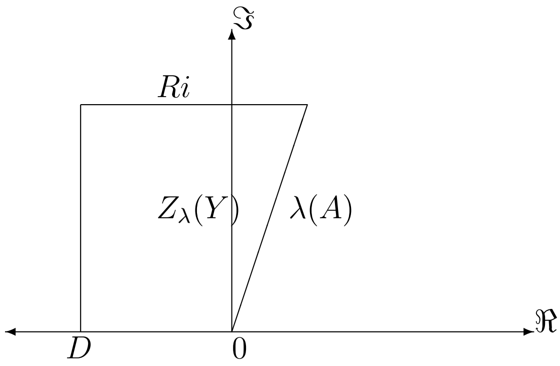
Proof. If $\max\{\lambda(Y) \mid Y \subset A, Y \text{torsion-free}\} = \lambda(A)$, we can take $X = A$. Otherwise $\{\lambda(Y) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\} \neq \emptyset$ and $\max\{\lambda(Y) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\} = \max\{\lambda(Y) \mid Y \subset A, Y \text{torsion-free}\}$ if a maximum exists. It therefore suffices to consider the set $\{\lambda(Y) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\}$. Firstly, observe that $Y \subset A$ forces

$$0 \leq \operatorname{rank}(\lambda_1^{\mathcal{A}}(Y)) \leq \operatorname{rank}(\lambda_1^{\mathcal{A}}(A)) \text{ and}$$
$$0 \leq \operatorname{rank}(\rho_2^{\mathcal{A}}(Y)) \leq \operatorname{rank}(\rho_2^{\mathcal{A}}(A)),$$

implying finiteness of $\{(C_1-1)\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y))+i(\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y))+\operatorname{rank}(\rho_2^{\mathcal{A}}(Y)) \mid Y \subset A, Y \text{torsion-free}\}$ as well as the inequality

$$\begin{aligned} 0 \leq \Im(Z_\lambda(Y)) &= \operatorname{rank}(\lambda_1^{\mathcal{A}}(Y)) + \operatorname{rank}(\rho_2^{\mathcal{A}}(Y)) \\ &\leq \operatorname{rank}(\lambda_1^{\mathcal{A}}(A)) + \operatorname{rank}(\rho_2^{\mathcal{A}}(A)) =: R. \end{aligned}$$

Since $\{\deg(\lambda_1^{\mathcal{A}}(Y)) \mid Y \subset A, Y \text{torsion-free}\}$ is bounded above by lemma 4.7.18, the expression $D_1(\deg(\lambda_1^{\mathcal{A}}(Y)))$ is bounded below, since $D_1 < 0$. Since the expression $(C_1-1)\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y))$ takes only finitely many values, this implies that $\Re(Z(Y)) = D_1(\deg(\lambda_1^{\mathcal{A}}(Y))) + (C_1-1)\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y))$ is bounded below by a $D \in \mathbb{R}$. Boundaries for $\{Z_\lambda(Y) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\}$ in the complex plain are therefore given by the enclosed area



We have that $|\deg(\lambda_1^{\mathcal{A}}(Y))| \gg 0$ implies $|D_1 \deg(\lambda_1^{\mathcal{A}}(Y))| \gg 0$. Since Z_λ is given by the formula

$$Z_\lambda(Y) = D_1 \deg(\lambda_1^{\mathcal{A}}(Y)) + (C_1-1)\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y)) + i(\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y)) + \operatorname{rank}(\rho_2^{\mathcal{A}}(Y)))$$

and we have already seen that the set $\{(C_1-1)\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y))+i(\operatorname{rank}(\lambda_1^{\mathcal{A}}(Y))+\operatorname{rank}(\rho_2^{\mathcal{A}}(Y)) \mid Y \subset A, Y \text{torsion-free}\}$ is finite, $|\deg(\lambda_1^{\mathcal{A}}(Y))| \gg 0$ would therefore imply that $|\Re(Z_\lambda(Y))| \gg 0$. This however – as we can see from the diagram – provides a contradiction. Hence, we obtain that the set $\{\deg(\lambda_1^{\mathcal{A}}(Y)) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\}$ is bounded. Since $\{\deg(\lambda_1^{\mathcal{A}}(Y)) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\} \subset \mathbb{Z}$ and is therefore discrete, it is hence finite. This – on the other hand – implies that the set $\{D_1 \deg(\lambda_1^{\mathcal{A}}(Y)) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\}$ is finite too. We conclude that $\{Z(Y) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\}$ too is finite. Therefore there is a $Z_\lambda(Y) \in \{Z_\lambda(Y) \mid Y \subset A, Y \text{torsion-free}, \lambda(Y) > \lambda(A)\}$ such that $\arg(Z_\lambda(Y))$ is maximal. Hence there is a Y such that $\lambda(Y) = \arg(Z_\lambda(Y))/\pi$ is maximal. □

This provides us with a HNF for torsion-free objects.

Lemma 4.7.20. *Let $X \in \mathcal{A}^{\uparrow}$ and $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve, assume that X is torsion free. There is a unique sequence*

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

where X_i is a torsion-free object such that

- $\lambda(A_i) \geq \lambda(U)$ *for any $U \subset A_i$ and*
- $\lambda(A_{i-1}) > \lambda(A_i)$

with $A_i = X_i/X_{i-1}$ for $i \in \{1, \dots n\}$ and the quotients A_i are also torsion-free.

Proof. We adapt the proof of [42, Proposition 5.4.2]. Using corollary 4.7.19 pick an torsion-free $X_1 \subset X$ such that $\lambda(X_1)$ is the maximum of $\{\lambda(Y) \mid Y \subset X, Y \text{torsion-free}\}$ and that additionally $\mathrm{rank}(\lambda_1^{\mathcal{A}}(X_1)) + \mathrm{rank}(\rho_2^{\mathcal{A}}(X_1))$ is maximal amongst the torsion-free subsheaves of maximal λ. If X/X_1 is not torsion-free, this means that $Q_1 = \rho_2^{\mathcal{A}}(X)/\rho_2^{\mathcal{A}}(X_1)$ or $Q_2 = \lambda_1^{\mathcal{A}}(X)/\lambda_1^{\mathcal{A}}(X_1)$ are not torsion free. Note, that quotients are taken componentwise (regarding the $\lambda_1^{\mathcal{A}}$ and the $\rho_2^{\mathcal{A}}$ component). Consider, on one hand, the exact sequence

$$0 \to X_1 \to X \to Q \to 0,$$

where $Q = Q_1 \xrightarrow{\varsigma} Q_2$ and ς is the induced map on the Quotients. Consider, on the other hand, the exact sequence

$$0 \to \mathrm{Tors}(Q) \to Q \xrightarrow{f} Q/\,\mathrm{Tors}(Q) \to 0 \tag{4.36}$$

that, combining them, provide us with the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X & \xrightarrow[q]{} & Q & \longrightarrow & 0 \\ & & & & \big\downarrow{=} & & \big\downarrow{f} & & \\ 0 & \longrightarrow & H & \xrightarrow[k]{} & X & \xrightarrow[f\circ q]{} & Q' & \longrightarrow & 0 \end{array}$$

where $Q' = Q/\,\mathrm{Tors}(Q)$ and $H \in \mathcal{A}^{\uparrow}$ the kernel of $f \circ q$ with k being its canonical embedding. The canonical completion of this diagram provides us with a map $g : X_1 \to H$. By the snake lemma this map is injective and moreover we have $\mathrm{coker}(g) = \ker(f)$. On the other hand, we see from (4.36) that $\ker(f) = \mathrm{Tors}(Q)$ which means that the canonical sequence

$$0 \to \ker(g) \to X_1 \xrightarrow{g} H \to \mathrm{coker}(g) \to 0$$

for the morphism g becomes

$$0 \to X_1 \xrightarrow{g} H \to \operatorname{Tors}(Q) \to 0$$

and since $\operatorname{rank}(\lambda_1^{\mathcal{A}}(\operatorname{Tors}(Q))) = \operatorname{rank}(\rho_2^{\mathcal{A}}(\operatorname{Tors}(Q))) = 0$ provided by the fact that both $\lambda_1^{\mathcal{A}}(\operatorname{Tors}(Q))$ and $\rho_2^{\mathcal{A}}(\operatorname{Tors}(Q))$ are torsion sheaves, we obtain that $\operatorname{rank}(\lambda_1^{\mathcal{A}}(X_1)) = \operatorname{rank}(\lambda_1^{\mathcal{A}}(H))$ and $\operatorname{rank}(\rho_2^{\mathcal{A}}(X_1)) = \operatorname{rank}(\rho_2^{\mathcal{A}}(H))$ using the additivity of rank on exact sequences. Furthermore, the exact sequence

$$0 \to \lambda_1^{\mathcal{A}}(X_1) \to \lambda_1^{\mathcal{A}}(H) \to \lambda_1^{\mathcal{A}}(\operatorname{Tors}(Q)) \to 0$$

gives $\deg(\lambda_1^{\mathcal{A}}(H)) = \deg(\lambda_1^{\mathcal{A}}(X_1)) + \deg(\lambda_1^{\mathcal{A}}(\operatorname{Tors}(Q)))$ and moreover we have $\deg(\lambda_1^{\mathcal{A}}(\operatorname{Tors}(Q))) \geq 0$ since $\lambda_1^{\mathcal{A}}(\operatorname{Tors}(Q))$ is torsion, which – as we will prove following – implies that we must have $\lambda(H) \geq \lambda(X_1)$ using $D_1 < 0$ and, since $\lambda(X_1)$ was chosen to be maximal therefore $\lambda(H) = \lambda(X_1)$. If $\lambda(H) < \lambda(X_1)$, this would imply $\deg(\lambda_1^{\mathcal{A}}(H)) < \deg(\lambda_1^{\mathcal{A}}(X_1))$, since other then on the ranks, which, as we have previously seen are equal, λ only depends on the degree of the $\lambda_1^{\mathcal{A}}$-component. We can therefore replace X_1 by H which is a maximal torsion-free subobject that has a torsion-free quotient $Q' = Q/\operatorname{Tors}(Q)$. This allows us to conclude in the same way in which it is done in the proof of [42, Proposition 5.4.2]. □

Definition 4.7.21. Let $X \in \mathcal{A}^{\uparrow}$ where $\mathcal{A} = \operatorname{Coh}(C)$ and C is an elliptic curve and assume that X is torsion free. Then define $\lambda_+(X) = \lambda(A_1)$ and $\lambda_-(X) = \lambda(A_n)$ where A_i are the HN-factors introduced in lemma 4.7.20.

We are now ready to provide the data we need to tilt. First we need to introduce new terminology.

Lemma 4.7.22. *Let $\mathcal{A} = \operatorname{Coh}(C)$ where C is a smooth projective curve. Let $E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}^{\uparrow}$. There is a short exact sequence*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(E)_1 & \longrightarrow & E_1 & \longrightarrow & F(E)_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle t} & & {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
0 & \longrightarrow & T(E)_2 & \longrightarrow & E_2 & \longrightarrow & F(E)_2 & \longrightarrow & 0
\end{array}$$

with $T(E)_i$ torsion and $F(E)_i$ torsion-free for $i \in \{1, 2\}$ in $\mathcal{A}^{\uparrow}$.

Proof. We obtain t via $\operatorname{Hom}_{\mathcal{A}^{\uparrow}}(T(E)_1, T(E)_2) \cong \operatorname{Hom}_{\mathcal{A}^{\uparrow}}(T(E)_1, E_2)$ from $\operatorname{Hom}_{\mathcal{A}^{\uparrow}}(T(E)_1, F(E)_2) = 0$, since $T(E)_1$ is a torsion and $F(E)_2$ a torsion-free sheaf. Then, f is well-known to exist as a morphism in the abelian category $\mathcal{A}$ and hence as an object in $\mathcal{A}^{\uparrow}$. Or, in the more abstract language of triangulated categories that we choose to follow whenever possible, f is given by [32, Section 1.1, (TR3)]. □

This allows us to introduce

Definition 4.7.23. Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve and $E \in \mathcal{A}^{\uparrow}$. Set

$$T(E) = T(E)_1 \xrightarrow{t} T(E)_2$$
$$F(E) = F(E)_1 \xrightarrow{f} F(E)_2$$

from lemma 4.7.22.

As well as

Definition 4.7.24. Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve and set $\phi = 3/4$. Define $\mathcal{T}$ and $\mathcal{F}$ to be the full subcategories of $\mathcal{A}^{\uparrow}$ given by: $E \in \mathcal{T}$ if the HN-factors A_i of $F(E)$ (see lemma 4.7.20) satisfy $\lambda(A_i) > \phi$ and $\mathbb{K}(E) \in \mathcal{A}$. Moreover $E \in \mathcal{F}$ if $\lambda_1(E)$ is torsion-free and the HN-factors A_i of $F(E)$ satisfy $\lambda(A_i) \leq \phi$.

In each of these cases we assume that the condition on the HN-factors is automatic if $F(E) = 0$.

Lemma 4.7.25. *If for $(X \xrightarrow{f} Y) \in \mathcal{A}^{\uparrow}$ the functor $\mathbb{K}$ exists and $\mathbb{K}(X \xrightarrow{f} Y) \in \mathcal{A}$, then and only then f is an epimorphism in $\mathcal{A}$.*

Proof. The assumption $\mathbb{K}(X \xrightarrow{f} Y) \in \mathcal{A}$ provides that all objects in the exact triangle

$$\mathbb{K}(X \xrightarrow{f} Y) \to X \xrightarrow{f} Y \xrightarrow{+} \tag{4.37}$$

are in $\mathcal{A}$. Therefore 4.37 provides us with the short exact sequence

$$0 \to \mathbb{K}(X \xrightarrow{f} Y) \to X \xrightarrow{f} Y \to 0$$

in $\mathcal{A}$, making f an epimorphism.

If, on the other hand, f is an epimorphism, then

$$\ker(X \xrightarrow{f} Y) \to X \xrightarrow{f} Y \xrightarrow{+}$$

is an exact triangle, providing $\mathbb{K}(X \xrightarrow{f} Y) \cong \ker(X \xrightarrow{f} Y) \in \mathcal{A}$. □

Remark 4.7.26. We obtain the analogous statement for f a monomorphism and $\mathbb{K}(X \xrightarrow{f} Y)[1] \in \mathcal{A}$.

We are going to use the following fact throughout the subsection.

Lemma 4.7.27. *For any $E \in \mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve, we have that $\lambda_1(E) = 0$ implies $E \in \mathcal{F}$.*

Proof. Assuming $\lambda_1(E) = 0$, any subobject S of E fulfils $\lambda_1(S) = 0$. This implies, that every HNF-factor H of E must fulfil $\lambda_1(H) = 0$ too. Therefore $Z(H) = i\,\mathrm{rank}(\rho_2(H))$ which gives $\lambda(H) = 1/2 \leq 3/4 = \phi$. Therefore, by definition 4.7.24, we have $E \in \mathcal{F}$. □

Remark 4.7.28. We therefore obtain an equivalent definition to the one of definition 4.7.24, given by $E \in \mathcal{F}$ if $\lambda_1(E)$ is torsion-free and the HN-factors A_i of $F(E)$ satisfy $\lambda(A_i) \leq \phi$ or if $\lambda_1(E) = 0$.

Lemma 4.7.29. *The subcategories $(\mathcal{T}, \mathcal{F})$ from definition 4.7.24 form a torsion pair on $\mathcal{A}^\uparrow$.*

Proof. At first we prove the Hom-vanishing of definition 4.7.1. We will demonstrate that $\mathrm{Hom}_{\mathcal{A}^\uparrow}(E, F) = 0$ for $E \in \mathcal{T}$ and $F \in \mathcal{F}$ if E, F torsion-free. It suffices to prove this under the additional assumption that E, F are λ-semistable as the result follows then via the HNF. We hence have $\lambda(U) \leq \lambda(E)$ for any subobject $U \hookrightarrow E$ with $Z(U) \neq 0$ by definition of λ-stability. By lemma 4.7.14 we now too obtain $\lambda(V) \geq \lambda(E)$ for any quotient $E \twoheadrightarrow V$ of E with $Z(E) \neq 0$. For $\xi : E \to F$ we obtain $E \twoheadrightarrow \mathrm{im}(\xi)$ and $\mathrm{im}(\xi) \hookrightarrow F$. Hence $\xi = 0$ as otherwise $\mathrm{im}(\xi) \neq 0$ and since $\mathrm{im}(\xi) \subset F$ is torsion-free, $\lambda(\mathrm{im}(\xi))$ is defined and from $E \twoheadrightarrow \mathrm{im}(\xi)$ we obtain $\lambda(E) \leq \lambda(\mathrm{im}(\xi))$ as well as $\lambda(\mathrm{im}(\xi)) \leq \lambda(F)$ such that

$$\lambda(E) \leq \lambda(\mathrm{im}(\xi)) \leq \lambda(F) < \lambda(E)$$

where $\lambda(F) < \lambda(E)$ is given by E, F torsion-free and $E \in \mathcal{T}, F \in \mathcal{F}$.

If we drop the assumption of E, F torsion-free, consider for an object $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{T}$ and $G = G_1 \to G_2 \in \mathcal{F}$, the short exact sequences

$$0 \to T(E) \to E \to F(E) \to 0$$

and

$$0 \to T(G) \to G \to F(G) \to 0$$

provided by lemma 4.7.22. The definition of $\mathcal{F}$ provides $F(G) \in \mathcal{F}$ and $\lambda_1(G)$ torsion-free, which implies $\lambda_1(T(G)) = 0$. Therefore we only have to consider the case where $G = (0 \to H)$ with $H \in \mathcal{A}$ and the case where G_1, G_2 are torsion-free.

Assuming G_1 and G_2 to be torsion-free and hence $G_1 \to G_2 \in \mathcal{F}$, we see that $\mathrm{Hom}_{\mathcal{A}^\uparrow}(F(E), G) = 0$ since $\lambda_-(F(E)) > \lambda_+(G)$. Since we too have $\mathrm{Hom}_{\mathcal{A}^\uparrow}(T(E), G) = 0$ provided by the fact that G is torsion-free, we now obtain $\mathrm{Hom}_{\mathcal{A}^\uparrow}(E, G) = 0$.

If, on the other hand, $G = (0 \to H)$, we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, G) &= \mathrm{Hom}_{\mathcal{D}^\uparrow}(E, i_2(\rho_2(G))) = \mathrm{Hom}_{\mathcal{D}}(\mathbb{K}(E)[1], \rho_2(G)) \\ &= \mathrm{Hom}_{\mathcal{D}}(\mathbb{K}(E), \rho_2(G)[-1]) \subset \mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, \mathcal{A}[-1]) = 0.\end{aligned}$$

As a result of the consideration these two cases we obtain $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, G) = 0$.

We now have to find the short exact sequence from definition 4.7.1 for an arbitrary $E \in \mathcal{A}$. We will use that – by the slope phase correspondence (lemma 2.4.8) – any non-zero $E \in \mathcal{T}$ satisfies

$$\frac{-\Re(Z(F(E)))}{\Im(Z(F(E)))} = \frac{-D_1 d_1 - (C_1 - 1) r_1}{r_1 + r_2} > -\cot(3\pi/4) = 1$$

or, in other words,

$$-D_1 d_1 - C_1 r_1 - r_2 > 0 \tag{4.38}$$

where $d_i = \deg(F(E)_i)$ and $r_i = \mathrm{rank}(F(E)_i)$ for $i \in \{1, 2\}$.

We will first prove the existence of the short exact sequence from definition 4.7.1 for a torsion-free object E. By 4.7.20, there is a short exact sequence $0 \to \widetilde{T} \to E \to \widetilde{F} \to 0$, with $\widetilde{T}$ torsion-free such that $\lambda_-(\widetilde{T}) > 3/4$ and $\widetilde{F} = F_1 \xrightarrow{f'} F_2 \in \mathcal{F}$ is also torsion-free and $\lambda_+(\widetilde{F}) \leq 3/4$. At first we will prove that if $\widetilde{T} = T_1 \xrightarrow{t'} T_2$ is not surjective, then $\mathrm{coker}(t')$ is a torsion sheaf. By lemma 4.7.20, it suffices to prove the statement for a λ-semistable object $\widetilde{T}$ with $\lambda(\widetilde{T}) > 3/4$. If $\mathrm{coker}(t') \neq 0$, then we also have $t' \neq 0$, because, as $T_2 \neq 0$ for t' not surjective, $t' = 0$ would yield that $0 \to T_2$ is a subobject and a quotient of $\widetilde{T}$, which, by the λ-semistability of $\widetilde{T}$ and lemma 4.7.17 would imply $\lambda(\widetilde{T}) \leq \lambda(0 \to T_2) = 1/2 < 3/4 < \lambda(\widetilde{T})$. This is a contradiction. Now, this implies $\mathrm{im}(t'), \mathrm{coker}(t') \neq 0$, and as a consequence we obtain the morphisms

$$\begin{array}{ccc} T_1 & \twoheadrightarrow & T_1 \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle 0} \\ T_2 & \twoheadrightarrow & \mathrm{coker}(t') \end{array} \quad \text{and} \quad \begin{array}{ccc} T_1 & \hookrightarrow & T_1 \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t'} \\ \mathrm{im}(t') & \hookrightarrow & T_2 \end{array}$$

in $\mathcal{A}^\uparrow$. Now, if $\mathrm{rank}(\mathrm{coker}(t')) > 0$, then by λ-semistability of $\widetilde{T}$, we obtain via $T_2 \twoheadrightarrow \mathrm{coker}(t') \twoheadrightarrow F(\mathrm{coker}(t'))$ that

$$\begin{array}{ccc} T_1 & \twoheadrightarrow & 0 \\ {\scriptstyle t'}\downarrow & & \downarrow \\ T_2 & \twoheadrightarrow & F(\mathrm{coker}(t')) \end{array} \quad .$$

We use lemma 4.7.17 once again to see that $\lambda(\widetilde{T}) \leq 1/2 < 3/4 < \lambda(\widetilde{T})$, which gives us a contradiction. Therefore, we have that $\mathrm{coker}(t')$ is a torsion sheaf.

Now, consider the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1 & \longrightarrow & E_1 & \longrightarrow & F_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle t'} & & {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{im}(t') & \longrightarrow & E_2 & \longrightarrow & F_2' & \longrightarrow & 0. \end{array} \tag{4.39}$$

We will prove $T' = T_1 \xrightarrow{t'} \operatorname{im}(t') \in \mathcal{T}$, which means proving the condition on the λ-semistable factors of $T_1 \xrightarrow{t'} \operatorname{im}(t') \in \mathcal{T}$. Let us consider the last short exact sequence in its HN-decomposition $0 \to S \to T' \twoheadrightarrow A \to 0$ with $A = A_1 \to A_2$ λ-semistable and torsion-free. We want to show that $\lambda(A) > 3/4$. Note that S is also a subobject of $\widetilde{T}$, as a consequence we consider the short exact sequence $0 \to S \to \widetilde{T} \to \widetilde{T}/S \to 0$. We have that $\widetilde{T}/S$ is a quotient of $\widetilde{T}$ and therefore $\lambda(F(\widetilde{T}/S)) > 3/4$ by lemma 4.7.17, such that we obtain $3/4 < \lambda(F(\widetilde{T}/S)) = \lambda(A)$ via the fact that $\lambda_1(\widetilde{T}/S) = \lambda_1(A)$ and $\operatorname{rank}(\rho_2(\widetilde{T}/S)) = \operatorname{rank}(\rho_2(A))$ provided by $\operatorname{coker}(t')$ torsion. Therefore, we have $T_1 \xrightarrow{t'} \operatorname{im}(t') \in \mathcal{T}$ by lemma 4.7.25. Moreover, we prove that $\widetilde{F}$ is the torsion-free part of $F_1 \xrightarrow{f'} F_2'$ by considering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}(t') & \longrightarrow & E_2 & \longrightarrow & F_2' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & {\scriptstyle =}\downarrow & & {\scriptstyle\alpha}\downarrow & & \\ 0 & \longrightarrow & T_2 & \longrightarrow & E_2 & \longrightarrow & F_2 & \longrightarrow & 0. \end{array}$$

where i is the canonical embedding of the image. Applying the snake lemma we obtain the exact sequence

$$0 \to \operatorname{coker}(t') \to F_2' \xrightarrow{\alpha} F_2 \to 0.$$

Since F_2 torsion-free and $\operatorname{coker}(t')$ either torsion or 0, we obtain $F(F_2') = F_2$ and $T(F_2') = \operatorname{coker}(t')$. This implies that $F_1 \xrightarrow{f'} F_2' \in \mathcal{F}$. Therefore, if $E_1 \to E_2$ is torsion-free, the triangle (4.39) gives us the decomposition of E in $(\mathcal{T}, \mathcal{F})$.

We are now in a position to prove the existence of the short exact sequence from definition 4.7.1 for an arbitrary $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}$. Our main tool is the snake lemma. Consider the short exact sequence

$$0 \to T(E) \to E \to F(E) \to 0$$

from lemma 4.7.22. We obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T(E) & \longrightarrow & \widetilde{E} & \longrightarrow & T' & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow{\scriptstyle c} & & \downarrow & & \downarrow \\
0 & \longrightarrow & T(E) & \longrightarrow & E & \longrightarrow & F(E) & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E/\widetilde{E} & \xrightarrow{\cong} & F' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{4.40}$$

with exact rows and columns in $\mathcal{A}^{\uparrow}$, where $T' \in \mathcal{T}, F' \in \mathcal{F}$. The object $\widetilde{E} = \widetilde{E}_1 \xrightarrow{\widetilde{\varphi}} \widetilde{E}_2$ is the fibre-product of E and T' over $F(E)$ such that the exactness of

$$0 \to T(E) \to E \to F(E) \to 0$$

provides the morphism $T(E) \to \widetilde{E}$ and subsequently the exactness of

$$0 \to T(E) \to \widetilde{E} \to T' \to 0.$$

The snake lemma then provides the exactness of

$$0 \to \widetilde{E} \xrightarrow{c} E \to E/\widetilde{E} \to 0$$

as well as $E/\widetilde{E} \cong F'$. Now define $\widetilde{T} = \widetilde{E}_1 \xrightarrow{\widetilde{\varphi}} \operatorname{im}(\widetilde{\varphi})$ and $\widetilde{F} = \widetilde{E}/\widetilde{T} = 0 \to \operatorname{coker}(\widetilde{\varphi})$, which means that

$$0 \to \widetilde{T} \to \widetilde{E} \to \widetilde{F} \to 0 \tag{4.41}$$

is exact. We will now prove that $\widetilde{T} \in \mathcal{T}$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(\widetilde{T}) & \longrightarrow & \widetilde{T} & \longrightarrow & F(\widetilde{T}) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow \\
0 & \longrightarrow & T(E) & \xrightarrow{b_1} & \widetilde{E} & \longrightarrow & T' & \longrightarrow & 0
\end{array} \tag{4.42}$$

of exact sequences in $\mathcal{A}^{\uparrow}$, where we obtain α, β via the fact that T, F are functors. By the snake lemma we obtain the exact sequence

$$0 \to \ker(\beta) \xrightarrow{l} \operatorname{coker}(\alpha) \xrightarrow{m} \widetilde{F} \xrightarrow{n} \operatorname{coker}(\beta) \to 0. \tag{4.43}$$

Since $\ker(\beta)$ is torsion-free and $\operatorname{coker}(\alpha)$ is torsion we obtain $l = 0$. The snake lemma provides us with the key diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(E)_1 & \longrightarrow & \widetilde{E_1} & \longrightarrow & T_1' & \longrightarrow & \\
 & & {\scriptstyle t}\downarrow & & {\scriptstyle \widetilde{\varphi}}\downarrow & & {\scriptstyle t'}\downarrow & & \\
0 & \longrightarrow & T(E)_2 & \xrightarrow{a_3} & \widetilde{E_2} & \longrightarrow & T_2' & \longrightarrow & \\
 & & {\scriptstyle a_1}\downarrow & & {\scriptstyle a_4}\downarrow & & \downarrow & & \\
 & & \operatorname{coker}(t) & \xrightarrow{a_2} & \operatorname{coker}(\widetilde{\varphi}) & \longrightarrow & 0 & &
\end{array} \tag{4.44}$$

with exact rows and columns – note that the surjectivity of t' is provided by $T' \in \mathcal{T}$. Since a_1, a_2 and hence $a_2 \circ a_1$ is surjective, so is $a_4 \circ a_3$. Now, letting $b_2 : \widetilde{E} \to \widetilde{F}$ be the morphism provided by (4.41) we see from $\rho_2(\widetilde{T}) = \operatorname{im}(\widetilde{\varphi})$ that $\rho_2^{\mathcal{A}}(b_2 \circ b_1) = a_4 \circ a_3$ making $\rho_2^{\mathcal{A}}(b_2 \circ b_1)$ surjective. Since $\widetilde{F}_1 = \widetilde{E}_1/\widetilde{E}_1 = 0$, we also obtain $\lambda_1^{\mathcal{A}}(b_2 \circ b_1)$ and therefore $b_2 \circ b_1$ surjective. The canonical morphism $T(E) \xrightarrow{\eta} \operatorname{coker}(\alpha)$ fulfils $m \circ \eta = b_2 \circ b_1$, we obtain that m is surjective. This gives $n = 0$.

Now we obtain from the exactness of (4.43) combined with $l = 0$ that $\ker(\beta) = 0$ and with $n = 0$ that $\operatorname{coker}(\beta) = 0$ as well. Hence, β is an isomorphism, providing us with $F(\widetilde{T}) \cong T' \in \mathcal{T}$. Since $F(\widetilde{T}) \in \mathcal{T}$, the condition on the HN-factors of $\widetilde{T}$ is met by definition. Since moreover $\widetilde{E}_1 \xrightarrow{\widetilde{\varphi}} \operatorname{im}(\widetilde{\varphi})$ is clearly surjective, we obtain $\widetilde{T} \in \mathcal{T}$.

It is now our task to prove the existence of an $\widetilde{F}' \in \mathcal{F}$ such that

$$0 \to T \to E \to \widetilde{F}' \to 0$$

is exact. The surjectivity of $a_4 \circ a_3$ in (4.44) provides us with an additional fact – since $T(E)_2$ is by definition a torsion sheaf, so is $\operatorname{coker}(\widetilde{\varphi})$. Hence, by definition of $\mathcal{T}$, $\widetilde{F} = (0 \to \operatorname{coker}(\widetilde{\varphi}))$. From the application of the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{T} & \longrightarrow & \widetilde{E} & \longrightarrow & \widetilde{F} & \longrightarrow & 0 \\
\downarrow & & \| & & {\scriptstyle c}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow \\
0 & \longrightarrow & \widetilde{T} & \longrightarrow & E & \longrightarrow & \widetilde{F}' & \longrightarrow & 0
\end{array}$$

of exact sequences (where $\widetilde{F}' = E/\widetilde{T}$ and γ is provided by [32, Section 1.1, (TR3)]), we obtain $\operatorname{coker}(c) = \operatorname{coker}(\gamma)$. Moreover, we have $\operatorname{coker}(c) \cong F'$ by (4.40) and therefore $F' = \operatorname{coker}(\gamma)$. Since $F' \in \mathcal{F}$, we have that $\lambda_1(F')$ is torsion-free. Now, as $\lambda_1(\widetilde{F}) = 0$ we have $\lambda_1(\widetilde{F}') \cong \lambda_1(F')$ and therefore

$\lambda_1(\widetilde{F'})$ torsion-free. Since, moreover, $\widetilde{F}$ is torsion, we have $F(\widetilde{F'}) = F(F')$ which implies that the condition on the Harder-Narashiman factors is met. Therefore, we obtain $\widetilde{F'} \in \mathcal{F}$ and the proof is finished. □

Lemma 4.7.30. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. The set*

$$\mathcal{H}(C_1, D_1) = \{E \in \mathcal{D}^{\uparrow} \mid H^i(E) = 0 \text{ if } i \notin \{1, 0\}, H^1(E) \in \mathcal{T} \text{ and } H^0(E) \in \mathcal{F}\}$$

is the heart of a bounded t-structure on $\mathcal{D}^{\uparrow}$.

Proof. This follows from the combination of lemmas 4.7.7, 4.7.29 and the application of the shift-functor to the resulting heart. □

We complete the data that we need for a stability condition.

Lemma 4.7.31. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Let $M = \begin{bmatrix} -A_1 & B_1 \\ -D_1 & C_1 \end{bmatrix}$, $\det(M) > 0, \det(M + I) > 0, A_1, B_1 \in \mathbb{R}$ and C_1, D_1 are the same as in definition 4.7.11. The group homomorphism $Z : \mathbb{Z}^4 \to \mathbb{C}$*

$$\begin{aligned} Z(Y) = {} & A_1 \deg(\lambda_1(Y)) + B_1 \operatorname{rank}(\lambda_1(Y)) - \deg(\rho_2(Y)) \\ & + i(D_1 \deg(\lambda_1(Y)) + C_1 \operatorname{rank}(\lambda_1(Y)) + \operatorname{rank}(\rho_2(Y))) \end{aligned}$$

is a stability function on $\mathcal{H}(C_1, D_1)$.

Proof. We need to prove that the image of $E \in \mathcal{H}(C_1, D_1)$ under Z lies in the strict upper half plane $\mathbb{H} \cup \mathbb{R}_{<0}$. By virtue of lemma 4.7.9 we let $E \in \mathcal{T}$ and consider the short exact sequence $0 \to T(E) \to E \to F(E) \to 0$, where $F(E) \in \mathcal{T}$. Indeed, because of the right exactness of coker$(-)$ we obtain that $\mathbb{K}(E) \in \mathrm{Coh}(C)$ implies that $\mathbb{K}(F(E)) \in \mathrm{Coh}(C)$.

We prove now that $Z(E[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. To that end, we will first show that $Z(F(E)[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. We only need to prove this for $F(E)$ λ-semistable – it is seen from the fact that (4.38) holds for $F(E)$, which implies that we obtain $\Im(Z(F(E)[-1])) > 0$. Additionally we see that $Z(T(E)[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Indeed, as $\operatorname{rank}(T(E)_1) = \operatorname{rank}(T(E)_2) = 0$, then $\deg(T(E)_1) \geq 0$. If $T(E)_1 \neq 0$ we have $\deg(T(E)_1) > 0$ and moreover that $\Im(Z((T(E)[-1])) = -\deg(T(E)_1)D_1 > 0$, since $D_1 < 0$. If $\deg(T(E)_1) = 0$ then $T(E_1) = 0$, and $F(E_1) \cong E_1$, thus $\Im(Z_r(F(E)[-1])) = \Im Z(E[-1]) > 0$. Since Z is additive with respect to short exact sequences, we obtain that $Z(E[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$.

We now show that $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ holds for $E \in \mathcal{F}$ as well. As $\lambda_1(E)$ is a torsion-free sheaf, we have $\lambda_1(T(E)) = 0$ and therefore $T(E) = 0 \to T(E)_2$, where $T(E)_2$ is a torsion sheaf. By lemma 4.7.27, $T(E) \in \mathcal{F}$ and $F(E) \in \mathcal{F}$, which means that we only have to prove the statement for $T(E)$ and $F(E)$. Clearly $Z(T(E)) = -\deg(T(E)_2) < 0$, as $T(E)_2$ is a torsion sheaf, or, if $\deg(T(E)_2) = 0$, we have $E \cong F(E)$ anyway.

Now consider λ-semistable $F(E) = F(E)_1 \xrightarrow{f} F(E)_2 \in \mathcal{F}$. If $\lambda(F(E)) < 3/4$, then we obtain $D_1(\deg(F(E)_1)) + C_1(\mathrm{rank}(F(E)_1)) + \mathrm{rank}(F(E))_2 > 0$ in analogy to (4.38) and $Z(E)$ lies in $\mathbb{H} \cup \mathbb{R}_{<0}$. We therefore have to consider the case where $\lambda(F(E)) = 3/4$, in other words where we now have the equation $D_1(\deg(F(E)_1)) + C_1(\mathrm{rank}(F(E)_1)) + \mathrm{rank}(F(E))_2 = 0$ for a λ-semistable object $F(E)$. We set $\deg(F(E))_i = d_i$ and $\mathrm{rank}(F(E))_i = r_i$ and aim to prove that $A_1d_1 + B_1r_1 - d_2 < 0$. First note that if $F(E)_1 = 0$, then $0 = D_1d_1 + r_1C_1 + r_2 = r_2$ which implies $F(E) = 0$. If $F(E)_2 = 0$, then $D_1d_1 + r_1C_1 = 0$ and $A_1d_1 + B_1r_1 - d_2 = A_1d_1 + B_1r_1 < 0$ given by the condition $\det(M) > 0$.

Therefore, we must now conduct our proof for $F(E)_1$ and $F(E)_2 \neq 0$. However, this implies $\mathrm{rank}(\mathrm{coker}(f)) = 0$. To see this, consider the short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(E)_1 & \xrightarrow{id} & F(E)_1 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{im}(f) & \longrightarrow & F(E)_2 & \longrightarrow & \mathrm{coker}(f) & \longrightarrow & 0.
\end{array}$$

If $\mathrm{rank}(\mathrm{coker}(f)) \neq 0$, then by lemma 2.4.22 and by the λ-semistability of $F(E)$, we obtain the contradiction $3/4 = \lambda(F(E)) \leq 1/2$. This implies on one hand that $\lambda(F(E)_1 \to \mathrm{im}(f)) = \lambda(F(E))$ which gives $E_1 \to \mathrm{im}(f) \in \mathcal{F}$ since $E_1 \to \mathrm{im}(f) \in \mathcal{F}$ is a subobject of $F(E)$ and on the other hand that $[F(E)] = [F(E)_1 \to \mathrm{im}(f)] + (0,0,0,d_2'')$ in the numerical Grothendieck group, where $d_2'' = \deg(\mathrm{coker}(f)) \geq 0$. If we let $d_1'' = \deg(\mathrm{im}(f))$, we see that

$$A_1d_1 + B_1r_1 - d_2 = A_1d_1 + B_1r_1 - d_1'' - d_2'' \leq A_1d_1 + B_1r_1 - d_1''$$

and can therefore reduce our investigation to objects $F(E)_1 \xrightarrow{f} F(E_2) \in \mathcal{F}$ with $\mathrm{coker}(f) = 0$. In other words, we have $F(E) = F(E)_1 \twoheadrightarrow F(E)_2$ and hence obtain $r_1 = \mathrm{rank}(F(E)_1) \geq \mathrm{rank}(F(E)_2) = r_2$.

Since $K = i_1(\mathbb{K}(F(E))) = \ker(f) \to 0$ is a subobject of $F(E)$ in $\mathcal{A}^{\uparrow}$ and $\mathcal{F}$ is closed under subobjects by lemma 4.7.3, we see that $K \in \mathcal{F}$. Since $[i_1(\mathbb{K}(E))] = [K] = (r_1 - r_2, d_1 - d_2, 0, 0)$, it follows from $K \in \mathcal{F}$ that $\Im(Z(K)) = D_1(d_1 - d_2) + C_1(r_1 - r_2) \geq 0$ and hence that $-D_1(d_1 - d_2) - C_1(r_1 - r_2) \leq 0$. Since $D_1d_1 + r_1C_1 + r_2 = 0$, we obtain $-D_1(d_1 - d_2) - C_1(r_1 - r_2) + D_1d_1 + r_1C_1 + r_2 \leq 0$ and therefore $D_1d_2 \leq -(C_1 + 1)r_2$. Moreover,

$D_1 d_1 + r_1 C_1 + r_2 = 0$ obviously implies $d_1 = \frac{r_2 + C_1 r_1}{-D_1}$ and we obtain

$$A_1 d_1 + B_1 r_1 - d_2 = (\frac{1}{-D_1})(A_1 r_2 + r_1 A_1 C_1 - r_1 B_1 D_1 + D_1 d_2).$$

Since $-D_1 > 0$, we must show that $A_1 r_2 + r_1 A_1 C_1 - r_1 B_1 D_1 + D_1 d_2 < 0$. We have

$$\begin{aligned} A_1 r_2 + r_1 A_1 C_1 - r_1 B_1 D_1 + D_1 d_2 &\leq A_1 r_2 + r_1 A_1 C_1 - r_1 B_1 D_1 - (C_1 + 1) r_2 \\ &= r_2(A_1 - C_1 - 1) - r_1(\det(M)) \\ &\leq r_2(A_1 - C_1 - 1) - r_2(\det(M)) \\ &= r_2(-\operatorname{Tr}(M) - 1 - \det(M)) \\ &= r_2(-\det(M + I)) < 0, \end{aligned}$$

using that $-r_1 \leq -r_2, \det(M) > 0$ and $\det(M + I) > 0$.

Since Z is additive with respect to short exact sequences, we obtain that $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ for $E \in \mathcal{F}$ and, since we have seen earlier that this holds true for $E \in \mathcal{T}[-1]$ also, we obtain $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ for arbitrary $E \in \mathcal{H}(C_1, D_1)$. □

The following lemma is based on ideas of [17, Proposition 7.1].

Lemma 4.7.32. *Let $\mathcal{A} = \operatorname{Coh}(C)$ where C is an elliptic curve. If we have $A_1, B_1, C_1, D_1 \in \mathbb{Q}$ and Z as in lemma 4.7.31.Then the pair $(Z, \mathcal{H}(C_1, D_1))$ is a pre-stability condition on $\mathcal{D}^{\uparrow}$.*

Proof. By [7, Proposition B.2], in order to see that the HN-property is fulfilled, we – on one hand – need to prove that $\{\Im(Z(E)) \mid E \in \mathcal{H}(C_1, D_1)\}$ is a discrete subgroup of $\mathbb{R}$. But since $C_1, D_1 \in \mathbb{Q}$, we have $D_1 = \frac{\alpha}{m}, C_1 = \frac{\beta}{m}$ for $\alpha, \beta \in \mathbb{Z}$ and suitable $m \in \mathbb{Z}$.Therefore

$$D_1 \deg(\lambda_1(E)) + C_1 \operatorname{rank}(\lambda_1(E)) + \operatorname{rank}(\rho_2(E)) = \frac{\alpha \deg(\lambda_1(E)) + \beta \operatorname{rank}(\lambda_1(E)) + m \operatorname{rank}(\rho_2(E))}{m}$$

and since $\deg(\lambda_1(E)), \operatorname{rank}(\lambda_1(E)), \operatorname{rank}(\rho_2(E)) \in \mathbb{Z}$ we therefore obtain that $\alpha \deg(\lambda_1(E)) + \beta \operatorname{rank}(\lambda_1(E)) + m \operatorname{rank}(\rho_2(E)) \in \mathbb{Z}$. Hence, since m is given by C_1, D_1 and therefore fixed, the subgroup $\{\Im(Z(E))\}$ is indeed discrete.

Moreover, we must prove for $E \in \mathcal{H}(C_1, D_1)$ and an ascending sequence

$$0 \subset L_1 \subset L_2 \ldots \subset L_i \subset \ldots \subset E$$

of subobjects of E, where L_i belongs to the full subcategory $\mathcal{P}'(1)$ of objects with phase one, the sequence stabilises (note that $\mathcal{P}'(1)$ only becomes a slice as a result of this lemma). To prove that the sequence stabilises, we show that $\mathcal{P}'(1) \subset \mathcal{A}^{\uparrow}$ which is noetherian by lemma 2.2.2, giving the desired result. As $L_i \in \mathcal{H}(C_1, D_1) = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$, applying lemma 4.7.9, we can use the exact sequence

$$0 \to F_i \to L_i \to T_i[-1] \to 0, \tag{4.45}$$

with $F_i \in \mathcal{F}$ and $T_i \in \mathcal{T}$. Now, since F_i and $T_i[-1]$ are in $\mathcal{H}(C_1, D_1)$, we obtain that $Z(F_i)$ and $Z(T_i[-1])$ are in the upper half plane which provides $\Im Z(F_i) \geq 0$ and $\Im Z(T_i[-1]) \geq 0$. Since $\Im Z(L_i) = 0$ we obtain $\Im Z(T_i[-1]) = 0$ (and of course $\Im Z(F_i) = 0$ also), by the exact sequence (4.45), which gives $T_i = 0$. To see this use the torsion pair from lemma 4.7.22. Note that $\Im Z(T(T_i)) \leq 0$ with equality holding only if $T(T_i) = 0$ and $\Im Z(F(T_i)) \leq 0$ with equality holding only if $F(T_i) = 0$ provided by (4.38) since $F(T_i)$ is a quotient of T_i. This gives $\Im Z(T_i) < 0$ and hence a contradiction, unless $T_i = 0$. Therefore $\mathcal{P}'(1) \subset \mathcal{F} \subset \mathcal{A}^{\uparrow}$, which finishes the proof. □

Lemma 4.7.33. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Let $\sigma = (Z, \mathcal{H}(C_1, D_1))$ be a pre-stability condition defined in 4.7.31. Then we obtain that $i_1(\mathbb{C}(x))[-1], \Delta(\mathbb{C}(x))[-1]$ and $i_2(\mathbb{C}(x))$ are in $\mathcal{H}(C_1, D_1)$ and $i_2(\mathbb{C}(x))$ is stable of phase one.*

Proof. Since $Z(i_2(\mathbb{C}(x))) = -1$, and $i_2(\mathbb{C}(x))$ is a simple object in $\mathcal{F}$, $i_2(\mathbb{C}(x))$ is σ-stable of phase one. We have $i_1(\mathbb{C}(x))[-1], \Delta(\mathbb{C}(x))[-1] \in \mathcal{H}(C_1, D_1)$ by lemmas 4.7.7 and 4.7.9, since $i_1(\mathbb{C}(x)), \Delta(\mathbb{C}(x)) \in \mathcal{T}$. □

We now define a torsion pair that we are going to need subsequently.

Lemma 4.7.34. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. The pair $(Z_1, \mathcal{A})$ with $Z_1(E) = D_1 \deg(E) + (C_1 - 1)\operatorname{rank}(E) + i \operatorname{rank}(E)$ for C_1, D_1 like in definition 4.7.11 is a stability condition on $\mathcal{D}$.*

Proof. The corresponding Matrix is $\begin{pmatrix} -D_1 & C_1 - 1 \\ 0 & 1 \end{pmatrix}$ (and hence $\det(M) = -D_1 > 0$) for which we choose the unique f such that $f(0) = 0$ providing us with an element $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$which finishes the proof. □

Definition 4.7.35. Let $\mathcal{A} = \mathrm{Coh}(C)$, $\mathrm{Coh}(C)$ a smooth projective curve. Define $\mathcal{T}_1 = \mathcal{P}_1(\frac{3}{4}, 1]$ and $\mathcal{F}_1 = \mathcal{F}_1(0, \frac{3}{4})$ where $\mathcal{P}_1$ is the slicing that corresponds to the stability condition $(Z_1, \mathcal{A})$ with $Z_1(E) = D_1 \deg(E) + (C_1 - 1)\operatorname{rank}(E) + i \operatorname{rank}(E)$ for C_1, D_1 like in definition 4.7.11.

Remark 4.7.36. Note, that equivalently $B \in \mathcal{T}_1$ if the HN-factors of its torsion-free part with regard to $Z_1(E) = D_1 \deg(E) + (C_1 - 1)\operatorname{rank}(E) +$

$i\,\mathrm{rank}(E)$ satisfy

$$\frac{-\Re(Z_1(B))}{\Im(Z_1(B))} = \frac{-D_1\deg(B) - (C_1-1)\,\mathrm{rank}(B)}{\mathrm{rank}(B)} > -\cot(3\pi/4) = 1$$

and $B \in \mathcal{F}_1$ if it is a torsion-free sheaf, whose HN-factors satisfy

$$\frac{-\Re(Z_1(B))}{\Im(Z_1(B))} = \frac{-D_1\deg(\lambda_1(B)) - (C_1-1)\,\mathrm{rank}(\lambda_1(B))}{\mathrm{rank}(\lambda_1(B))} \leq -\cot(3\pi/4) = 1.$$

Lemma 4.7.37. *The pair $(\mathcal{T}_1, \mathcal{F}_1)$ defined in 4.7.35 is a torsion pair on $\mathcal{A}$.*

Proof. As mentioned in Remark 4.7.6, this is a standard-construction of torsion pair. □

We use this to obtain the following important lemma after introducing new notation.

Definition 4.7.38. On $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ for $\mathcal{A} = \mathrm{Coh}(C)$, C a smooth projective curve, define $\mathcal{A}^r := \mathcal{P}_\mu(r, r+1]$.

Lemma 4.7.39. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. We have that*

$$i_2(\mathcal{A}) \subset \mathcal{H}(C_1, D_1), i_1(\mathcal{A}^r) \subset \mathcal{H}(C_1, D_1) \text{ and } \Delta(\mathcal{A}^{r_3}) \subset \mathcal{H}(C_1, D_1), \quad (4.46)$$

where $\cot(r\pi) = \frac{C_1}{D_1}, \cot(r_3\pi) = \frac{C_1+1}{D_1}$ with $r, r_3 \in (-1, 0)$ such that $\mathcal{A}^r = \langle \mathcal{F}_1, \mathcal{T}_1[-1]\rangle$.

Proof. Since $\lambda_1 \circ i_2 = 0$ we have $i_2(\mathcal{A}) \subset \mathcal{F} \subset \mathcal{H}(C_1, D_1)$ by lemma 4.7.27.

The strategy to prove that both other inclusions hold is to make use of the torsion-pair defined in 4.7.35. We will prove $i_1(\mathcal{T}_1) \subset \mathcal{T}$ as well as $i_1(\mathcal{F}_1) \in \mathcal{F}$. We proceed similarly for $\Delta(\mathcal{A}^r) \subset \mathcal{H}(C_1, D_1)$, using a torsion pair analogous to $(\mathcal{T}_1, \mathcal{F}_1)$ (the corresponding matrix is now $\begin{pmatrix} -D_1 & C_1 - 1 \\ 0 & 2 \end{pmatrix}$).

To see that $i_1(\mathcal{A}^r) \subset \mathcal{H}(C_1, D_1)$, let $E \in \mathcal{A}^r$ be a μ-semistable object. Without loss of generality we may assume E to be torsion-free as otherwise we would simply work with $F(i_1(E))$. If $i_1(E)$ had a λ-destabilising subobject S, we would obtain

$$\frac{-D_1\deg(E) - (C_1-1)\,\mathrm{rank}(E)}{\mathrm{rank}(E)} < \frac{-D_1\deg(S) - (C_1-1)\,\mathrm{rank}(S)}{\mathrm{rank}(S)}$$

and since $-D_1 > 0$, hence

$$\frac{\deg(E)}{\mathrm{rank}(E)} < \frac{\deg(F)}{\mathrm{rank}(F)}$$

which provides a contradiction implying that $i_1(E)$ is λ-semistable also. Moreover, since Z_1 of definition 4.7.35 fulfils $Z_1 = Z_\lambda \circ i_1$, $E \in \mathcal{P}_1(t)$ implies $\lambda(i_1(E)) = t$.

If $A \in \mathcal{T}_1$, note that $\mathbb{K}(i_1(A)) = A \in \mathcal{A}$. Now consider its HNF $A_1, \ldots, A_n$. Since $A_1, \ldots, A_n$ are μ-semistable and in $\mathcal{T}_1$ we have $A_1, \ldots, A_n \in \mathcal{P}_1(\frac{3}{4}, 1]$ and hence $\lambda(i_1(A_1)), \ldots, \lambda(i_1(A_n)) \in (\frac{3}{4}, 1]$. Since $A_1, \ldots, A_n$ are also torsion-free However, $\mathcal{T}$ is extension closed and we obtain $A \in \langle A_1, \ldots, A_n \rangle \subset \mathcal{T}$.

Let now $A \in \mathcal{F}_1$. Since $\mathcal{P}_1(1) \subset \mathcal{T}_1$ in $\mathcal{A}$, the torsion-free criterion on the objects in $i_1(\mathcal{F}_1)$ is fulfilled – if $A = \lambda_1(i_1(A))$ was not torsion-free, then there would be a non-zero morphism from a torsion subobject $S \in \mathcal{P}_1 \in \mathcal{T}_1$ onto $A \in \mathcal{F}_1$ which is impossible. Now consider the HNF $A_1, \ldots, A_n$ of A and repeat the argument used in the case of $\mathcal{T}_1$.

Therefore we obtain $i_1(\mathcal{T}_1) \subset \mathcal{T}$ and $i_1(\mathcal{F}_1) \subset \mathcal{F}$. This implies $i_1(\mathcal{A}^r) \subset \mathcal{H}(C_1, D_1)$.

To see that $\Delta(\mathcal{A}^{r_3}) \subset \mathcal{H}(C_1, D_1)$, we consider a torsion pair given by $(\mathcal{T}_3, \mathcal{F}_3) = (\mathcal{P}(r_3{+}1, 1], \mathcal{P}(0, r_3{+}1])$, $\phi \in \mathbb{R}$ on $\mathcal{A}$, such that $\mathcal{A}^{r_3} = \langle \mathcal{F}_3, \mathcal{T}_3[-1] \rangle$. Let $E \in \mathcal{A}^{r_3}$ be a μ-semistable object. Note that we can assume μ-semistability without loss of generality because otherwise we only consider its last or, respectively, its first HN-factor. We can additionally assume E to be torsion-free as otherwise we simply work with $F(\Delta(E))$. We have that $\mathbb{K}(\Delta(E)) = 0 \in \mathcal{A}$. Let A_m be the last factor in the λ-HNF of $\Delta(E)$. We need to prove that $\lambda_-(\Delta(E)) = \lambda(A_m) > 3/4$, that is $-D_1 d_1' - C_1 r_1' - r_2' > 0$ (see (4.38)) where $\operatorname{rank}((A_m)_i) = r_i'$ and $\deg((A_m)_i) = d_i'$ with $A_m = (A_m)_1 \twoheadrightarrow (A_m)_2$. Note that the surjectivity is provided by the fact that A_m is a quotient of $\Delta(E)$. Since A_m being a quotient of $\Delta(E)$ also implies $\lambda_1(\Delta(E)) \twoheadrightarrow (A_m)_1$, we obtain from the μ-stability of $\lambda_1(\Delta(E))$ that

$$\frac{d_1'}{r_1'} = \mu((A_m)_1) \geq \mu(\lambda_1(\Delta(E))) = \frac{\deg(\lambda_1(\Delta(E)))}{\operatorname{rank}(\lambda_1(\Delta(E)))}$$

which, in combination with the fact that $\Delta(E) \in \Delta(\mathcal{T}_3)$ and we therefore obtain $\lambda_1(\Delta(E)) \in \mathcal{T}_3$, provides

$$\frac{-D_1 d_1' - C_1 r_1'}{r_1'} \geq \frac{-D_1 \deg(\lambda_1(\Delta(E))) - C_1 \operatorname{rank}(\lambda_1(\Delta(E)))}{\operatorname{rank}(\lambda_1(\Delta(E)))} > 1$$

via $\frac{d_1}{r_1} > -\frac{C_1+1}{D_1}$. This gives $-D_1 d_1' - C_1 r_1' - r_1' > 0$ and since $(A_m)_1 \twoheadrightarrow (A_m)_2$ implies $r_1' \geq r_2'$, we conclude

$$-D_1 d_1' - C_1 r_1' - r_2' \geq -D_1 d_1' - C_1 r_1' - r_1 > 0.$$

Hence, $\Delta(\mathcal{T}_3) \in \mathcal{T}$ and similarly, now using the first semistable quotient A_1 in the HNF of E which – since it is the first quotient – is also a subobject of $\Delta(E)$, we obtain $\Delta(\mathcal{F}_3) \subset \mathcal{F}$ which, as before, finishes the proof. □

4.8 Pre-stability conditions in Θ_{12}

This subsection aims to introduce the important result that pre-stability conditions in $\sigma \in \Theta_{ij}$ are always constructed by CP-gluing using one of the semiorthogonal decompositions $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle, \langle, \mathcal{D}_3, \mathcal{D}_1 \rangle, \langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ or by tilting in the sense of lemma 4.7.31 up to the $\mathrm{GL}_2^+(\mathbb{R})$-action. We will establish theorem 4.8.36 via the investigation of Θ_{12}.

We first characterise the hearts of the pre-stability conditions in terms of the stability of the skyscraper sheaves, by studying pre-stability conditions satisfying that one of the three embeddings of the skyscraper is stable of phase one. We then distinguish between the case where one certain other embedding is not σ-stable and one where it is. Where the non-stability of the embedding in question will turn out to be resulting in a CP-glued pre-stability condition, the other ones will turn out to be pre-stability conditions of the form of lemma 4.7.31. We adapt the theory developed in [17, Proposition 10.1].

The following is a special case of the theory developed in corollary 4.2.24.

Definition 4.8.1. Define

- $\mathcal{H}_{12} = \{E \in \mathcal{D}^\uparrow \mid \lambda_1(E) \in \mathcal{A}, \rho_2(E) \in \mathcal{A}\}$ and therefore to be the heart obtained by CP-gluing using two copies of $\mathcal{A}$, with regard to the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$,
- $\mathcal{H}_{31} = \{E \in \mathcal{D}^\uparrow \mid \rho_2(E) \in \mathcal{A}, \mathbb{K}(E)[1] \in \mathcal{A}\}$ and therefore to be the heart obtained by CP-gluing using two copies of $\mathcal{A}$, with regard to the semiorthogonal decomposition $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$,
- $\mathcal{H}_{23} = \{E \in \mathcal{D}^\uparrow \mid \mathbb{K}(E) \in \mathcal{A}, \lambda_1(E) \in \mathcal{A}\}$ and therefore to be the heart obtained by CP-gluing using two copies of $\mathcal{A}$, with regard to the semiorthogonal decomposition $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$.

Remark 4.8.2. Note that $\mathcal{H}_{12}$ from definition 4.8.1 is equal to $\mathcal{A}^\uparrow$.

Lemma 4.8.3. *Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve, and $\sigma = (Z, \mathcal{H})$ be a pre-stability condition and assume that there are exact functors $i_* : \mathcal{D} \to \mathcal{D}^\uparrow$, $j_* : \mathcal{D} \to \mathcal{D}^\uparrow$, $l_* : \mathcal{D} \to \mathcal{D}^\uparrow$, $j^! : \mathcal{D}^\uparrow \to \mathcal{D}$, $l^! : \mathcal{D}^\uparrow \to \mathcal{D}$, $j^* : \mathcal{D}^\uparrow \to \mathcal{D}$ with $i_*[-1] \dashv j^* \dashv j_* \dashv j^! \dashv l_* \dashv l^! \dashv i_*$ where j_* is an embedding,* *such that we have* $l_*(\mathbb{C}(x)), i_*(\mathbb{C}(x))$ *are* σ*-stable and* $j_*(\mathbb{C}(x))$ *is* σ*-stable of phase one. Assume additionally, that there are non-zero morphisms*

$$\begin{aligned} i_*(\mathbb{C}(x)) &\to j_*(\mathbb{C}(x)[1]), \\ j_*(\mathbb{C}(x)) &\to l_*(\mathbb{C}(x)) \\ \textit{and } l_*(\mathbb{C}(x)) &\to i_*(\mathbb{C}(x)). \end{aligned} \tag{4.47}$$

If $E \in \mathcal{H}$*, then* $H^k(j^!(E)) = H^k(l^!(E)) = 0$*, unless* $k \in \{0, 1\}$*. Also* $H^k(j^*(E)) = 0$*, unless* $k \in \{-1, 0\}$*. Moreover,* $H^{-1}(j^*(E))$ *and* $H^0(l^!(E))$ *torsion-free.*

Proof. First note that $i_*(\mathbb{C}(x))[-1], l_*(\mathbb{C}(x))[-1] \in \mathcal{H}$. To see this, consider the morphisms of (4.47), which provide $1 < \phi_\sigma(l_*(\mathbb{C}(x))) < \phi_\sigma(i_*(\mathbb{C}(x))) < 2$.

It suffices to conduct our proof for E stable. Otherwise we consider its JHF with last exact triangle

$$E_{k-1} \to E_k \to A_k \xrightarrow{+}$$

and using the exactness of $F \in \{j^!, l^!, j^*, \}$ we obtain the exact triangle

$$F(E_{k-1}) \to F(E_k) \to F(A_k) \xrightarrow{+}$$

and hence the exact sequence

$$H^i(F(E_{k-1})) \to H^i(F(E_k)) \to H^i(F(A_k))$$

where we obtain $H^i(F(A_k)) = 0$ directly and $H^i(F(E_{k-1})) = 0$ by induction. We will continuously apply [19, Proposition 5.4] and proceed as follows.

To see that $H^k(j^!(E)) = 0$, unless $k = 0, 1$ we will prove that

$$\mathrm{Hom}^j(j^!(E), \mathbb{C}(x)) = 0 \text{ for } j \notin \{-1, 0\} \text{ for } j^!(E) \neq \mathbb{C}(x).$$

If $j^!(E) \cong \mathbb{C}(x)$ then the statement obviously holds true anyway. We have

$$\mathrm{Hom}^j(j^!(E), \mathbb{C}(x)) = \mathrm{Hom}^j(E, l_*(\mathbb{C}(x))) \text{ which is zero if } j \leq 2$$

because of $E \in \mathcal{H} = \mathcal{P}(0, 1]$ and $\mathbb{C}(x) \in \mathcal{P}(1, 2)$ where $\mathcal{P}$ is the slicing of σ. On the other hand,

$$\begin{aligned} \mathrm{Hom}^j(j^!(E), \mathbb{C}(x)) &= \mathrm{Hom}^{1-j}(\mathbb{C}(x), j^!(E))^* \\ &= \mathrm{Hom}^{1-j}(j_*(\mathbb{C}(x)), E)^* = 0 \text{ if } 1 - j \leq 0 \end{aligned}$$

and $E \notin \mathcal{P}(1)$, since $j_*(\mathbb{C}(x)) \in \mathcal{P}(1)$ by assumption. If, however $E \in \mathcal{P}(1)$, then $\mathrm{Hom}^{1-j}(j_*(\mathbb{C}(x)), E) \neq 0$ would imply $j_*(\mathbb{C}(x)) \cong E$ since E

was assumed to be stable. Since j_* is an embedding we obtain $j^!(E) \cong j^!(j_*(\mathbb{C}(x))) \cong \mathbb{C}(x))$ and the statement holds true as well.

To see that $H^k(l^!(E)) = 0$ for $k \notin \{-1, 0\}$ proceed similar as before. However, in this case this will also imply $H^0(l^!(E))$ torsion-free since we can prove the Hom-vanishing without the consideration of $l^!(E) = \mathbb{C}(x)$. We have

$$\mathrm{Hom}^j(l^!(E), \mathbb{C}(x)) = \mathrm{Hom}^j(E, i_*(\mathbb{C}(x))) = 0 \text{ for } j \leq -2$$

and

$$\begin{aligned}\mathrm{Hom}^j(l^!(E), \mathbb{C}(x)) &= \mathrm{Hom}^{1-j}(\mathbb{C}(x), l^!(E))^* \\ &= \mathrm{Hom}^{1-j}(l_*(\mathbb{C}(x)), E)^* = 0 \text{ for } 1-j \leq 0.\end{aligned}$$

To see that $H^k(j^*(E)) = 0$, unless $k \in \{-1, 0\}$ and $H^{-1}(j^*(E))$ torsion-free we use

$$\mathrm{Hom}^j(j^*(E), \mathbb{C}(x)) = \mathrm{Hom}(E, j_*(\mathbb{C}(x))) = 0 \text{ for } j \leq -1$$

as well as

$$\begin{aligned}\mathrm{Hom}^j(j^*(E), \mathbb{C}(x)) &= \mathrm{Hom}^{1-j}(\mathbb{C}(x), j^*(E))^* \\ &= \mathrm{Hom}^{1-j}(i_*[1]\mathbb{C}(x), E)^* = 0 \text{ if } 1-j \leq -1,\end{aligned}$$

or, in other words, $2 \leq j$.

□

Remark 4.8.4. We need the level of generality of lemma 4.47 to not only obtain the first part of lemma 4.8.5 but the analogous statements for the situation where

$$j^* = \lambda_1, j_* = i_1, j^! = \mathbb{K}, l_* = i_2[1], l^! = \rho_2[-1], i_* = \Delta[1]$$

and the situation where

$$j^* = \rho_2, j_* = \Delta, j^! = \lambda_1, l_* = i_1, l^! = \mathbb{K}, i_* = i_2[1]$$

because we subsequently need this to prove lemma 4.9.31.

Lemma 4.8.5. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve and $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$ and assume that $\Delta(\mathbb{C}(x)), i_1(\mathbb{C}(x))$ are σ-stable and $i_2(\mathbb{C}(x))$ is σ-stable of phase one. Then, for $E \in \mathcal{D}^{\uparrow}$ we have*

1. *If $E \in \mathcal{H}$, then $H^i(\rho_2(E)) = H^i(\lambda_1(E)) = 0$, unless $i = 0, 1$. Also $H^i(\mathbb{K}(E)[1]) = 0$, unless $i = -1, 0$. Moreover, $H^{-1}(\mathbb{K}(E)[1]), H^0(\lambda_1(E))$ are torsion-free.*

2. *If E is stable of phase one, then either $E = i_2(T)$, where $T \in \mathcal{A}$ a torsion sheaf, or $E \in \mathcal{H}_{12}$ with $H^0(\mathbb{K}(E)[1]) = 0$ and we have that $\lambda_1(E)$ and $\rho_2(E)$ are torsion-free.*

3. $\mathcal{H}_{12} \subset \mathcal{P}_\sigma(0, 2]$

4. *The pair $\mathcal{T} = \mathcal{H}_{12} \cap \mathcal{P}_\sigma(1, 2]$ and $\mathcal{F} = \mathcal{H}_{12} \cap \mathcal{P}_\sigma(0, 1]$ defines a torsion pair of $\mathcal{H}_{12}$. Moreover, the heart $\mathcal{H}$ is the corresponding tilt.*

Proof. There are morphisms

$$\begin{aligned} i_1(\mathbb{C}(x)) &\to i_2(\mathbb{C}(x)[1]), \\ i_2(\mathbb{C}(x)) &\to \Delta(\mathbb{C}(x)) \\ \text{and } \Delta(\mathbb{C}(x)) &\to i_1(\mathbb{C}(x)), \end{aligned}$$

such that we can apply lemma 4.8.3 to see that part one holds.

We now proceed to prove the second part. Let $E \in \mathcal{P}_\sigma(1)$ a stable object, which is not isomorphic to $i_2(T)$, where T is a torsion sheaf. Since $\phi_\sigma(E) = 1$, like the proof of lemma 4.8.3 continuously applying [19, Proposition 5.4], we have

$$H^i(\lambda_1(E)) = 0 \text{ unless } i = 0 \text{ and } H^i(\rho_2(E)) = 0 \text{ unless } i = 0$$

as well as $\lambda_1(E)$ and $\rho_2(E)$ torsion-free.

For the third part assume $E \in \mathcal{H}_{12} \subset \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$, where $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is the standard t-structure on $\mathcal{D}^\uparrow$. If $F \in \mathcal{P}_\sigma(2, \infty)$, then, by the first part, $F \in \mathcal{D}^{\leq -1}$. Consequently, we have $0 = \operatorname{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = \operatorname{Hom}(\mathcal{D}^{\leq -2}, \mathcal{D}^{\geq -1})$, therefore $\operatorname{Hom}_{\mathcal{D}^\uparrow}(F, E) = 0$. Analogously, we have that if $B \in \mathcal{P}_\sigma(\leq 0)$, then $B \in \mathcal{D}^{\geq 1}$. Now, since $\operatorname{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$, then $\operatorname{Hom}_{\mathcal{D}^\uparrow}(E, B) = 0$. It follows that $E \in \mathcal{P}_\sigma(0, 2]$.

We now prove the fourth part. Let $E \in \mathcal{H}_{12}$, by the third part of the statement, there is an exact triangle $A \to E \to B \xrightarrow{+}$, where $A \in \mathcal{P}_\sigma(1, 2]$ and $B \in \mathcal{P}_\sigma(0, 1]$. After applying λ_1 we obtain a long exact cohomology-sequence and by part one we have $H^{-1}(\lambda_1(A)) = H^1(\lambda_1(B)) = 0$. This implies that $\lambda_1(A), \lambda_1(B) \in \mathcal{A}$. Analogously, we have $\rho_2(A), \rho_2(B) \in \mathcal{A}$ and we obtain $A, B \in \mathcal{H}_{12}$.

□

With the aid of lemmas 4.8.6 and 4.8.7, we now prove the following crucial fact, proposition 4.8.11. We will use the stability function Z_μ from definition 2.5.38.

Lemma 4.8.6. *If $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve, $F \in \{i_1, i_2, \Delta\}$ and σ is a pre-stability condition on $\mathcal{D}^{\uparrow}$ such that we have $F(\mathbb{C}(x))$ and $F(\mathcal{L})$ are σ-stable for any line-bundle $\mathcal{L}$, then there exists $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $\sigma g = (Z', \mathcal{H}')$ satisfies $F(\mathcal{P}_\mu(0,1]) = F(\mathcal{A}) \subset \mathcal{H}'$ and $Z'(F(E)) = Z_\mu(E)$ for all $E \in \mathcal{D}$.*

Proof. We have that

$$\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}(x)) \neq 0 \tag{4.48}$$

and – since the Serre functor $S_{\mathcal{D}}$ is equal to [1] provided by the fact that C is elliptic – additionally that $\mathrm{Hom}(\mathbb{C}(x), \mathcal{O}_C[1]) = \mathrm{Hom}(\mathcal{O}_C, \mathbb{C}(x))$ and therefore

$$\mathrm{Hom}(\mathbb{C}(x), \mathcal{O}_C[1]) = 0. \tag{4.49}$$

The assumed stability of $F(\mathbb{C}(x))$ and $F(\mathcal{O}_C)$, now provides the inequality $\phi_\sigma(F(\mathcal{O}_C)) < \phi_\sigma(F(\mathbb{C}(x))) < \phi_\sigma(F(\mathcal{O}_C)) + 1$. Therefore, there is an orientation preserving transformation $M : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying that

$$(A, D) \mapsto (-1, 0) \text{ and } (B, C) \mapsto (0, 1), \tag{4.50}$$

where $Z(F(\mathbb{C}(x)) = A + Di$ and $Z(F(\mathcal{O}_C)) = B + Ci$. There is an increasing function $f : \mathbb{R} \to \mathbb{R}$ that is compatible with $M = T^{-1}$ and that satisfies $f(x+1) = f(x) + 1$ with $f(1) = \phi_\sigma(F(\mathbb{C}(x)))$, $f(1/2) = \phi_\sigma(F(\mathcal{O}_C))$. The existence of this f is granted by the fact that $M(Z(F(\mathcal{O}_C))) = (0, 1)$ and $M(Z(F(\mathbb{C}(x)))) = (-1, 0)$. We obtain $(T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. The stability condition $\sigma' = \sigma(T, f)$ satisfies $i_1(\mathcal{A}) \subset \mathcal{H}'$, where $\sigma' = (Z', \mathcal{H}')$. Indeed, we have

$$F(\mathbb{C}(x)) \in \mathcal{P}(\phi_\sigma(F(\mathbb{C}(x)))) = \mathcal{P}(f(1)) = \mathcal{P}'(1),$$
$$F(\mathcal{L}) \in \mathcal{P}(\phi_\sigma(F(\mathcal{L}))) = \mathcal{P}(f(t_{\mathcal{L}})) = \mathcal{P}'(t_{\mathcal{L}}),$$

with $t_{\mathcal{L}} := \phi_{\sigma'}(F(\mathcal{L})) \in (0, 1)$ since (4.48) and (4.49) provide the inequality $\phi_{\sigma'}(F(\mathbb{C}(x))) - 1 < \phi_{\sigma'}(F(\mathcal{L})) < \phi_{\sigma'}(F(\mathbb{C}(x)))$. Therefore, all point sheaves and line bundles in $\mathcal{A}$ are mapped into $\mathcal{P}'(0, 1]$ by F. Since any object in $\mathcal{A}$ admits a filtration with quotients either isomorphic to point sheaves or to line bundles, we obtain $F(\mathcal{A}) \subset \mathcal{H}'$.

Finally we require $Z'(F(E)) = Z_\mu(E)$ for all $E \in \mathcal{D}$. This is provided by (4.50). □

Lemma 4.8.7. *Assume $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve, $F \in \{i_1, i_2, \Delta\}$ and $\sigma = (Z, \mathcal{H})$ a pre-stability condition on $\mathcal{D}^{\uparrow}$ such that for $Z \circ F = Z_\mu$, $F(\mathcal{P}_\mu(0, 1]) = F(\mathcal{A}) \subset \mathcal{H}$ and $F(X)$ is σ-stable for all stable $X \in \mathrm{Coh}(C)$. For any $(T, f) = g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ we have that $\sigma g = (Z', \mathcal{H}')$ satisfies $F(\mathcal{P}_\mu(r, r+1]) = F(\mathcal{A}^r) \subset \mathcal{H}'$, where $r = f(0)$.*

Proof. Because $F(X)$ is σ-stable and $Z(F(X)) = Z_\mu(X)$, we get that X and $F(X)$ have the same phase up to the addition of an even number. But we assumed $F(\mathcal{P}_\mu(0,1]) \subset \mathcal{H}$, so the phase of $F(X)$ lies in the interval $(0,1]$. As the phase of X is in the same interval, they must agree. This shows that $F(\mathcal{P}_\mu(t)) \subset \mathcal{P}(t)$, where $\mathcal{P}$ is the slicing of σ. If $\mathcal{P}'$ is the slicing of σg, then $\mathcal{H}' = \mathcal{P}'(0,1] = \mathcal{P}(r, r+1]$ and the result follows now. □

For simplicities sake we introduce the following notation.

Notation 4.8.8. Assume $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve. Let $Y \in \mathcal{A}^\dagger$. Denote

$$\begin{aligned} d_1 = \deg(\lambda_1(Y))), r_1 = \mathrm{rank}(\lambda_1(Y))) \\ d_2 = \deg(\rho_2(Y)), r_2 = \mathrm{rank}(\rho_2(Y)). \end{aligned} \tag{4.51}$$

Definition 4.8.9. Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve and $\sigma \in \Theta_{12}$. Assume that $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ was applied to σ such that there are stability conditions $\sigma_1 = \sigma_\mu(T, f) = (Z_1, \mathcal{A}^r), r = f(0) > -1, (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $\sigma_2 = (Z_\mu, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$ with $i_1(\mathcal{A}^r) \subset \mathcal{H}$, $i_2(\mathcal{A}) \subset \mathcal{H}$, and $Z\big|_{\mathcal{D}_1} = Z_1$ and $Z\big|_{\mathcal{D}_2} = Z_\mu$. We refer to the conditions above as "normalised stability conditions".

Notation 4.8.10. We denote $M := T^{-1}$ and $M := \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$.

Proposition 4.8.11. *Assume $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve.*

1. *Every $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit on Θ_{12} (with Θ_{12} defined in 4.5.27) contains exactly one normalised element σ_N.*
2. *Let $(Z, \mathcal{H}) = \sigma \in \Theta_{12}$ and $\sigma = \sigma_\mu(T_2, f_2) = \sigma_N g_2$ where σ_N is the normalised stability condition corresponding to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of σ then $i_1(\mathcal{A}^{f''(f_2(0))}) \subset \mathcal{H}$ and $i_2(\mathcal{A}^{f_2(0)}) \subset \mathcal{H}$ where f'' corresponds to σ_N.*

Proof. We will start by proving the second statement as it contains the first as a special case if we additionally prove $r > -1$. Pick σ in the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit. By lemma 4.8.6 where $F = i_1$, we can pick $g' \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $i_1(\mathcal{A}) \subset \mathcal{H}'$ and $Z' \circ i_1 = Z_\mu$ and $Z'' \circ i_2 = Z_\mu$, where $\sigma' \circ g' = \sigma' = (Z', \mathcal{H}')$. Now, again by lemma 4.8.6 where we now let $F = i_2$, we can pick $g'' \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $i_2(\mathcal{A}) \subset \mathcal{H}''$, where $\sigma \circ g'' = \sigma'' = (Z'', \mathcal{H}'')$. By lemma 4.8.7, letting $F = i_1$, we obtain $i_1(\mathcal{A}^r) = i_1(\mathcal{P}_\mu(r, r+1]) \subset \mathcal{H}''$ as well. Now, considering $\sigma'' \circ g_2$ for any $g_2 \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, provides $i_2(\mathcal{A}^{f_2(0)}) =$

$i_2(\mathcal{P}_\mu(f_2(0), f_2(0)+1]) \subset \mathcal{H}'''$ via lemma 4.8.7 with $F = i_2$, where $\mathcal{H}'''$ is the heart corresponding to $\sigma'' \circ g_2 = \sigma' \circ g'' \circ g_2$. Considering $\sigma' \circ g'' \circ g_2$ provides $i_1(\mathcal{A}^{f'' \circ f_2(0)}) = i_1(\mathcal{P}_\mu(f'' \circ f_2(0), f'' \circ f_2(0)+1]) \subset \mathcal{H}'''$ via lemma 4.8.7 with $F = i_1$. Letting $g_1 = g' \circ g''$ and $f'' = f_1$ we obtain $\mathcal{H}''' = \mathcal{H}$ and the result for arbitrary $g_2 \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ follows.

If, on the other hand, $g_2 = \mathbb{1}_{\widetilde{\mathrm{GL}}_2^+(\mathbb{R})}$ and hence $\mathcal{H}'' = \mathcal{H}'''$, we must prove $r > -1$. Lemma 4.5.34 provides $r \geq -1$. Therefore, all that is left to prove is that $r \neq -1$. Assume that $r = -1$. Then if $Z'' \circ i_1(r_1, d_1) = Ad_1 + Br_1 + (Cr_1 + Dd_1)i$, we get $D = 0$ since $f''(0) \in \mathbb{Z}$ is equivalent to $D = 0$ by the compatibility of T and f that provides the equation $\exp(i\pi f(0)) = \frac{C+Di}{|C+Di|}$ where $T^{-1} = M = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$. Hence $Z(i_1(\mathbb{C}(x))) \in \mathbb{R}$ such that

$$\phi_{\sigma''}(i_1(\mathbb{C}(x))) \in \mathbb{Z}. \tag{4.52}$$

Consider the non-zero morphism $i_1(\mathbb{C}(x)) \to i_2(\mathbb{C}(x))[1]$ which provides us with

$$\phi_{\sigma''}(i_1(\mathbb{C}(x)) < \phi_{\sigma''}(i_2(\mathbb{C}(x))[1] = \phi_{\sigma''}(i_2(\mathbb{C}(x)) + 1 = 2. \tag{4.53}$$

since we have $\phi_{\sigma''}(i_2(\mathbb{C}(x)) = 1$. Since $r = -1$ we obtain $(r, r+1] = (-1, 0]$ such that $\mathbb{C}(x)[-1] \in \mathcal{P}_\mu(r, r+1]$ and therefore $i_1(\mathbb{C}(x)[-1]) \in \mathcal{H} = \mathcal{P}(0,1]$. This gives $i_1(\mathbb{C}(x)) \in \mathcal{P}(1,2]$ and therefore $\phi_{\sigma''}(i_1(\mathbb{C}(x))) > 1$. Combined with $\phi_{\sigma''}(i_1(\mathbb{C}(x))) < 2$ by (4.53) we therefore have $\phi_{\sigma''}(i_1(\mathbb{C}(x))) \notin \mathbb{Z}$, contradicting (4.52). □

Remark 4.8.12. In the following we will use the notation $\sigma_1 = \sigma_\mu(T, f)$ and $M := T^{-1} = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$ and $r = f(0)$ for stability conditions of the kind of definition 4.8.9.

Proposition 4.8.11 puts us into the position to determine by which construction a normalised stability condition with $r \geq 0$ was obtained.

Lemma 4.8.13. *Let* $\mathcal{A} = \mathrm{Coh}(C)$*,* C *a smooth projective curve and* $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$*, such that there are stability conditions*

$$\sigma_1 = (Z_1, \mathcal{A}^r) = \sigma_\mu(T_1, f_1) \text{ and } \sigma_2 = (Z_\mu, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$$

with $i_1(\mathcal{A}^r) \subset \mathcal{H}$*,* $i_2(\mathcal{A}) \subset \mathcal{H}, Z\big|_{\mathcal{D}_1} = Z_1$ *and* $Z\big|_{\mathcal{D}_2} = Z_\mu$*. If* $f_1(0) = r \geq 0$*, then* σ *is obtained by CP-gluing stability conditions* (σ_1, σ_μ) *via the semiorthogonal decomposition* $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$*.*

Proof. Since σ_1 and σ_2 satisfy gluing conditions $f_1(0) \geq 0$, (see lemma 3.2.31 where $\alpha = f_1(0)$ and $\beta = 0$) there is a pre-stability condition $\widetilde{\sigma} = (\mathcal{H}_{\widetilde{\sigma}}, Z_{\widetilde{\sigma}})$ that is obtained by CP-gluing stability conditions (σ_1, σ_μ) via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ on $\mathcal{D}^\uparrow$. Because of $i_1(\mathcal{A}^r) \subset \mathcal{H}$, $i_2(\mathcal{A}) \subset \mathcal{H}$, $Z\big|_{\mathcal{D}_1} = Z_1$, we have $\mathcal{H}_{\widetilde{\sigma}} \subset \mathcal{H}$ and since $\mathcal{H}_{\widetilde{\sigma}}$ and $\mathcal{H}$ are hearts of bounded t-structures this implies $\mathcal{H}_{\widetilde{\sigma}} = \mathcal{H}$. On the other hand, Z and $Z_{\widetilde{\sigma}}$ are uniquely determined by Z_1 and Z_μ anyway. □

We will now deal with the case where $f_1(0) = r < 0$ broken down into a series of lemmas.

Lemma 4.8.14. *Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve and $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$, be a normalised pre-stability condition on $\mathcal{D}^\uparrow$. We have $r < 0$ if and only if $\Delta(\mathbb{C}(x))$ is σ-stable.*

Proof. We will start by proving that $\Delta(\mathbb{C}(x))$ σ-stable is equivalent to having $\phi_\sigma(i_1(\mathbb{C}(x)) > 1$. To see this, consider two cases.

1. If $\Delta(\mathbb{C}(x))$ is σ-stable, then the exact triangle

$$i_2(\mathbb{C}(x)) \to \Delta(\mathbb{C}(x)) \to i_1(\mathbb{C}(x)) \xrightarrow{+}$$

   provides $\phi_\sigma(i_2(\mathbb{C}(x))) < \phi_\sigma(\Delta(\mathbb{C}(x))) < \phi_\sigma(i_1(\mathbb{C}(x)))$. And since $\phi_\sigma(i_2(\mathbb{C}(x))) = 1$ by assumption, we have $\phi_\sigma(i_1(\mathbb{C}(x))) > 1$.

2. If $\Delta(\mathbb{C}(x))$ is not σ-stable, then the exact triangle

$$i_2(\mathbb{C}(x)) \to \Delta(\mathbb{C}(x)) \to i_1(\mathbb{C}(x)) \xrightarrow{+}$$

   is the JHF of $\Delta(\mathbb{C}(x))$ if $\Delta(\mathbb{C}(x))$ is σ-semistable, in which case we have $\phi_\sigma(i_2(\mathbb{C}(x))) = \phi_\sigma(\Delta(\mathbb{C}(x))) = \phi_\sigma(i_1(\mathbb{C}(x)))$ and the HNF if $\Delta(\mathbb{C}(x))$ is not σ-semistable, in which case we have $1 = \phi_\sigma(i_2(\mathbb{C}(x))) > \phi_\sigma(i_1(\mathbb{C}(x)))$. Therefore $\phi_\sigma(i_1(\mathbb{C}(x)) \leq \phi_\sigma(i_2(\mathbb{C}(x)) = 1$.

Hence, we have established that $\Delta(\mathbb{C}(x))$ σ-stable is equivalent to having $\phi_\sigma(i_1(\mathbb{C}(x)) > 1$.

We have $i_1(\mathcal{P}_\mu(r, r+1]) \subset \mathcal{P}(0, 1]$ and hence $i_1(\mathcal{P}_\mu(r+n, r+n+1]) \subset \mathcal{P}(n, n+1]$ for any $n \in \mathbb{Z}$ as well. There is a unique $n \in \mathbb{Z}$ for which we have $1 \in (r+n, r+n+1]$. Since we know $\mathbb{C}(x) \in \mathcal{P}_\mu(1)$, we obtain $i_1(\mathbb{C}(x)) \in \mathcal{P}(n, n+1]$. This is equivalent to $\phi_\sigma(i_1(\mathbb{C})(x) \in (n, n+1]$. However, $\phi_\sigma(i_1(\mathbb{C})(x) > 1$ if and only if $n \geq 1$. Since $1 \in (r+n, r+n+1]$ we now have $1 > r + n \geq r + 1$ if and only if $\phi_\sigma(i_1(\mathbb{C})(x) > 1$, if and only if $r < 0$. □

Lemma 4.8.15. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve and $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$ be a normalised pre-stability condition on $\mathcal{D}^{\uparrow}$. If we have $r = f_1(0) < 0$ and $\sigma_1 = \sigma_\mu(T_1, f_1) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and let $M = T_1^{-1}$, then $\det(M + I) > 0$.*

Proof. Let $M = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$. If $\mathrm{Tr}(M) \geq 0$, we have $\det(M+I) > 0$ via $\det(M+I) = \det(M) + \mathrm{Tr}(M) + 1 > 0$. If $-A + C = \mathrm{Tr}(M) < 0$, then $\Delta(\mathcal{O}_C)$ not σ-stable would imply $\phi_\sigma(i_1(\mathcal{O}_C)) \leq \phi_\sigma(i_2(\mathcal{O}_C))$. We have $\phi_\sigma(i_2(\mathcal{O}_C)) = 1/2$ since σ is normalised, such that $\phi_\sigma(i_1(\mathcal{O}_C)) = (0, 1/2]$, such that $B, C \geq 0$. Hence, via $D < 0$ and $-AC + BD = \det(M) > 0$ we obtain $-AC > 0$ such that $-A + C > 0$ contradicting the assumption. Since $\Delta(\mathbb{C}(x))$ is also stable, by lemma 4.8.14, we obtain $\phi_\sigma(\Delta(\mathcal{O}_C)) < \phi_\sigma(\Delta(\mathbb{C}(x))) < \phi_\sigma(\Delta(\mathcal{O}_C)) + 1$. Since $Z(\Delta(\mathbb{C}(x))) = A - 1 + iD$, $Z(\Delta(\mathcal{O}_C)) = B + i(C+1)$ and we also have $Z(\Delta(\mathbb{C}(x))) = m_{\Delta(\mathbb{C}(x))} \exp(i\pi\phi_\sigma(\Delta(\mathbb{C}(x))))$ and $Z(\Delta(\mathcal{O}_C)) = m_{\Delta(\mathcal{O}_C)} \exp(i\pi\phi_\sigma(\Delta(\mathcal{O}_C)))$, we therefore obtain

$$\begin{aligned} \det(M+I) &= BD - (A-1)(C+1) \\ &= m_{\Delta(\mathbb{C}(x))} \cos(\pi\phi_\sigma(\Delta(\mathcal{O}_C))) m_{\Delta(\mathbb{C}(x))} \sin(\pi\phi_\sigma(\Delta(\mathbb{C}(x)))) - \\ &\quad m_{\Delta(\mathbb{C}(x))} \sin(\pi\phi_\sigma(\Delta(\mathcal{O}_C))) m_{\Delta(\mathbb{C}(x))} \cos(\pi\phi_\sigma(\Delta(\mathbb{C}(x)))) \\ &= m_{\Delta(\mathbb{C}(x))} m_{\Delta(\mathcal{O}_C)} \sin((\phi_\sigma(\Delta(\mathbb{C}(x))) - \phi_\sigma(\Delta(\mathcal{O}_C)))\pi) > 0 \end{aligned}$$

(since obviously $m_{\Delta(\mathbb{C}(x))}, m_{\Delta(\mathcal{O}_C)} > 0$). □

Lemma 4.8.16. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve and $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$ be a normalised pre-stability condition on $\mathcal{D}^{\uparrow}$ and assume that $\Delta(\mathbb{C}(x))$ is σ-stable. Then σ is given by a pair constructed in lemma 4.7.31.*

Proof. Since σ is normalised in Θ_{12} we have that $i_1(\mathbb{C}(x))$ is σ-stable and the object $i_2(\mathbb{C}(x))$ and $i_2(\mathcal{O}_C)$ are in $\mathcal{H}$ and are also σ-stable with

$$Z([i_2(\mathbb{C}(x))]) = -1 \text{ and } Z([i_2(\mathcal{O}_C)]) = i.$$

By proposition 4.8.11, in combination with the assumption that σ is normalised, there are stability conditions $\sigma_1 = (Z_1, \mathcal{A}^r)$ and $\sigma_2 = (Z_\mu, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$, such that $i_1(\mathcal{A}^r) \subset \mathcal{H}$ and $i_2(\mathcal{A}) \subset \mathcal{H}$, with $Z\big|_{\mathcal{D}_1} = Z_1$ and $Z\big|_{\mathcal{D}_2} = Z_\mu$. Lemma 4.8.14 implies $-1 < f_1(0) < 0$, where $\sigma_1 = (T_1, f_1)\sigma_\mu \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. We start by noting that Z can be written in the way of lemma 4.7.31 as it is completely determined by Z_1 and Z_μ and therefore, with the notation of (4.51), has the form

$$Z(r_1, d_1, r_2, d_2) = Ad_1 + Br_1 - d_2 + i(Cr_1 + Dd_1 + r_2)).$$

Let M be defined by $M = T_1^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$. Since σ_1 is a stability condition, we have that $\det(M) > 0$. The exact triangle $i_2(\mathbb{C}(x)) \to \Delta(\mathbb{C}(x)) \to i_1(\mathbb{C}(x)) \xrightarrow{+}$ has all involved objects stable and therefore $1 = \phi_\sigma(i_2(\mathbb{C}(x))) < \phi_\sigma(\Delta(\mathbb{C}(x))) < \phi_\sigma(i_1(\mathbb{C}(x))) < \phi_\sigma(i_2(\mathbb{C}(x))) + 1 = 2$. This implies $D = m_{i_1(\mathbb{C}(x))} \sin(\phi_\sigma(i_1(\mathbb{C}(x)))\pi) < 0$, since, by definition, we have $m_{i_1(\mathbb{C}(x))} > 0$.

Now consider the torsion pair $\langle \mathcal{T}, \mathcal{F} \rangle = \mathcal{A}^\uparrow$ given in lemma 4.8.5. We are going to show that it equals the torsion pair $(\mathcal{T}^{'}, \mathcal{F}^{'})$ given by definition 4.7.24. It is enough to prove that $\mathcal{T}^{'} \subseteq \mathcal{T}$ and $\mathcal{F}^{'} \subseteq \mathcal{F}$.

We take a torsion-free λ-semistable object $E = E_1 \to E_2 \in \mathcal{T}'$, which by definition satisfies $\lambda(E) > 3/4$. We have the existence of a short exact sequence

$$0 \to T \to E \to F \to 0, \tag{4.54}$$

with $T \in \mathcal{T}$ and $F = F_1 \to F_2 \in \mathcal{F}$. Since $E \in \mathcal{T}'$ and – hence – we have $E = (E_1 \twoheadrightarrow E_2)$ such that $F = (F_1 \twoheadrightarrow F_2)$ via $E \twoheadrightarrow F$, we can assume $F \neq (0 \to G_2)$, where G_2 is a torsion sheaf. We apply lemma 4.7.14 to see that the short exact sequence (4.54) gives us $\lambda(E) \leq \lambda(F)$, provided by the λ-semistability of E which implies $\lambda(T) \leq \lambda(E)$, hence $\lambda(F) > \frac{3}{4}$. On the other hand, we have $\Im(Z(F)) = \Re(Z_\lambda(F)) + \Im(Z_\lambda(F))$, which is greater or equal to zero if $\Re(Z_\lambda(F)) \geq -\Im(Z_\lambda(F))$. The latter, however, is equivalent to $\lambda(F) \leq \frac{3}{4}$, providing us with a contradiction and this implies that $F = (0 \to G_2)$ where G_2 is torsion which is another contradiction such that $F = 0$ and hence $E \in \mathcal{T}$.

Taking a λ-semistable torsion-free object $E = E_1 \to E_2 \in \mathcal{F}'$, we have $\lambda(E) \leq 3/4$ by definition. Firstly, consider the case $\lambda(E) < 3/4$. There is a short exact sequence given by $0 \to T \to E \to F \to 0$, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Since $T \in \mathcal{T} \subset \mathcal{P}(1, 2]$ we have $\Im(Z(T)) \leq 0$. this is equivalent to $\Re(Z_\lambda(T)) + \Im(Z_\lambda(T))$ such that $\lambda(T) \geq 3/4$. If $T \neq 0$, the λ-semistability of E implies $3/4 \leq \lambda(T) \leq \lambda(E) < 3/4$, which gives a contradiction and hence $T = 0$ and we obtain $E \in \mathcal{F}$.

Take a torsion-free λ-semistable object $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{F}'$ with $\lambda(E) = 3/4$, we consider the short exact sequence $0 \to T \to E \to F \to 0$, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The inequality $3/4 \leq \lambda(T) \leq \lambda(E) = 3/4$ holds such that $3/4 = \lambda(T)$. We get $T[-1] \in \mathcal{P}(1)$, where $\mathcal{P}$ is the slicing corresponding to σ. This is because $\lambda(T) = \frac{3}{4}$ is equivalent to $\Im(Z'(T)) = 0$ and since $T \in \mathcal{P}(1, 2]$, we obtain $T \in \mathcal{P}(2)$. To see this consider that $T \in \mathcal{P}(1, 2]$ provides $T[-1] \in \mathcal{H}$. We have $\Im(Z(T[-1])) = -\Im(Z(T)) = 0$. Any HN-factor L of $T[-1] \in \mathcal{H}$ is also in $\mathcal{H}$ and therefore $\Im(Z(L)) \geq 0$. Since $T[-1]$ is the sum of its HN-factors in the Grothendieck group, we obtain $\Im(Z(L)) \geq 0$. We obtain $L \in \mathcal{P}(1)$ which is extension closed such that

$T[-1] \in \mathcal{P}(1)$ and therefore $T \in \mathcal{P}(2)$. Moreover by lemma 4.8.5, we have that $\mathcal{P}(1) \subseteq \mathcal{A}^{\dagger}$. Hence, $T[-1] \in \mathcal{A}^{\dagger}$ and since – at the same time – we have $T \in \mathcal{A}^{\dagger}$ this implies $T = 0$.

Note that for all torsion-free object $E \in \mathcal{T}'$, we obtain $E \in \mathcal{T}$ via its λ-HNF – which works analogously for $E \in \mathcal{F}'$. This result extends to any object in $\mathcal{T}$ and $\mathcal{F}$. In consequence, $\mathcal{T} = \mathcal{T}'$, $\mathcal{F} = \mathcal{F}'$ and therefore $\mathcal{H} = \mathcal{H}(C_1, D_1)$.

We can now apply lemma 4.7.31 letting $C = C_1$ and $D = D_1$. This finished the proof. □

Lemma 4.8.17. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve and $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$, such that there are stability conditions*

$$\sigma_1 = (Z_1, \mathcal{A}^r) = (T_1, f_1) \text{ and } \sigma_2 = (Z_\mu, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$$

with $i_1(\mathcal{A}^r) \subset \mathcal{H}$, $i_2(\mathcal{A}) \subset \mathcal{H}$, $Z\big|_{\mathcal{D}_1} = Z_1$ and $Z\big|_{\mathcal{D}_2} = Z_\mu$. If $f_1(0) = r$, with $-1 < r < 0$, then σ is given by a pair constructed in lemma 4.7.31.

Proof. This combines lemmas 4.8.16 and 4.8.14. □

Proposition 4.8.18. *Let σ be a pre-stability condition in Θ_{12}. There is an element $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that σg is given by a CP-glued pre-stability condition or one constructed by tilting in lemma 4.7.31.*

Proof. Apply proposition 4.8.11, then the result follows by lemmas 4.8.13 and 4.8.17. □

We will finish this section by adding to our description of $\mathrm{pre\,Stab}(\mathcal{D}^{\dagger})$ provided in theorem 4.5.29, by studying which stability conditions in a given Θ_{ij} do actually belong to the subset of stability conditions glued via either of the semiorthogonal decompositions $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ or $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$. The aim – therefore – is to refine the result of proposition 4.8.18 by establishing theorem 4.8.36.

Notation 4.8.19. For a Matrix $M = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$, denote

1. its characteristic polynomial $x^2 - \mathrm{Tr}(M)x + \det(M)$ (where $\mathrm{Tr}(M)$ is the trace of M), by $p_M(x)$ and

2. the discriminant $\mathrm{Tr}(M)^2 - 4\det(M)$ of $p(x)$ by $\mathrm{Discr}(M)$.

Definition 4.8.20. Define

- Θ_i to be the set of pre-stability conditions, for $i = 1, 2$ or 3, which are, up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, CP-glued with respect to the semiorthogonal decomposition $\langle \mathcal{D}_i, {}^{\perp}\mathcal{D}_i \rangle$, and

- Γ to be the set of pre-stability conditions, which up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action is given by lemma 4.7.31 with $\mathrm{Discr}(M) < 0$.

By studying the discriminant of M, we will investigate if the corresponding σ is in either of the Θ_i, for $i \in \{1, 2, 3\}$.

Lemma 4.8.21. *We have*

- $\Theta_1 \subset \Theta_{12}$,

- $\Theta_2 \subset \Theta_{23}$ *and*

- $\Theta_3 \subset \Theta_{31}$.

Proof. Firstly,

$$\Theta_1 \subset \widetilde{\Theta}_{12}, \Theta_2 \subset \widetilde{\Theta}_{23} \text{ and } \Theta_3 \subset \widetilde{\Theta}_{31}$$

follows from [21, Proposition 2.2(3)].

For $\sigma \in \mathrm{PreStab}(\mathcal{D}^{\uparrow})$ assume $E \in \mathcal{A}$ stable and $i_1(E)$ strictly σ-semistable, then $\phi(i_1(E)) = \phi(i_2(E)[1])$ from the JHF (lemma 4.5.31). If σ CP-glued from $\sigma_1 = \sigma_\mu(T_1, f_1)$ and $\sigma_2 = \sigma_\mu(T_2, f_2)$, then $f_1(0) \geq f_2(0)$ (condition for CP-gluing). Now, for $\phi := \phi(i_1(E)) = \phi(i_2(E)) + 1$, using [21, Proposition 2.2(3)], we obtain $E \in \mathcal{P}_1(\phi) = \mathcal{P}_\mu(f_1(\phi))$ as well as $E \in \mathcal{P}_2(\phi - 1) = \mathcal{P}_\mu(f_2(\phi) - 1)$ such that $f_1(\phi) = f_2(\phi) - 1$. Let $n \in \mathbb{Z}$ such that $\phi \in (n, n+1]$, then we obtain $f_1(n) < f_1(\phi) = f_2(\phi) - 1 \leq f_2(n+1) - 1 = f_2(n)$ such that we now obtain $f_1(0) < f_2(0)$ which is a contradiction such that $i_1(E)$ must be σ-stable. A similar argument leads to $i_2(E)$ σ-stable such that $\Theta_1 \subset \Theta_{12}$. Again by similar arguments we obtain $\Theta_2 \subset \Theta_{23}$ and $\Theta_3 \subset \Theta_{31}$. □

Definition 4.8.22. For $\sigma_3 = ((M + I)^{-1}, f_3)$ define

$$\begin{aligned} f_{12}(\sigma)(x) &= f_1(x) - x \text{ and hence also} \\ f_{23}(\sigma)(x) &= x - f_3(x) \text{ and} \\ f_{31}(\sigma)(x) &= f_3(x) - f_1(x). \end{aligned}$$

Definition 4.8.23. Let $(\mathcal{A}^{\uparrow})^l$ be the unique heart of a bounded t-structure obtained by CP-gluing via $\langle i_1(\mathcal{D}), i_2(\mathcal{D}) \rangle$ from two copies of hearts $\mathcal{A}^l$ on $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ for $\mathcal{A} = \mathrm{Coh}(C)$, C a smooth projective curve, where $\mathcal{A}^l = \mathcal{P}_\mu(l, l+1]$ by definition 4.7.38 with $l = r$.

Lemma 4.8.24. *Let $\sigma \in \Theta_{12}$ normalised with corresponding M. If $f_1(0) < 1$ and we additionally assume that we have $\operatorname{Discr}(M) \geq 0$ and also the eigenvalues of M are positive, then there is $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that $\sigma g = (Z, (\mathcal{A}^{\uparrow})^l)$, with $l \in \mathbb{R}$ and $-1 < l \leq 1$. Moreover we have $\sigma \in \Theta_1$.*

Proof. The eigenvalues are assumed to be real (by the assumption that we have $\operatorname{Discr}(M) \geq 0$), positive numbers. The same follows for T_1 corresponding to f_1. Let $\lambda \in \mathbb{R}$ be an eigenvalue of T_1 and $v \in \mathbb{R}^2 \cong \mathbb{C}$ its corresponding eigenvector, in other words, $T_1 v = \lambda v$. We consider the polar coordinates of $v = (m\cos(\phi), m\sin(\phi))$ with $\phi \in (-\pi, \pi]$ and $m \in \mathbb{R}_{>0}$ and claim that $\sigma g \in \Theta_1$, where $g = (K_\phi, f_\phi) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. (For the definition of K_ϕ see A.2.1 and lemma A.2.2 for its implications, f_ϕ fulfils $f_\phi(0) = \frac{\phi}{\pi}$.) In other words, we want to prove that σg is obtained by CP-gluing via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. We consider $\sigma_1 g = \sigma_\mu(T_1 K_\phi, f_1 \circ f_\phi)$ where $\sigma_1 = \sigma_\mu(T_1, f_1)$.

Using the compatibility between $f_1 \circ f_\phi$ and $T_1 K_\phi$ over S^1 and the fact that v is an eigenvector, we will now prove that $f_1 \circ f_\phi(0) = \phi/\pi$. As a representative for the positive real axis we choose the vector $v_0 = (1, 0)$ Since $K_\phi v_0$ is $(\cos(\phi), \sin(\phi))$ (K_ϕ is the rotation-by-ϕ-matrix), we therefore have $K_\phi v_0 = \frac{1}{m} v$ providing us with $T_1 K_\phi v_0 = \frac{\lambda}{m} v$. This means that $T_1 K_\phi$ maps the positive real axis onto the ray trough v, in other words, $\exp(i\pi f_1 f_\phi(0)) = \exp(i\phi)$. This implies $f_1 f_\phi(0) = \phi/\pi + 2k$ for some $k \in \mathbb{Z}$. We now have to prove $k = 0$. From $\phi \in (-\pi, \pi]$ we obtain $-1 < \phi/\pi \leq 1$. Since f_1 is an increasing function and $-1 < f_1(0)$ this implies $-2 < f_1(-1) < f_1(\phi/\pi) = \phi/\pi + 2k = f_1(\phi/\pi) \leq f_1(1) < 2$. Therefore we obtain $k \in \{-1, 0, 1\}$. If $k = 1$ we obtain $\phi/\pi + 2 < 2$ which implies $\phi/\pi < 0$. Then $f_1(\phi/\pi) < f_1(0) < 1$ and hence $\phi/\pi + 2 = f(0) = f_1(\phi/\pi) < 1$ which gives $\phi/\pi < -1$, providing a contradiction. Similarly, $k = -1$ gives a contradiction via $\phi/\pi > 1$ such that we obtain $k = 0$ and hence $f_1 \circ f_\phi(0) = \phi/\pi$.

We now consider $\sigma_2 g = \sigma_\mu(K_\phi, f_\phi)$. We obtain $f_{12}(\sigma g)(0) = 0$. By lemma 3.1.16, we obtain that $\sigma g = (Z, (\mathcal{A}^{\uparrow})^l), l = \phi/\pi \in (-1, 1]$ is glued from $\sigma_1 g$ and $\sigma_2 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. $\square$

Lemma 4.8.25. *Let $\mathcal{A} = \operatorname{Coh}(C)$ where C is an elliptic curve and $\sigma = (Z, \mathcal{H}) \in \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow}), Z \circ i_1 = M Z_\mu, Z \circ i_2 = Z_\mu, M = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}, \det(M) > 0, \det(M + I) > 0$ and*

$$i_1(\mathcal{P}_\mu(r, r+1]) \subset \mathcal{H}, i_2(\mathcal{P}_\mu(0, 1]) \subset \mathcal{H}, \Delta(\mathcal{P}_\mu(r_3, r_3 + 1]) \subset \mathcal{H}$$

where $-1 < r < r_3 < 0$.

Then, for $p_M(x) = -Dx^2 - (A + C)x - B$

- *if* $0 < \phi_\mu \leq r$ *then* $i_1(X), i_2(X)$ *are* σ*-stable and* $\Delta(X)$ *is* σ*-stable if and only if* $p_M(\mu(X)) > 0$,
- *if* $r+1 < \phi_\mu(X) \leq r_3 + 1$ *then* $i_1(X), \Delta(X)$ *are* σ*-stable and* $i_2(X)$ *is* σ*-stable if and only if* $p_M(\mu(X)) > 0$,
- *if* $r_3 + 1 < \phi_\mu(X) \leq 1$ *then* $i_2(X), \Delta(X)$ *are* σ*-stable and* $i_1(X)$ *is* σ*-stable if and only if* $p_M(\mu(X)) > 0$.

Proof. For $X \in \mathcal{A}$ stable we have

$$\begin{aligned} i_1(X)\sigma - \text{stable} &\iff \phi_\sigma(\Delta(X)) < \phi_\sigma(i_2(X)) + 1 \\ i_2(X)\sigma - \text{stable} &\iff \phi_\sigma(i_1(X)) < \phi_\sigma(\Delta(X)) + 1 \\ \Delta(X)\sigma - \text{stable} &\iff \phi_\sigma(i_2(X)) < \phi_\sigma(i_1(X)). \end{aligned} \tag{4.55}$$

Moreover recall that $Z(r_1, d_1, r_2, d_2) = Ad_1 + Br_1 - d_2 + i(Dd_1 + Cr_1 + r_2)$. To prove the first part we start with the claimed equivalence and see that $0 < \phi_\mu(X) \leq r + 1$ if and only if $\mu(X) \leq \frac{C}{-D}$ if and only if $-D\mu(X) \leq C$ (since$-D > 0$) if and only if $-Dd_X \leq Cr_X$ (since $\mu(x) = \frac{d_X}{r_X}$ and $r_X > 0$) if and only if $0 \leq Dd_X + Cr_X$ if and only if $0 \leq \Im(Z(i_1(X)))$. Now we have that $\Delta(X)$ σ-stable if and only if $\phi_\sigma(i_2(X)) < \phi_\sigma(i_1(X))$ (by (4.55)) if and only if $\mu_\sigma(i_2(X)) < \mu_\sigma(i_1(X))$ (by the slope phase correspondence) if and only if $\frac{\Re(Z(i_2(X)))}{-\Im(Z(i_2(X)))} < \frac{\Re(Z(i_1(X)))}{-\Im(Z(i_1(X)))}$ (where we can assume $\Im(Z(i_2(X))), \Im(Z(i_1(X))) \neq 0$ as otherwise the σ-stability of $\Delta(X)$ is automatic). This now is equivalent to $\Re(Z(i_2(X)))(-\Im(Z(i_1(X)))) < \Re(Z(i_1(X)))(-\Im(Z(i_2(X))))$, which is equivalent to $d_X(Dd_X + Cr_X) < -r_X(Ad_X + Br_X)$, which is equivalent to $-D(\frac{d_X}{r_X})^2 - (A + C)\frac{d_X}{r_X} - B > 0$, which is equivalent to $-D(\mu(X))^2 - (A + C)\mu(X) - B > 0$, which is equivalent to $p_M(\mu(X)) > 0$.

To prove the stability of $i_1(X), i_2(X)$ on the other hand consider that $0 < \phi_\mu(X) \leq r+1$ in combination with $i_1(\mathcal{P}_\mu(r, r+1]) \subset \mathcal{H}$ and $\Delta(\mathcal{P}_\mu(r_3, r_3 + 1]) \subset \mathcal{H}$ implies $i_1(X) \in \mathcal{H}$ and and $\Delta(X) \in \mathcal{H}$. Moreover we obviously have $i_2(X) \in \mathcal{H}$. Therefore $\phi_\sigma(\Delta(X)) < \phi_\sigma(i_2(X)) + 1$ which implies that $i_1(X)$ is σ-stable by (4.55) and $\phi_\sigma(i_1(X)) < \phi_\sigma(\Delta(X)) + 1$ which implies that $i_2(X)$ is σ-stable, again by (4.55).

To prove the second part we start with the claimed equivalence and see that $r + 1 < \phi_\mu(X) \leq r_3 + 1$ if and only if $\frac{C}{-D} < \mu(X) \leq \frac{C+1}{-D}$ if and only if $D\mu(X) + C < 0 \leq D\mu(X) + C + 1$ (since $-D > 0$) if and only if $Dd_X + Cr_X < 0 \leq Dd_X + (C+1)r_X$ (since $\mu(x) = \frac{d_X}{r_X}$ and $r_X > 0$) if and only if $\Im(Z(i_1(X))) < 0 \leq \Im(Z(\Delta(X)))$. Now we have that $i_2(X)$ is σ-stable if and only if $\phi_\sigma(i_1(X)) < \phi_\sigma(\Delta(X)) + 1$ (by (4.55)) if and only if $\mu_\sigma(i_1(X)) < \mu_\sigma(\Delta(X)[1])$ (by the slope phase correspondence) if and only if $\frac{\Re(Z(i_1(X)))}{-\Im(Z(i_1(X)))} <$

$\frac{-\Re(Z(\Delta(X)[1]))}{\Im(Z(\Delta(X)[1]))} = \frac{\Re(Z(\Delta(X)))}{-\Im(Z(\Delta(X)))}$ (where we can assume $\Im(Z(\Delta(X))) \neq 0$ as otherwise the σ-stability of $i_2(X)$ is automatic). This now is equivalent to $\Re(Z(i_1(X)))(-\Im(Z(\Delta(X)))) < -\Im(Z(i_1(X)))\Re(Z(\Delta(X)))$, which is equivalent to $(Ad_X + Br_X)(-(Dd_X + (C+1)r_X)) < -(Dd_X + Cr_X)(Ad_X + Br_X)$, which is equivalent to $p_M(\mu(X)) > 0$.

To prove the stability of $i_1(X)$ and $\Delta(X)$ on the other hand consider that $r + 1 < \phi_\mu(X) \leq r_3 + 1$ in combination with $i_1(\mathcal{P}_\mu(r, r+1]) \subset \mathcal{H}$ and $\Delta(\mathcal{P}_\mu(r_3, r_3+1]) \subset \mathcal{H}$ implies $i_1(X) \in \mathcal{H}[1]$ (such that $i_1(X)[-1] \in \mathcal{H}$) and $\Delta(X) \in \mathcal{H}$. Moreover we obviously have $i_2(X) \in \mathcal{H}$. Therefore $\phi_\sigma(\Delta(X)) < \phi_\sigma(i_2(X)) + 1$ which implies that $i_1(X)$ is σ-stable by (4.55) and $\phi_\sigma(i_2(X)) < \phi_\sigma(i_1(X))$ which implies that $\Delta(X)$ is σ-stable, again by (4.55).

Proving the third part, finally, is similar to the previous two cases where we now use $r_3 + 1 < \phi_\mu(X) \leq 1$ equivalent to $\frac{C+1}{-D} < \mu(X)$ which is equivalent to $\Im(Z(\Delta(X))) < 0$. □

Lemma 4.8.25 has the following useful implication.

Corollary 4.8.26. *Let* $\mathcal{A} = \mathrm{Coh}(C)$ *where* C *is an elliptic curve,* $\sigma = (Z, \mathcal{H}) \in \mathrm{pre\,Stab}(\mathcal{D}^\uparrow), Z \circ i_1 = M Z_\mu, Z \circ i_2 = Z_\mu, M = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}, \det(M) > 0, \det(M + I) > 0$ *and*

$$i_1(\mathcal{P}_\mu(r, r+1]) \subset \mathcal{H}, i_2(\mathcal{P}_\mu(0,1]) \subset \mathcal{H}, \Delta(\mathcal{P}_\mu(r_3, r_3+1]) \subset \mathcal{H}$$

where $r, r_3 \in (-1, 0)$.

Then $\mathrm{Discr}(M) < 0$ *implies that for any stable* $X \in \mathcal{A} = \mathcal{P}_\mu(0,1]$ *we have* $i_1(X), i_2(X), \Delta(X)$ σ*-stable.*

Proof. We have $r < r_3$ since $C + 1 + iD \in \mathbb{R}_{>0} \exp(i\pi f_3(0))$ because of the compatibility of f_3 and T_3, while $f_1(0) = r \in (-1, 0)$ and $C + iD \in \mathbb{R}_{>0} \exp(i\pi f_1(0))$ implies $D < 0$ such that $r = f_1(0) < f_3(0) = r_3$. Therefore lemma 4.8.25 applies and additionally $D < 0$ provides that $p_M(\xi)$ for the polynomial p_M of lemma 4.8.25 is positive for $\xi >> 0$ (the corresponding parabola is opened above). Since $\mathrm{Discr}(M) < 0$, we additionally obtain that $p_M(x)$ has no zeroes and therefore $p_M(x) > 0$ which by lemma 4.8.25 finishes the proof. □

Lemma 4.8.27. *If* $\mathcal{A} = \mathrm{Coh}(C), C$ *is an elliptic curve,* $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$ *a normalised pre-stability condition where we assume that the embedding* $i_1(\mathcal{P}(r, r+1]) \in \mathcal{H}$ *satisfies* $-1 < r < 0$ *and* $X \in \mathcal{A}$ *stable with* $i_1(X)[-1] \in \mathcal{H}$*, then* $\Delta(X)$ *is stable.*

Proof. If we assume $\Delta(X)$ not stable, lemma 4.5.31 provides us with the HNF/JHF triangle

$$i_2(X) \to \Delta(X) \to i_1(X) \xrightarrow{+},$$

therefore with $\phi_\sigma(i_2(X)) \geq \phi_\sigma(i_1(X))$, which implies $\phi_\sigma(i_1(X)) - \phi_\sigma(i_2(X)) \leq 0$. As $0 < \phi_\sigma(i_2(X)) \leq 1$, this gives $\phi_\sigma(i_1(X)) \leq 1$, contradicting our hypothesis since $i_1(X)[-1] \in \mathcal{H}$ implies $1 < \phi_\sigma(i_1(X)) \leq 2$. □

Remark 4.8.28. Note that the discriminant of the quadratic polynomial $Dd^2 + (A+C)d + B$ is given by $(A+C)^2 - 4BD = \mathrm{Discr}(M)$, in other words, it equals that of $p(x)$.

Lemma 4.8.29. *Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve, $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$ a normalised pre-stability condition with corresponding M and eigenvalues λ_1, λ_2 of M. Assume $r < 0$ and $\mathrm{Discr}(M) \geq 0$. If $X \in \mathcal{A}$ stable, $\lambda_1, \lambda_2 < 0$ and $i_1(X) \in \mathcal{H}$, then*

$$-D\mu(X)^2 - (A+C)\mu(X) - B > 0.$$

Proof. We consider the polynomial $q(x) = Dx^2 + (A+C)x + B$. As the discriminant of $q(x)$ is $\mathrm{Discr}(M) \geq 0$, the polynomial $q(x)$ has real roots $\mu_1, \mu_2 \in \mathbb{R}$. Assume that $\mu_1 \leq \mu_2$. Since for $i \in \{1,2\}$ we have $\lambda_i = \frac{1}{2}(C - A \pm \sqrt{\mathrm{Discr}(M)})$ and $\mu_i = \frac{-A-C\pm\sqrt{\mathrm{Discr}(M)}}{2D}$ we have $\mu_i = \frac{\lambda_i}{D} - \frac{C}{D}$. Since $\lambda_i < 0$ and $r < 0$ provides $D < 0$ this implies $\mu_i > -\frac{C}{D}$. Since $D < 0$ the parabola $q(x)$ is opened at the bottom such that $q(x) < 0$ for all $x < \mu_1 \leq \mu_2$. Since we have $\mu(x) \leq -\frac{C}{D}$ by lemma 4.8.25 we have $\mu(x) \leq -\frac{C}{D} < \mu_1$ such that $q(\mu(X)) < 0$. The proof is finished. □

Corollary 4.8.30. *If $\mathcal{A} = \mathrm{Coh}(C), C$ is an elliptic curve, $\sigma = (Z, \mathcal{H}) \in \Theta_{12}$ a normalised pre-stability condition with corresponding M and eigenvalues λ_1, λ_2 of M where we assume that the embedding $i_1(\mathcal{P}(r, r+1]) \in \mathcal{H}$ satisfies $-1 < r < 0$, $\lambda_1, \lambda_2 < 0$, then $\Delta(X)$ is σ-stable for all stable $X \in \mathcal{A}$. Moreover, σ is in Θ_{23} and Θ_{31}.*

Proof. From $-1 < r < 0$ we obtain that either $i_1(X)[-1] \in \mathcal{H}$, in which case we obtain the result from lemma 4.8.27, or $i_1(X) \subset \mathcal{H}$, in which case it follows by applying lemma 4.8.29 and after that lemma 4.8.25. Now, $\sigma \in \Theta_{23} \cap \Theta_{31}$ follows by definition. □

Lemma 4.8.31. *If $\mathcal{A} = \mathrm{Coh}(C), C$ is an elliptic curve, $\sigma = (Z, \mathcal{H})$ a normalised pre-stability condition where we assume that the embedding $i_1(\mathcal{P}(r, r+1]) \in \mathcal{H}$ satisfies $-1 < r < 0$, $\lambda_1, \lambda_2 < 0$, then there is a $t \in \mathbb{R}$ with $f_{31}(\sigma)(t) = 1$ such that $\sigma \in \Theta_3$.*

Proof. Assume there is a $t \in \mathbb{R}$ with $f_{31}(\sigma)(t) = 1$. By lemma 4.8.21 we have $\Theta_3 \subset \Theta_{31}$ and corollary 4.8.30 implies $\sigma \in \Theta_{31}$. Moreover, the assumption is, by definition of f_{31} made in 4.8.9, that $f_3(t) - f_1(t) = 1$ and therefore $f_1(t) = f_3(t) - 1$. Proving $\sigma \in \Theta_3$ means finding $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that σg satisfies the gluing conditions in question. Let $g = (K_{t\pi}, f_{t\pi}) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ (see section A.2 for the definition of $K_{t\pi}$) and obtain

$$f_1 \circ f_{t\pi}(0) = f_1(t) = f_3(t) - 1 \text{ as well as } f_3 \circ f_{t\pi}(0) = f_3(t) \text{ which implies}$$
$$f_{31}(\sigma g)(0) = f_3 \circ f_{t\pi}(0) - f_1 \circ f_{t\pi}(0) = f_3(t) - (f_3(t) - 1) = 1.$$

Hence, the condition for CP-gluing via the semiorthogonal decomposition $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ is fulfilled and like in the proof of lemma 4.8.13, we obtain $\sigma \in \Theta_3$. □

Remark 4.8.32. Note that lemma 4.8.31 in fact is an "if and only if". The general idea how to prove this is that one considers a $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that for σg the condition for CP-gluing via the semiorthogonal decomposition $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ is fulfilled and where, since non-rotations do not change the heart, $g = (K_{l\pi}, f_{l\pi}), l \in \mathbb{R}$ can be chosen. One can subsequently prove that we obtain $f_3(l) = f_3 \circ f_{l\pi}(0) \geq f_1 \circ f_{l\pi}(0) = l + 1$. We have $f_{31}(\sigma)(0) = d < 1$ by assumption and $f_{31}(\sigma)(1) = e \geq 1$ which, in combination with the fact that f_{31} is continuous, by the intermediate value theorem, provides a $t \in \mathbb{R}$ such that $f_{31}(\sigma)(t) = 1$, such that the proof is finished. However we do not need this implication here.

Lemma 4.8.33. *Let* $\mathcal{A} = \mathrm{Coh}(C)$, *where* C *is an elliptic curve,* $\sigma = (Z, \mathcal{H}) = \sigma_\mu(T, f)$ *a normalised pre-stability condition with corresponding* M *and eigenvalues* λ_1, λ_2 *of* M *and* $\mathrm{Discr}(M) \geq 0$ *as well as* $\det(M+I) > 0$. *If we assume that the embedding* $i_1(\mathcal{P}(r, r+1]) \in \mathcal{H}$ *satisfies* $-1 < r < 0$, $\lambda_1, \lambda_2 < 0$, *then we have*

1. *if* $0 > \lambda_1, \lambda_2 > -1$, *then* $\sigma \in \Theta_3$.
2. *if* $\lambda_1, \lambda_2 < -1$, *then* $\sigma \in \Theta_2$.

Proof. We will prove that $0 > \lambda_1, \lambda_2 > -1$ implies $\sigma \in \Theta_3$. The case of $\sigma \in \Theta_2$ for $\lambda_1, \lambda_2 < -1$ is similar. Let f_3 such that $(T_3, f_3) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $f_3(0) \in (-1, 1], \sigma_3 = \sigma_\mu(T_3, f_3)$ and β be an eigenvalue of $T_3 = (M + I)^{-1}$ (which exists since $\det(M + I) > 0$ such that $M + I$ is invertible), then $\beta = \frac{1}{\lambda+1}$ where λ is an eigenvalue of M. Therefore $\beta > 1$ and – in particular – positive. Let $w = m(\cos(\theta), \sin(\theta)), \theta \in (-\pi, \pi]$ be an eigenvector of T_3 which has β as its eigenvalue. Then, by linearity, $v = (\cos(\theta), \sin(\theta)), \theta \in (-\pi, 0]$

is an eigenvector. Note that v being an eigenvector of T_3 is an eigenvector of T as well. Consider $\sigma_3 g$ where $g = (K_\theta, f_\theta) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. We investigate $f_3 \circ f_\theta(0) = f_3(\frac{\theta}{\pi})$ The compatibility between Matrix and function now gives $\beta \exp(i\theta) = d \exp(i\pi f_3(\frac{\theta}{\pi})), d \in \mathbb{R}_{>0}$. Since $\beta > 0$ we obtain $\exp(i\theta) = \exp(i\pi f_3(\frac{\theta}{\pi}))$ which gives $i\theta + 2ki\pi = i\pi f_3(\frac{\theta}{\pi})$ for a $k \in \mathbb{Z}$. Define $t = \frac{\theta}{\pi}$ such that we have $f_3(t) = t + 2k$.

Now let $\sigma_1 = \sigma_\mu(T, f_1)$ and consider $\sigma_1 g$. Similar to before we obtain $1/\lambda \exp(it\pi) = e \exp(i\pi f_1(t)), e \in \mathbb{R}_{>0}$ using $Tv = \frac{1}{\lambda} v$. Since $\lambda < 0$ we obtain

$$- \exp(it\pi) = \exp(i\pi f_1(t)) \text{ and hence } -1 = \frac{\exp(i\pi f_1(t))}{\exp(it\pi)}, \text{ such that}$$
$$-1 = \exp(i\pi f_1(t) - it\pi). \text{ Hence } -1 = \exp(i\pi(f_1(t) - t))$$

such that we now end up with $f_1(t) = t + 2l + 1$ for an $l \in \mathbb{Z}$.

Hence, we now have $f_{31}(\sigma)(t) = f_3(t) - f_1(t) = 2(k - l) - 1$. We obtain from $\theta \in [\pi, 0)$ that $-1 < t \leq 0$ and applying f_1 and f_3 to this gives, since both are increasing functions, $f_1(-1) < f_1(t) \leq f_1(0)$ as well as $f_3(-1) < f_3(t) \leq f_3(0)$. We have $f_1(0) = r < 0$ and $f_1(-1) = r - 1 > -2$ since $-1 < r < 0$ as well as $f_3(0) = r_3 < 0$ and $f_3(-1) = r_3 - 1 > -2$ since $-1 < r_3 < 0$, which is seen from the fact that $M + I = \begin{pmatrix} 1 - A & B \\ -D & C + 1 \end{pmatrix}$ such that $C + 1 + iD \in \mathbb{R}_{>0} \exp(i\pi f_3(0))$ because of the compatibility of f_3 and T_3, while $f_1(0) = r \in (-1, 0)$ and $C + iD \in \mathbb{R}_{>0} \exp(i\pi f_1(0))$ implies $D < 0$ such that $f_3(0) \in (-1, 1]$ now gives $f_3(0) \in (-1, 0)$. Using $f_1(t) = t + 2l + 1$ and $f_3(t) = t + 2k$ we now arrive at $-2 < t + 2l + 1 < 0$ and at $-2 < t + 2k < 0$, therefore $-2 - 2l - 1 < t < -2l - 1$ and $-2 - 2k < t < -2k$. Since $-1 < t \leq 0$ we now have $-1 < -2l - 1$ and $-2 - 2l - 1 < 0$, which provides $-\frac{3}{2} < l < 0$. Therefore $l = -1$ and similarly we see that $k = 0$. hence, $f_{31}(\sigma)(t)) = 2(k - l) - 1 = 1$ and by lemma 4.8.31 the proof is finished. $\square$

Lemma 4.8.34. *If* $\det(M + I) > 0$ *and* $\mathrm{Discr}(M) \geq 0$ *then for the eigenvalues* λ_1, λ_2 *of* M *with* $\lambda_1, \lambda_2 < 0$ *we have either* $\lambda_1, \lambda_2 < -1$ *or* $\lambda_1, \lambda_2 > -1$.

Proof. We have $\lambda_1 = \frac{1}{2}(\mathrm{Tr}(M) - \sqrt{\mathrm{Discr}(M)})$ as well as $\lambda_2 = \frac{1}{2}(\mathrm{Tr}(M) + \sqrt{\mathrm{Discr}(M)})$. Now, $0 < \det(M + I) = \det(M) + 1 + \mathrm{Tr}(M)$ such that $-\det(M) < 1 + \mathrm{Tr}(M)$. Hence,

$$\begin{aligned} \mathrm{Discr}(M) &= \mathrm{Tr}(M)^2 - 4\det(M) < \mathrm{Tr}(M)^2 + 4(\mathrm{Tr}(M) + 1) \\ &= \mathrm{Tr}(M)^2 + 4\,\mathrm{Tr}(M) + 4 = (-\mathrm{Tr}(M) - 2)^2 = (\mathrm{Tr}(M) + 2)^2. \end{aligned}$$

Since $0 \leq \mathrm{Discr}(M)$ we obtain $\mathrm{Tr}(M) \neq 2$ and distinguish two cases.

1. If
$$\mathrm{Tr}(M) < -2$$
then $-\mathrm{Tr}(M) - 2 > 0$ and $\sqrt{\mathrm{Discr}(M)} < -\mathrm{Tr}(M) - 2$ such that we have $\lambda_1 \leq \lambda_2 < -1$.

2. If
$$\mathrm{Tr}(M) > -2$$
then $-\mathrm{Tr}(M) + 2 > 0$ and $\sqrt{\mathrm{Discr}(M)} < \mathrm{Tr}(M) + 2$ such that we have $-1 < \lambda_1 \leq \lambda_2$.

□

Lemma 4.8.35. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. We have $\Gamma \subset \Theta_{12} \cap \Theta_{23} \cap \Theta_{31}$.*

Proof. We can apply corollary 4.8.26 since for $\sigma \in \Gamma$ we have $\mathrm{Discr}(M) < 0$ and therefore the conditions of corollary 4.8.26 are fulfilled by lemma 4.7.39. □

Theorem 4.8.36. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. We have*
$$\operatorname{pre}\mathrm{Stab}(\mathcal{D}^{\uparrow}) = \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Gamma.$$

Proof. By theorem 4.5.29, we have that $\sigma \in \Theta_{12} \cup \Theta_{23} \cup \Theta_{31}$. Via (4.35) we obtain that
$$S_{\mathcal{D}^{\uparrow}}(\Theta_{12}) \subset \Theta_{23}, S_{\mathcal{D}^{\uparrow}}(\Theta_{23}) \subset \Theta_{31}, S_{\mathcal{D}^{\uparrow}}(\Theta_{31}) \subset \Theta_{12} \tag{4.56}$$
and additionally, using $(S_{\mathcal{D}^{\uparrow}}\sigma)g = S_{\mathcal{D}^{\uparrow}}(\sigma g), g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ provided by the fact that $S_{\mathcal{D}^{\uparrow}}$ is an autoequivalence, that
$$S_{\mathcal{D}^{\uparrow}}(\Theta_1) \in \Theta_2, S_{\mathcal{D}^{\uparrow}}(\Theta_2) \in \Theta_3, S_{\mathcal{D}^{\uparrow}}(\Theta_1) \in \Theta_1. \tag{4.57}$$
Now, let $\sigma' \in \operatorname{pre}\mathrm{Stab}(\mathcal{D}^{\uparrow}) = \Theta_{12} \cup \Theta_{23} \cup \Theta_{31}$. If $\sigma' \in \Theta_{12}$, define $\sigma'' := \sigma'$. If $\sigma' \in \Theta_{31}$ define $\sigma'' := S_{\mathcal{D}^{\uparrow}}(\sigma')$. If $\sigma' \in \Theta_{23}$ define $\sigma'' := (S_{\mathcal{D}^{\uparrow}})^2(\sigma')$. Hence $\sigma'' \in \Theta_{12}$ by (4.56). Let $\sigma = \sigma'' g, g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that σ normalised – its existence is granted by proposition 4.8.11. In other words, $\sigma = (Z, \mathcal{H})$ such that $Z \circ i_1 = M Z_\mu, Z \circ i_2 = Z_\mu, \det(M) > 0, -1 < r$ and $i_1(\mathcal{P}(r, r+1]) \subset \mathcal{H}$ as well as $i_2(\mathcal{P}(0, 1]) \subset \mathcal{H}$. We can reduce our investigation to these pre-stability conditions, because the sets Θ_i and Γ, for $i = 1, 2, 3$ are defined up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. If $f(0) \geq 0$, then by lemma 4.8.13, we obtain $\sigma \in \Theta_1$. If $-1 < f(0) < 0, \sigma$ is obtained by tilting – but might yet still be in any of the

Θ_i also. We distinguish two cases – one where $\mathrm{Discr}(M) \geq 0$ and one where $\mathrm{Discr}(M) < 0$. If $\mathrm{Discr}(M) \geq 0$, with positive eigenvalues then by lemma 4.8.24 we get $\sigma \in \Theta_1$. If the eigenvalues are smaller than -1, then we have that $\sigma \in \Theta_3$ by lemma 4.8.33. If the eigenvalues are between 0 and -1, which by lemma 4.8.34 is the only remaining case then, again by lemma 4.8.33, we obtain that $\sigma \in \Theta_2$. In other words, $\sigma \in \Theta_1 \cup \Theta_2 \cup \Theta_3$ and, applying g^{-1}, we have $\sigma'' \in \Theta_1 \cup \Theta_2 \cup \Theta_3$. Therefore, by (4.57), we obtain $\sigma' \in \Theta_1 \cup \Theta_2 \cup \Theta_3$.

Finally, if $\mathrm{Discr}(M) < 0$ then by lemma 4.8.14, $\Delta(\mathbb{C}(x))$ is σ-stable ($r < 0$) and therefore by lemma 4.8.16, $\sigma \in \Gamma$, implying $\sigma'' \in \Gamma$ by the definition of Γ. Since $\Gamma \subset \Theta_{12} \cap \Theta_{23} \cap \Theta_{31}$ by lemma 4.8.35, we have $\sigma \in \Theta_{12} \cap \Theta_{23} \cap \Theta_{31}$ which implies $\sigma \in \Theta_{12}$, therefore $\sigma'' = \sigma'$ by definition of σ'', in other words $\sigma' \in \Gamma$. The proof is finished. □

We will finish this subsection by introducing the useful proposition 4.8.38 following.

Lemma 4.8.37. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Then*

$$(\Theta_1 \cup \Theta_2 \cup \Theta_3) \cap \Gamma = \emptyset.$$

Proof. Let $\sigma \in (\Theta_1 \cup \Theta_2 \cup \Theta_3) \cap \Gamma$, then as $\sigma \subset \Gamma \subset \Theta_{12} \cap \Theta_{23} \cap \Theta_{31}$ by lemma 4.8.35 we have $\sigma \in \Theta_{12}$ and can without loss of generality assume σ to be normalised ($\Theta_1, \Theta_2, \Theta_3$ and Γ defined up to the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action). Also we obtain from lemmas 4.8.14 and 4.8.15 that $r < 0$ and that $\det(M + I) > 0$. Hence f_1, f_2 and f_3 are defined with $M_1 = M, M_2 = I$ and $M_3 = M + I$.

We now distinguish the cases $\sigma \in \Theta_1$, $\sigma \in \Theta_2$ and $\sigma \in \Theta_3$.

1. If $\sigma \in \Theta_1$ then there is a $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that σg has a heart obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and hence $\sigma_1 g$ and $\sigma_2 g$ (where $\sigma_1, \sigma_2 \in \mathrm{Stab}(\mathcal{D})$ are the usual stability conditions associated with σ) satisfy the CP-gluing condition with regard to $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Without loss of generality we can assume $g = (K_{\pi t}, f_{\pi t})$. Then the hearts of $\sigma_1 g$ and $\sigma_2 g$ are $\mathcal{A}^{f_1(t)}$ and $\mathcal{A}^{f_2(t)}$ where f_i are associated to σ_i. The gluing condition is now $f_1(t) \geq f_2(t)$. We also have $f_1(0) = r < f_2(0) = 0$ ($\sigma \in \Gamma$ and normalised). The continuity of $f_1(x) - f_2(x)$ now implies the existence of an $s \in \mathbb{R}$ such that $f_1(s) = f_2(s) = s$. The compatibility condition of $(M_1^{-1}, f_1) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ means that there is a $\lambda \in \mathbb{R}$ such that
$$M_1 \exp(i\pi f_1(s)) = \lambda \exp(i\pi s) (\text{this is } Mv = \lambda)$$
and hence
$$M_1 \exp(i\pi s) = \lambda \exp(i\pi s) (Mv = \lambda)$$

which means that M_1 has (at least) one real eigenvalue and therefore, by definition $\sigma \notin \Gamma$.

2. If $\sigma \in \Theta_2$ we proceed similar as before to obtain $f_2(t) \geq f_3(t) + 1$. From lemmas 4.8.16 and 4.7.39 we obtain $f_1(0), f_3(0) \in (-1, 0)$. This implies $D < 0$ and because $f_1(0), f_3(0)$ are the directions of $C + Di$ and $C + 1 + Di$, respectively, we have

$$-1 < f_1(0) < f_3(0) < 0. \tag{4.58}$$

   Hence $0 = f_2(0) < f_3(0) + 1$ and and continuity of $f_2(x) - f_3(x)$ gives $s \in \mathbb{R}$ with $f_2(s) = f_3(s) + 1$ such that $f_3(s) = s - 1$. The compatibility condition of $(M_3^{-1}, f_3) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ means that there is a $\lambda \in \mathbb{R}$ such that

$$M_1 \exp(i\pi f_1(s)) = \lambda \exp(i\pi s)(Mv = \lambda)$$

   and, similar to the previous case we obtain $(M + I)v = \lambda v$ and hence $Mv = (-1 - \lambda)v$ such that $-1 - \lambda \in \mathbb{R}$ is an eigenvalue of M and we argue as before.

3. If $\sigma \in \Theta_3$ we use $f_3(t) \geq f_1(t) + 1$ as well as $f_3(0) < f_1(0) + 1$ from (4.58) to obtain $s \in \mathbb{R}$ with $f_3(s) = f_1(s) + 1$. Since $(M_3^{-1}, f_3), (M_1^{-1}, f_1) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ there are positive real numbers α, β such that

$$(M + I) \exp(i\pi f_3(s)) = \alpha \exp(i\pi s) \text{ and } M \exp(i\pi f_1(s)) = \beta \exp(i\pi s).$$

   Writing $v = \exp(i\pi f_1(s))$ and $\lambda = \frac{\alpha}{\beta} \in \mathbb{R}_{>0}$ and we obtain $(M + I)(-v) = \lambda M v$ such that $Mv = -\frac{1}{\lambda+1} M$. Hence, since $\lambda > 0$ we – once again – found a real eigenvalue of M.

We obtain $\sigma \notin \Gamma$ if $\sigma \in \Theta_i$ for $i \in \{1, 2, 3\}$ and the proof is finished. □

Proposition 4.8.38. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Then*

$$S_{\mathcal{D}^\uparrow}(\Gamma) \subset \Gamma$$

Proof. Assume $\sigma \in \Gamma$. If $S_{\mathcal{D}^\uparrow}(\sigma) \notin \Gamma$ then $S_{\mathcal{D}^\uparrow}(\sigma) \in \Theta_1 \cup \Theta_2 \cup \Theta_3$ and hence $\sigma \in \Theta_1 \cup \Theta_2 \cup \Theta_3$ such that $\sigma \in (\Theta_1 \cup \Theta_2 \cup \Theta_3) \cap \Gamma = \emptyset$ by lemma 4.8.37, we obtain a contradiction. □

4.9 Support property

The support property (definition 2.5.43) depends on a quadratic form. We will now prove its holding for CP-glued pre-stability conditions. We start by proving it for pre-stability conditions under a stronger orthogonality condition.

Lemma 4.9.1. *If a pair $\sigma = (Z, \mathcal{H})$ is a pre-stability condition on $\mathcal{D}^{\uparrow}$ obtained by CP-gluing from stability conditions σ_1, σ_2 with regard to $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, satisfying that $\mathrm{Hom}^{\leq 1}_{\mathcal{D}^{\uparrow}}(i_1(\mathcal{H}_1), i_2(\mathcal{H}_2)) = 0$ (or equivalently $f(0) \geq 1$), then it satisfies the support property and as consequence, is a Bridgeland stability condition.*

Proof. We use the notation $Z_1([E]) = Z_1([\lambda_1(E)])$ and $Z_2([E]) = Z_2([\rho_2(E)])$. We can linearly extend Z and the homomorphism induced by the exact functors λ_1, ρ_2 to $\mathcal{N}(\mathcal{D}^{\uparrow}) \otimes \mathbb{R} \cong \mathbb{R}^4$. We define the quadratic form

$$Q : \mathcal{N}(\mathcal{D}^{\uparrow}) \otimes \mathbb{R} \to \mathbb{R} \text{ as } Q(v) = \Im(Z_1(v))\Im(Z_2(v)) + \Re(Z_1(v))\Re(Z_2(v)),$$

where $\Re(v)$ and $\Im(v)$ are the real and the imaginary part of $Z_i(v) \in \mathbb{C}, i \in \{1, 2\}$ respectively. By the linearity of Z, it is clear that Q is a quadratic form. We first show that it is negative definite on $\ker(Z) = \{v \in \mathbb{R}^4 \mid \Re(Z_1(v)) = -\Re(Z_2(v)) \text{ and } \Im(Z_1(v)) = -\Im(Z_2(v))\}$. To see this, we now consider $Q(v) = -\Im(Z_1(v))^2 - \Re(Z_1(v))^2 \leq 0$, for $v \in \ker(Z)$. If, on the other hand, $Q(v) = 0$, we obtain $Z_2(v) = Z_1(v) = 0$. This implies $v = 0$.

Let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ be a σ-stable object – by lemma 2.5.46 it is enough to show that $Q(E) \geq 0$ for σ-stable objects. Let $[\varphi]$ be the morphism in $\mathrm{Hom}_{\mathcal{D}}(E_1, E_2)$ that is induced by the object $E_1 \xrightarrow{\varphi} E_2 \in \mathcal{D}^{\uparrow}$. Since σ is a CP-glued pre-stability condition, by the definition of a heart obtained by CP-gluing from $(\mathcal{H}_1, \mathcal{H}_2)$ via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, we have that $E_1 \in \mathcal{H}_1$ and $E_2 \in \mathcal{H}_2$. Since $[\varphi] \in \mathrm{Hom}_{\mathcal{D}}(E_1, E_2) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(E_1), i_2(E_2)[1])$ and – by hypothesis – we have $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(E_1), i_2(E_2)[1]) = 0$, we obtain that $[\varphi] = 0$. Therefore, applying corollary 4.5.6 we have $E = i_1(\lambda_1(E)) \oplus i_2(\rho_2(E))$ and this gives us $E = i_1(\lambda_1(E)) \oplus i_2(\rho_2(E)) \cong i_1(E_1) \oplus i_2(E_2)$. Since E is stable, this implies either $i_1(\lambda_1(E)) = 0$ such that $Z_1(E) = 0$ or $i_2(\rho_2(E))$ such that $Z_2(E) = 0$ either of which provides $Q = 0$.

□

Lemma 4.9.2. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Let $\sigma = (Z, \mathcal{H})$ be a stability condition on $\mathcal{D}^{\uparrow}$ obtained from CP-gluing stability conditions σ_1, σ_2 with $\sigma_1 = \sigma_2 g$ via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Assume $f(0) = 0$. Let*

$$d_2 = -\Re(Z_2([\rho_2(E)])), d_1 = -\Re(Z_2([\lambda_1(E)])),$$
$$r_2 = \Im(Z_2([\rho_2(E)])) \text{ and } r_1 = \Im(Z_2([\lambda_1(E)])).$$

where Z_2 is the stability function associated with σ_2. If we have $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ with $r_1 > 0, r_2 > 0$ a σ-stable object and $[\varphi] \neq 0$, then

$$-B \geq (A + C)\mu_\sigma(E).$$

Proof. We have that φ is a complex-morphism over $\mathcal{A}$ and $[\varphi]$ the associated morphism in $\mathcal{D}$, in other words $[\varphi] \in \mathrm{Hom}_{\mathcal{D}}(\lambda_1(E), \rho_2(E))$. By lemma 3.1.5 we have $E_1 = \lambda_1(E) \in \mathcal{H}_1$ and $E_2 = \rho_2(E) \in \mathcal{H}_2$ where $\mathcal{H}_1, \mathcal{H}_2$ are the hearts associated with σ_1 and σ_2. Since $f(0) = 0$ we also have $\mathcal{H}_1 = \mathcal{H}_2$. We consider a short exact sequence in $\mathcal{H}_1$ constructed following. Since $\mathcal{H}_1$ is an abelian category, we can compute $\ker([\varphi]), \mathrm{coker}([\varphi]) \in \mathcal{H}_1$ and by the definition of $\mathcal{H}$, we have morphisms

$$E \twoheadrightarrow i_1(\mathrm{im}(\varphi)) \text{ and } i_2(\mathrm{im}(\varphi)) \hookrightarrow E. \tag{4.59}$$

Let $Z_2(\mathrm{im}([\varphi])) = -d_1'' + r_1'' i$. We have that $r_1'' \neq 0$, as $r_1'' = 0$, would imply $\phi_\sigma(i_2(\mathrm{im}([\varphi]))) = 1$, and by the σ-stability of E we would therefore get $1 = \phi_\sigma(i_2(\mathrm{im}([\varphi]))) \leq \phi_\sigma(E) \leq 1$, giving $\Im(Z(E)) = 0$. This – on the other hand – would imply that $r_1 = 0$ or $r_2 = 0$, which contradicts our assumption.

From (4.59), combined with the correspondence between slope and phase, we obtain

$$\frac{d_1''}{r_1''} \leq \mu_\sigma(E) \text{ and } \mu_\sigma(E) \leq \frac{-Ad_1'' - Br_1''}{Cr_1''},$$

therefore, using $-A, C > 0$, we have $\mu_\sigma(E) \leq \frac{-A}{C}\frac{d_1''}{r_1''} - \frac{B}{C} \leq \frac{-A}{C}\mu_\sigma(E) - \frac{B}{C}$, in other words, $\mu_\sigma(E) \leq \frac{-A}{C}\mu_\sigma(E) - \frac{B}{C}$ and conclude $\mu_\sigma(E)(A + C) \leq -B$. □

Lemma 4.9.3. *If, under the conditions of lemma 4.9.2 with $r_1 > 0, r_2 > 0$ and $[\varphi] \neq 0$, then*

$$-Ad_1 + d_2 - \mu_\sigma(E)(r_2 - Ar_1) \leq 0.$$

Proof. We have

$$\mu_\sigma(E) = \frac{-Ad_1 + d_2 - Br_1}{Cr_1 + r_2} = \frac{-Ad_1 + d_2}{Cr_1 + r_2} + \frac{-Br_1}{Cr_1 + r_2}.$$

By Lemma 4.9.2, we obtain

$$\mu_\sigma(E) \geq \frac{-Ad_1 + d_2}{Cr_1 + r_2} + (A + C)\frac{\mu_\sigma(E)r_1}{Cr_1 + r_2},$$

using $r_1 > 0, r_2 > 0$ and $C > 0$, which implies

$$\mu_\sigma(E)(r_2 - Ar_1) \geq -Ad_1 + d_2$$

and therefore gives the claim. □

Lemma 4.9.4. *If, under the conditions of lemma 4.9.3, $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ with $r_1 > 0, r_2 > 0$ with $[\varphi] \neq 0$ is a σ-stable object, then*

$$(r_2 - r_1)\mu_\sigma(E) \leq d_2 - d_1.$$

Proof. Recall our explanations regarding φ at the start of the proof of lemma 4.9.2. We first consider the case $\ker([\varphi]) = 0$, providing us with the exact sequence $0 \to \Delta(E_1) \to E \to i_2(\operatorname{coker}([\varphi])) \to 0$. By the σ-stability of E, we obtain $\mu_\sigma(E) \leq \frac{d_2-d_1}{r_2-r_1}$, which implies the desired result given by the fact that $\ker([\varphi]) = 0$ implies $r_2 \geq r_1$ and hence $r_2 - r_1 \geq 0$.

Next, we consider the case $\operatorname{coker}([\varphi]) = 0$, where we have the exact sequence $0 \to i_1(\ker([\varphi])) \to E \to \Delta(E_2) \to 0$. We obtain

$$\mu_\sigma(E) \geq \frac{-A(d_1 - d_2) - B(r_1 - r_2)}{C(r_1 - r_2)} = \frac{-A(d_1 - d_2)}{C(r_1 - r_2)} + \frac{-B}{C}.$$

Lemma 4.9.2 gives $\mu_\sigma(E) \geq \mu_\sigma(E) + \mu_\sigma(E)\frac{A}{C} - \frac{A(d_1-d_2)}{C(r_1-r_2)}$, Since $-A, C \geq 0$, (from $f(0) = 0$ and $D = 0$) we obtain

$$\mu_\sigma(E) \geq \frac{d_1 - d_2}{r_1 - r_2}.$$

Since now $r_{(\ker([\varphi]))} = r_1 - r_2$ where $r_{(\ker([\varphi]))} = \Im(Z \circ i_2(\ker([\varphi]))$ and $\ker([\varphi]) \in \mathcal{H}$, we have $r_1 - r_2 \geq 0$ and obtain $\mu_\sigma(E)(r_2 - r_1) \leq (d_2 - d_1)$.

It remains to consider the case where $\operatorname{coker}([\varphi]) \neq 0$ and $\ker([\varphi]) \neq 0$. From the morphism $i_1(\ker([\varphi])) \hookrightarrow E$ we obtain $\frac{-Ad_1'-Br_1'}{Cr_1'} \leq \mu_\sigma(E)$, where $Z_2(\ker([\varphi])) = -d_1' + r_1'i$. The previous quotient exists since $r_1' \neq 0$, as otherwise, if we had $r_1' = 0$, we would obtain $\phi(i_1(\ker([\varphi]))) = 1$ and by σ-semistability we have that $1 = \phi(i_1(\ker([\varphi]))) \leq \phi(E) \leq 1$, which, implying $r_1, r_2 = 0$, would give a contradiction.

Since $\frac{-Ad_1'-Br_1'}{Cr_1'} \leq \mu_\sigma(E)$ provides $\frac{-Ad_1'}{Cr_1'} \leq \mu_\sigma(E) + \frac{B}{C}$, we therefore obtain $\frac{-Ad_1' r_1}{r_1'} \leq \mu_\sigma(E)Cr_1 + Br_1$. We have

$$\begin{aligned}
\mu_\sigma(E)Cr_1 + Br_1 \leq \frac{-ACd_1r_1 - BCr_1^2 + Cd_2r_1 + BCr_1^2 + Br_1r_2}{Cr_1 + r_2} \\
\leq \frac{-ACd_1r_1 + Cd_2r_1 + Br_1r_2}{Cr_1 + r_2} \\
\leq \frac{(Cr_1 + r_2)(-Ad_1 + d_2) + Ad_1r_2 + Br_1r_2 - d_2r_2}{Cr_1 + r_2} \\
\leq -Ad_1 + d_2 - \mu_\sigma(E)r_2,
\end{aligned}$$

providing $\frac{-Ad_1'r_1}{r_1'} \leq \mu_\sigma(E)Cr_1 + Br_1 \leq -Ad_1 + d_2 - \mu_\sigma(E)r_2$. This gives

$$-Ad_1'r_1 + \mu_\sigma(E)r_2r_1' - r_1'(-Ad_1 + d_2) \leq 0. \tag{4.60}$$

If $r_2 - r_1'' \neq 0$, then the morphism $E \twoheadrightarrow i_2(\mathrm{coker}([\varphi]))$ gives us

$$\mu_\sigma(E) \leq \frac{d_2 - d_1''}{r_2 - r_1''}, \tag{4.61}$$

where $Z_2(\mathrm{im}([\varphi])) = -d_1'' + ir_1''$, as before.

As $d_1'' = d_1 - d_1'$ and $r_1'' = r_1 - r_1'$, multiplying (4.61) by $-Ar_1$ and adding (4.60), we obtain

$$Ar_1(d_2 - d_1) - r_1'(-Ad_1 + d_2) + \mu_\sigma(E)(-Ar_1(r_2 - r_1) + r_1'(-Ar_1 + r_2)) \leq 0.$$

By lemma 4.9.3, we have $d_1 - d_2 + \mu_\sigma(E)(r_2 - r_1) \leq 0$ and as a consequence

$$\mu_\sigma(E)(r_2 - r_1) \leq d_2 - d_1.$$

If $r_2 - r_1'' = 0$, then also $d_2 - d_1'' \geq 0$. Since $d_1'' = d_1 - d_1'$ and $r_1'' = r_1 - r_1'$, we obtain

$$-Ar_1(d_1 - d_1' - d_2) \leq 0.$$

Adding the previous inequality to (4.60) we have

$$\mu_\sigma(E)r_2r_1' - r_1'(-Ad_1 + d_2) - Ar_1(d_1 - d_2) \leq 0.$$

By Lemma 4.9.3, we obtain

$$\mu_\sigma(E)r_2r_1' - r_1'(\mu_\sigma(E)(r_2 - Ar_1)) - Ar_1(d_1 - d_2) \leq 0,$$

which, provided by $-A > 0$, gives $\mu_\sigma(E)(r_2 - r_1) \leq (d_2 - d_1)$ and therefore finishes the proof. □

Lemma 4.9.5. *Let $\sigma = (Z, \mathcal{H})$ be a pre-stability condition on $\mathcal{D}^\uparrow$ and assume that σ is obtained via CP-gluing stability conditions σ_1 and σ_2, using the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Assume that $\sigma_1 = \sigma_2 g$ where $g = (T, f)$ in $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, that satisfies $f(0) = 0$.*

Define r_i, d_i as in lemma 4.9.2, $x = \frac{d_1}{r_1}$ and $y = \frac{d_2}{r_2}$.. If there is a σ-semistable object $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ with $r_2 \geq r_1 > 0$ then $Cy + Ax \leq -B$. Moreover, if $[\varphi] \neq 0$ then $y - x \geq 0$.

Proof. Let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ be a σ-semistable object. Considering the short exact sequence $0 \to i_2(E_2) \to E \to i_1(E_1) \to 0$, the semistability of E and the correspondence between slope and phase provides us with

$$\frac{d_2}{r_2} \leq \mu_\sigma(E) \leq \frac{-Ad_1 - Br_1}{Cr_1}.$$

This implies $Cr_1d_2 + Ad_1r_2 \leq -Br_1r_2$, as $r_1, r_2, C > 0$. We obtain $Cy + Ax \leq -B$.

If $r_1 = r_2$ and $[\varphi] = 0$, we have $d_1 \leq d_2$ by lemma 4.9.4 and therefore $y \geq x$. Now assume $r_2 > r_1$. Since $[\varphi] \neq 0$, lemma 4.9.4 provides us with the inequality $\frac{-Ad_1 - Br_1 + d_2}{Cr_1 + r_2} \leq \frac{d_2 - d_1}{r_2 - r_1}$, hence $-B \leq \frac{Cd_2r_1 + Ad_1r_2 - (C+A)d_1r_1 + d_2r_1 - d_1r_2}{(r_2 - r_1)r_1}$.

From the inequality $Cy + Ax \leq -B$, we now obtain

$$Cy + Ax \leq \left(Ax + Cy + y - x - \frac{r_1}{r_2}x(A + C) \right) \frac{r_2}{r_2 - r_1}, \text{ implying}$$
$$0 \leq (y - x)(r_2 + Cr_1).$$

Since $r_2 + Cr_1 > 0$, we have proved $y - x \geq 0$. □

Lemma 4.9.5 requires the restriction $r_2 \geq r_1 > 0$. We require to prove the analogous statement for $r_1 \geq r_2 > 0$ or, using the language of lemma 4.9.5, $C_2r_1 + D_2d_1 \geq C_2r_2 + D_2d_2 > 0$. In order to do so, we follow ideas of [29]. The following definition makes sense due to [37, Remark 2.51].

Definition 4.9.6. Let C be a smooth projective curve. Define

$$\mathbb{D} = \mathcal{R}\operatorname{Hom}(-, \mathcal{O}_C) \colon \mathcal{D}^b(\operatorname{Coh}(C)) \to \mathcal{D}^b(\operatorname{Coh}(C))$$

as the right derived functor of $\mathbb{D}_0 = \operatorname{Hom}(-, \mathcal{O}_C)$.

It was proven in [40, Section 3.2] that $\mathbb{D}$ is an equivalence of categories. Additionally we have the following.

Lemma 4.9.7. *We have* $\mathbb{D}^2 = \mathrm{id}$.

Proof. See [40, Section 3.2]. □

We will discuss "locally free" objects in $\mathcal{A}^\uparrow$ in analogy to definition 4.7.15.

Definition 4.9.8. For $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ define $\delta(g)$ by

$$\delta(g) := (CTC^{-1}, h) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$$

where $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $h(t) = -f(t)$.

Hence, for $\sigma = \sigma_\mu g$ define $\sigma^\vee$ by

$$\sigma^\vee = \sigma_\mu \delta(g).$$

Lemma 4.9.9. *The map δ is a homomorphism of groups.*

Proof. Let $M := T^{-1}$. We have $M\exp(i\pi f(t)) \in \mathbb{R}_{\geq 0}\exp(i\pi t)$. Such that, substituting $-t$ for t we obtain $M\exp(i\pi f(-t)) \in \mathbb{R}_{\geq 0}\exp(i\pi - t)$. Moreover, $C = C^{-1}$ and $C\exp(i\pi s) = -\exp(-i\pi s)$ and hence $MC^{-1}\exp(-i\pi f(-t)) \in -\mathbb{R}_{>0}\exp(i\pi t)$ and $CMC^{-1}\exp(-i\pi f(-t)) \in \mathbb{R}_{>0}\exp i\pi t$. Therefore we have $CMC^{-1} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R}$. Since conjugation with C is a group homomorphism on $\mathrm{GL}_2^+(\mathbb{R}$ and composition with $-\operatorname{id}$ is a group homomorphism on the group of bijective maps from $\mathbb{R}$ to $\mathbb{R}$, we obtain that δ is a homomorphism of groups. □

Lemma 4.9.10. *If $\sigma = \sigma_\mu g$ is a stability condition on $\mathcal{D}$ with $g = (T, f)$, $M = T^{-1} = \begin{pmatrix} -A & -B \\ D & C \end{pmatrix}$ and heart $\mathcal{P}_\mu(f(0), f(1)]$, then $\sigma^\vee = \sigma_\mu\delta(g)$ with $\delta(g) = (T', f'), M' = (T')^{-1} = \begin{pmatrix} -A & -B \\ D & C \end{pmatrix}$ has $\mathcal{P}_\mu(-f(0), -f(-1)]$ as its heart.*

Proof. Let $\theta = f(0)$. We obtain the stability function of $\sigma^\vee$ by conjugation with C as defined in 4.9.8. Moreover, for $\mathcal{P}_\mu(\theta, \theta + 1]$ we obtain from $h(0) = -f(0) = -\theta$ that $\sigma^\vee$ has $\mathcal{P}_\mu(-\theta, 1-\theta]$ as heart and we have $\mathcal{P}_\mu(1-\theta, 2-\theta][-1] = \mathcal{P}_\mu(-\theta, 1-\theta]$. □

Lemma 4.9.11. *For all $t \in \mathbb{R}$ we have $\mathbb{D}\mathcal{P}_\mu(t) = \mathcal{P}_\mu(1-t)$.*

Proof. We have $\mathbb{D}(F[n]) = \mathbb{D}(F)[-n]$ for all $F \in \mathrm{Coh}(C)$ and $n \in \mathbb{Z}$ such that we can impose the restriction $t \in (0, 1]$.

If $T \in \mathcal{P}_\mu(1)$ torsion we have $\mathbb{D}(T) = T[-1] \in \mathcal{P}_\mu(0)$. If $E \in \mathcal{P}_\mu(t)$ is locally free, then $\mathbb{D}(E) = E^\vee$ and $Z_\mu(\mathbb{D}(E)) = -\deg(\mathbb{D}(E)) + \operatorname{rank}(\mathbb{D}(E)) = \deg(E) + i\operatorname{rank}(E) = -\overline{Z_\mu(E)}$. Since E is μ-semistable if and only if $E^\vee$ is μ-semistable we obtain $E^\vee \in \mathcal{P}_\mu(s)$ for some $s \in (0, 1)$ determined by $\overline{Z_\mu(E)}$. In other words, the complex number is mapped by a reflection on the imaginary axis which for $t \in (0, 1)$ swaps the phase t with $1-t$. Hence $E^\vee \in \mathcal{P}_\mu(1-t)$ and the proof is finished. □

Corollary 4.9.12. *We have $\mathbb{D}\mathcal{P}_\mu(\theta, \theta + 1) = \mathcal{P}_\mu(-\theta, -\theta + 1)$.*

Proof. This follows from 4.9.11 and the HNF since $\mathbb{D}$ is an exact functor. □

Definition 4.9.13. Let σ be obtained by CP-gluing stability conditions (σ_1, σ_2) on $\mathcal{D}$ with respect to $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and assume $\mathcal{H}_1 = \mathcal{H}_2$ where $\mathcal{H}_i$ is the heart corresponding to $\sigma_i, i \in \{1, 2\}$. We define the dual stability condition $\sigma^* =$ on $\mathcal{D}^\dagger$ as the stability condition obtained via CP-gluing from $\sigma_2^\vee, \sigma_1^\vee$ with respect to $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$.

Lemma 4.9.14. *Let $\sigma = (Z, \mathcal{H})$ be a pre-stability obtained by CP-gluing stability conditions (σ_1, σ_2) on $\mathcal{D}$ with respect to $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ such that $\sigma_1 = \sigma_2 g$, where $g = (T, f)$ and assume $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{P}_\mu(\theta, \theta+1], \theta \in \mathbb{R}$ where $\mathcal{H}_i$ is the heart corresponding to $\sigma_i, i \in \{1, 2\}$. If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{P}(0, 1)$ where $\mathcal{P}$ is the slicing of σ then $\lambda_1(E), \rho_2(E) \in \mathcal{P}_\mu(\theta, \theta+1)$.*

Proof. Define F_σ^θ to be $\mathcal{F}$ and T_σ^θ to be $\mathcal{T}$ from lemma 4.7.4 with $\phi = 1$. We obtain the short exact sequence

$$0 \to T_2 \to E_2 \to F_2 \to 0 \tag{4.62}$$

as well as the short exact sequence

$$0 \to T_1 \xhookrightarrow{f} E_1 \to F_1 \to 0 \tag{4.63}$$

with $T_1, T_2 \in T_\sigma^\theta$ and $F_1, F_2 \in F_\sigma^\theta$ in $\mathcal{H}_1 = \mathcal{H}_2$. If $T_2 \neq 0$ then $i_2(T_2) = (0 \to T_2)$, using that $i_2(\mathcal{P}(\theta+1)) \subset \mathcal{P}(1)$, is a non-zero subobject of E with $\phi_\sigma(i_2(T_2)) = \phi_\sigma(0 \to T_2) = 1$, contradicting $E \in \mathcal{P}(0, 1)$. Hence $T_2 = 0$, providing $E_2 \in F^\theta$. Hence, $E_2 \cong F_2$ provides $\mathrm{Hom}(T_1, E_2) = 0$ from the torsion pair such that the embedding f from the short exact sequence (4.63) provides us with the commutative diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{f} & E_1 \\ \downarrow & & \downarrow \varphi \\ 0 & \longrightarrow & E_2, \end{array}$$

in other words, with an embedding of $i_1(T_1) = (T_1 \to 0)$ into $E_1 \xrightarrow{\varphi} E_2$. We conclude – as in the previous case – that $T_1 = 0$ □

Definition 4.9.15. Define $\mathbb{D}_0^\uparrow : \mathcal{A}^\uparrow \to \mathcal{A}^\uparrow$ by

$$\mathbb{D}_0^\uparrow(E_1 \xrightarrow{\varphi} E_2) := (\mathbb{D}_0(E_2) \xrightarrow{\mathbb{D}(\varphi)} \mathbb{D}_0(E_1))$$

for all $E = (E_1 \xrightarrow{\varphi} E_2) \in \mathcal{A}^\uparrow$.

We require the right derived functor of $\mathbb{D}_0^\uparrow$ and will now prove that it exists.

Lemma 4.9.16. *Let $\mathcal{A} = \mathrm{Coh}(C)$, C a smooth projective curve. Every bounded object E in $\mathcal{C}(\mathcal{A}^\uparrow)$ is isomorphic to a complex $F \in \mathcal{C}(\mathcal{A}^\uparrow)$ of locally free sheaves.*

Proof. It is enough to show that for $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}^\uparrow$ there is a $G \in \mathcal{A}^\uparrow$ of locally free sheaves with $G \twoheadrightarrow E$. In $\mathcal{A}$, there are locally-free sheaves $F_i \in \mathcal{A}$ and surjective morphisms $F_i \xrightarrow{\pi_i} E_i$ for $i = 1, 2$, provided by [37, Remark 3.26]. Let now $G = (F_1 \xrightarrow{(\mathrm{id},0)} F_1 \oplus F_2)$. Since $F_1 \oplus F_2$ is also locally free, we have constructed G to be locally free. We have a surjective morphism $G \to E$ in $\mathcal{A}^\uparrow$ given by the diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\pi_1} & E_1 \\ \downarrow{\scriptstyle (\mathrm{id},0)} & & \downarrow{\scriptstyle \varphi} \\ F_1 \oplus F_2 & \xrightarrow{\varphi\circ\pi_1+\pi_2} & E_2 \end{array}$$

and the proof is finished. □

Lemma 4.9.17. *Let $\mathcal{A} = \mathrm{Coh}(C)$, C a smooth projective curve. The right derived functor of $\mathbb{D}_0^\uparrow$ exists.*

Proof. For any bounded acyclic object $E \in \mathcal{C}(\mathcal{A}^\uparrow)$ of locally free sheaves we have that $\mathrm{Hom}(E, \mathcal{O}_C)$ is acyclic. By lemma 4.9.16 the result now follows from [37, Remark 2.51]. □

We are therefore able to provide the following definition.

Definition 4.9.18. Define

$$\mathbb{D}^\uparrow := \mathcal{R}\mathbb{D}_0^\uparrow : \mathcal{D}^\uparrow \to \mathcal{D}^\uparrow$$

to be the right derived functor of $\mathbb{D}_0^\uparrow$.

Lemma 4.9.19. *Let $\mathcal{A} = \mathrm{Coh}(C)$, C an elliptic curve and let $\sigma = (\mathcal{P}, Z)$ be CP-glued from stability conditions (σ_1, σ_2) via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Assume that σ_1, σ_2 have hearts as in lemma 4.9.14. For $\sigma^* = (\mathcal{P}^*, Z^*)$ we have that $E = (E_1 \to E_2) \in \mathcal{P}(0, 1)$ σ-stable implies $\mathbb{D}^\uparrow(E) \in \mathcal{P}^*(0, 1)$ and $\mathbb{D}^\uparrow(E)$ σ^*-stable.*

Proof. By lemma 4.9.14 we have $\lambda_1(E), \rho_2(E) \in \mathcal{P}_\mu(\theta, \theta+1)$. Following from the analogous identities for $\mathbb{D}_0^\uparrow$ and $\mathbb{D}_0$ we obtain

$$\lambda_1 \circ \mathbb{D}^\uparrow = \mathbb{D} \circ \rho_2 \text{ and } \rho_2 \circ \mathbb{D}^\uparrow = \mathbb{D} \circ \lambda_1. \tag{4.64}$$

Together with lemma 3.1.5 this provides $\mathbb{D}^\uparrow(E) \in \mathcal{P}^*(0, 1]$. In addition, (4.64) provides, using that for a given stability condition σ, the stability function $Z_{\sigma^\vee}$ is obtained from Z_σ via conjugation with the matrix C from definition 4.9.8, that

$$\begin{aligned} Z_{\sigma^*}(\mathbb{D}^\uparrow(E)) &= Z_{\sigma^*}(i_1\lambda_1\mathbb{D}^\uparrow(E)) + Z_{\sigma^*}(i_2\rho_2\mathbb{D}^\uparrow(E)) \\ = Z_{\sigma_1^\vee}(\mathbb{D}(E_1)) + Z_{\sigma_2^\vee}(\mathbb{D}(E_2)) &= -\overline{Z_{\sigma_1}(E_1)} - \overline{Z_{\sigma_2}(E_2)} \\ &= -\overline{(Z_\sigma(i_1\lambda_1(E)) + Z_\sigma(i_2\rho_2(E))} = -\overline{Z_\sigma(E)}. \end{aligned} \tag{4.65}$$

This implies that $Z_{\sigma^*}(\mathbb{D}^{\uparrow}(E)) \in \mathbb{R}$ if and only if $Z_\sigma(E) \in \mathbb{R}$ if and only if $E \in \mathcal{P}(1)$. Therefore $Z_{\sigma^*}(\mathbb{D}^{\uparrow}(E)) \notin \mathbb{R}$. Hence σ^*-semistability of $\mathbb{D}^{\uparrow}(E)$ would imply $\phi_{\sigma^*}(\mathbb{D}^{\uparrow}(E)) < 1$ and hence $\mathbb{D}^{\uparrow}(E) \in \mathcal{P}^*(0,1)$.

It is therefore our task to prove $\mathbb{D}^{\uparrow}(E)$ σ^*-semistable. Let

$$0 \to G \to \mathbb{D}^{\uparrow}(E) \to Q \to 0$$

be an exact sequence in $\mathcal{P}^*(0,1]$. If $\phi_{\sigma^*}(Q) = 1$, we have $\phi_{\sigma^*}(\mathbb{D}^{\uparrow}(E)) \leq \phi_{\sigma^*}(Q)$. If $Q \in \mathcal{P}^*(0,1)$ then lemma 4.9.14 implies that $\lambda_1(Q), \rho_2(Q) \in \mathcal{P}_\mu(-\theta, -\theta+1)$. Hence, by corollary 4.9.12, $\mathbb{D}^{\uparrow}(Q) \in \mathcal{P}(0,1]$. Because $(\mathcal{P}^*(1), \mathcal{P}^*(0,1))$ is a torsion pair in $\mathcal{P}^*(0,1]$ and the torsion-free part in a torsion pair is always closed under taking subobjects, we have $G \in \mathcal{P}^*(0,1)$ and hence $\mathbb{D}^{\uparrow}(G) \in \mathcal{P}(0,1]$ by lemma 4.9.14, arguing as before. The exact triangle

$$\mathbb{D}^{\uparrow}(Q) \to E \to \mathbb{D}^{\uparrow}(G) \xrightarrow{+}$$

therefore is an exact sequence

$$0 \to \mathbb{D}^{\uparrow}(Q) \to E \to \mathbb{D}^{\uparrow}(G) \to 0$$

in $\mathcal{P}(0,1]$, using lemma 4.9.7, which implies $(\mathbb{D}^{\uparrow})^2 = \mathrm{id}$. The σ-stability of E provides $\phi_\sigma(\mathbb{D}^{\uparrow}(Q)) < \phi_\sigma(E)$. By (4.65) this is equivalent to $\phi_{\sigma^*}(\mathbb{D}^{\uparrow}(E)) < \phi_{\sigma^*}(Q)$ (because $\mathbb{D}^{\uparrow}(Q), E \in \mathcal{P}(0,1]$ and $Q, \mathbb{D}^{\uparrow}(E) \in \mathcal{P}^*(0,1]$). Hence, we get that $\mathbb{D}^{\uparrow}(E)$ is σ^*-semistable. This, in turn, implies $\mathbb{D}^{\uparrow}(E) \in \mathcal{P}(0,1)$ and even when $\phi_{\sigma^*}(Q) = 1$ we obtain $\phi_{\sigma^*}(\mathbb{D}^{\uparrow}(E)) < \phi_{\sigma^*}(Q)$. Therefore $\mathbb{D}^{\uparrow}(E)$ is σ^*-stable.

□

The previous lemma is indeed an equivalence but we only require the implication discussed, in order to prove the next lemma.

Lemma 4.9.20. *Let* $\mathcal{A} = \mathrm{Coh}(C)$*,* C *an elliptic curve and* $\sigma = (Z, \mathcal{H})$ *be a pre-stability condition on* $\mathcal{D}^{\uparrow}$ *obtained by CP-gluing stability conditions* σ_1, σ_2 *on* $\mathcal{D}$ *via* $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$*, with* $\sigma_1 = \sigma_2 g$*, were* $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ *and* $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$*. Assume* $f(0) = 0$*. Let* $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ *be a* σ*-stable object and define*

$$d_2 = -\Re(Z \circ i_2([\rho_2(E)])), d_1 = -\Re(Z \circ i_2([\lambda_1(E)])),$$
$$r_2 = \Im(Z \circ i_2([\rho_2(E)])), r_1 = \Im(Z \circ i_2([\lambda_1(E)])),$$
$$x := \frac{d_1}{r_1} \text{ and } y := \frac{d_2}{r_2}.$$

If $r_1 \geq r_2 > 0$ *then*

$$Cy + Ax + B \leq 0$$

and if, additionally, $[\varphi] \neq 0$ *then*

$$x \leq y.$$

Proof. We have $Z(E) = Z(i_1\lambda_1(E)) + Z(i_2\rho_2(E)) = Ar_1 + Bd_1 - d_2 + i(Cr_1 + r_2)$. As $f(0) = 0$ and $D = 0$ we have $C > 0$. From $r_1 > 0$ and $r_2 > 0$ we now get $\Im(Z(E)) = Cr_1 + r_2 > 0$, hence $\phi_\sigma(E) < 1$. Since σ is obtained by CP-gluing stability conditions σ_1, σ_2 via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and $\sigma_1 = \sigma_2 g$, we obtain that σ^* is obtained by CP-gluing stability conditions $\sigma_1^\vee, \sigma_2^\vee$ and therefore from $\sigma_2^\vee \delta(g), \sigma_2^\vee$ via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Let $g' := \delta(g)^{-1}$ and define $\sigma' = \sigma^* g'$. Because $\sigma_2^\vee g'$ and $\sigma^\vee$ have the same heart they satisfy the CP-gluing condition so that we can apply lemma 3.1.16 to obtain that σ' is obtained by CP-gluing stability conditions $\sigma_2^\vee g', \sigma_2^\vee$ via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$.

Let $E' := \mathbb{D}(E)$. By lemma 4.9.19, we get $E' \in \mathcal{P}^*(0,1)$ and E' σ^*-stable. As $\sigma' = \sigma^* g$, E' is σ'-stable too. Let $g' = (T', f')$, then $M' = (T^{-1})' = \begin{pmatrix} -A & -B \\ 0 & C \end{pmatrix}^{-1} = \frac{1}{-AC}\begin{pmatrix} C & B \\ 0 & -A \end{pmatrix} =: \begin{pmatrix} -A' & B' \\ 0 & C' \end{pmatrix}$. We write $\sigma' = (Z', \mathcal{H}')$ and define

$$\begin{aligned} d_2' = -\Re(Z' \circ i_2([\rho_2(E')])), d_1' = -\Re(Z' \circ i_2([\lambda_1(E')])), \\ r_2' = \Im(Z' \circ i_2([\rho_2(E')])), r_1' = \Im(Z' \circ i_2([\lambda_1(E')])), \\ x' := \frac{d_1'}{r_1'} \text{ and } y' := \frac{d_2'}{r_2'}. \end{aligned}$$

Because, σ is obtained by CP-gluing stability conditions $\sigma_2 g, \sigma_2$ on $\mathcal{D}$ via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and σ' is obtained by CP-gluing stability conditions $\sigma_2' g', \sigma_2'$ on $\mathcal{D}$ via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ we obtain that $Z \circ i_2$ and $Z' \circ i_2$ are, using that for a given stability condition σ, the stability function $Z_{\sigma^\vee}$ is obtained from Z_σ via conjugation with the matrix C from definition 4.9.8, that related by the formula

$$Z' \circ i_2 \mathbb{D}(X) = -\overline{Z \circ i_2(X)} \text{ for any } X \in \mathcal{D}.$$

Using $\lambda_1(E') = \lambda_1 \mathbb{D}^\uparrow(E) = \mathbb{D}\rho_2(E)$ and $\rho_2(E') = \rho_2 \mathbb{D}^\uparrow(E) = \mathbb{D}\lambda_1(E)$, we obtain

$$d_1' = -d_2, r_1' = r_2, d_2' = -d_1 \text{ and } r_2' = r_1.$$

In particular does the assumption $r_1 \geq r_2 > 0$ translates into $r_2' \geq r_1' > 0$. Moreover $x' = \frac{d_1'}{r_1'} = \frac{-d_2}{r_2} = -y$ and $y' = \frac{d_2'}{r_2'} = \frac{-d_1}{r_1} = -x$ and we see that $x \leq y$ if and only if $y' \leq x'$.

Finally

$$Cy+Ax+B = (-AC)(-A'(-x')-C'(-y')+B') = (-AC)(C'y'+A'y'+B').$$

We have $[\psi] \neq 0$ where $E' = \mathbb{D}^{\uparrow}(E) = (\mathbb{D}(E_2) \xrightarrow{\psi} \mathbb{D}(E_1))$, to see this we use that E σ-stable implies E' σ'-stable, which combined with $[\varphi] = 0$ would imply $E'_1 = \lambda_1\mathbb{D}^{\uparrow}(E) = 0$ or $E'_2 = \rho_2\mathbb{D}^{\uparrow}(E) = 0$ by corollary 4.5.6. Then $r'_1 = 0$ or $r'_2 = 0$ which contradicts $r'_2 \geq r'_1 > 0$. Because $-AC > 0$ and, $[\psi] \neq 0$, the proof is finished by applying lemma 4.9.5 to σ' and E'.

□

The previous series of lemmas now allows us to prove the support property for yet another type of stability conditions.

Lemma 4.9.21. *Let $\sigma = (Z, \mathcal{H})$ be a pre-stability condition obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2\rangle$ from stability conditions $\sigma_1 = (Z_1, \mathcal{H}_1), \sigma_2 = (Z_2, \mathcal{H}_2)$, such that there is $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ with $\sigma_1 = \sigma_2 g$ and $T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$.*

If $\mathcal{H}_1 = \mathcal{H}_2$, then $f(0) = 0$ and hence $D = 0, C > 0$.

Proof. Let $\sigma_2 = \sigma_\mu(T_2, f_2)$, then $H_2 = \mathcal{P}_\mu(f_2(0), f_2(1)]$. We have $\sigma_1 = \sigma_2 g = \sigma_\mu(T_2, f_2)(T, f) = \sigma_\mu(T_2T, f_2 \circ f)$. Therefore $H_1 = \mathcal{P}_\mu(f_2(f(0)), f_2(f(1))]$. From $\mathcal{H}_1 = \mathcal{H}_2$ we obtain $f_2(0) = f_2(f(0))$ which by the injectivity of f_2, provided by the invertibility of $M_2 = T_2^{-1}$ gives $f(0) = 0$. Hence $\exp(i\pi f(0)) \in \mathbb{R}$ and we obtain from

$$\exp(i\pi f(0)) = \frac{C + Di}{|C + Di|}. \tag{4.66}$$

that $D = 0$. Since $\exp(i\pi f(0)) = \exp(i\pi f(0)) = 1 > 0$ and $D = 0$ turns (4.66) into

$$\exp(i\pi f(0)) = \frac{C}{|C|}.$$

such that $C > 0$. □

Lemma 4.9.22. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Let $\sigma = (Z, \mathcal{H})$ be a pre-stability condition obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2\rangle$ from stability conditions $\sigma_1 = (Z_1, \mathcal{H}_1), \sigma_2 = (Z_2, \mathcal{H}_2)$, such that there is $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ with $\sigma_1 = \sigma_2 g$ and $T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$. Assume that $\mathcal{H}_1 = \mathcal{H}_2$, then σ satisfies the support property and therefore it is a Bridgeland stability condition.*

Proof. We have $f(0) = 0$ since $\mathcal{H}_1 = \mathcal{H}_2$. By lemma 4.9.21 we obtain $D = 0$, that is $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$. Because $C + iD = C \in \mathbb{R}_{>0}\exp(i\pi f(0)) = \mathbb{R}_{>0}$ we obtain $C > 0$. By lemma 4.9.21 $\mathcal{H}_1 = \mathcal{H}_2$ gives $D = 0$, that is $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$. Moreover, again by lemma 4.9.21 we have $A = 0$. Via $0 < \det(M) = -AC$ this implies $-A > 0$.

Recall that CP-gluing means that $\mathcal{H}$ is given by the equation

$$\mathcal{H} = \{X \in \mathcal{D}^{\uparrow} \mid \lambda_1(X) \in \mathcal{H}_1, \rho_2(X) \in \mathcal{H}_2\}$$

and $Z_1 = Z \circ i_1, Z_2 = Z \circ i_2$. From $\sigma_1 = \sigma_2 g$ we get $Z_1 = MZ_2$. For any $E \in \mathcal{D}^{\uparrow}$ we define

$$d_2 = -\Re(Z \circ i_2([\rho_2(E)])), d_1 = -\Re(Z \circ i_2([\lambda_1(E)])),$$
$$r_2 = \Im(Z \circ i_2([\rho_2(E)])) \text{ and } r_1 = \Im(Z \circ i_2([\lambda_1(E)])).$$

We now obtain $Zi_2\rho_2(E) = -d + ir$ and $Zi_1\lambda_1(E) = Ad_1 + Br_1 + iCr_1$ such that

$$Z(E) = Ad_1 + Br_1 - d_2 + i(Cr_1 + r_2).$$

From corollary 3.2.19 we obtain $i_1(\mathcal{H}) \subset \mathcal{H}$ as well as $i_2(\mathcal{H}) \subset \mathcal{H}$ such that $Zi_1\lambda_1(E), Zi_2\rho_2(E) \in \mathbb{H}$. In particular $r_2 \geq 0$ and $Cr_1 \geq 0$. Now $C > 0$ implies $r_1 \geq 0$ as well.

We want to show that σ fulfils the support property with regard to the quadratic form

$$Q(E) := -(Ad_1 + Br_1)(r_2 + \delta d_2) + Cr_1(r_2 - d_2)$$

for positive δ which, if B non-zero also fulfils $\delta \leq \frac{-AC}{B^2}$. Hence, we will subsequently check the conditions of definition 2.5.43.

Firstly we must prove that $Q\big|_{\ker(Z)}$ is negative definite. We have

$$\ker(Z) = \{v \mid Ad_1 + Br_1 = d_2, Cr_1 = -r_2\} \text{ for } v \in \mathcal{N}(\mathcal{D}^{\uparrow}) \otimes \mathbb{R} \cong \mathbb{R}^4$$

and hence $Q\big|_{\ker(Z)} = -d_2(r_2 + \delta d_2) - r_2(r_2 - d_2) = -r_2^2 - \delta d_2^2 \leq 0$. If indeed $-r_2^2 - \delta d_2^2 = 0$ then $r_2 = d_2 = 0$ since δ is positive and since $C \neq 0$ this implies $r_1 = 0$. Now $A \neq 0$ implies $d_1 = 0$ providing $v = 0$ such that $Q\big|_{\ker(Z)}$ is negative definite.

By lemma 2.5.46 we now have to prove $Q(E) \geq 0$ for $E = (E_1 \xrightarrow{\varphi} E_2) \in \mathcal{H}$ σ-stable. Note that it suffices to assume $E \in \mathcal{H}$, since $Q(E[n]) = Q(E)$ for all $n \in \mathbb{Z}$. We will conduct our proof by considering the following cases.

1. If $[\varphi] = 0 \in \mathrm{Hom}_{\mathcal{D}}(E_1, E_2)$, then $E \cong i_1(E_1) \oplus i_2(E_2)$ by corollary 4.5.6. The σ-stability of E now implies $E_1 = 0$ or $E_2 = 0$. Either implies $Q(E) = 0$.

2. If $r_1 = 0$, we have $\Im(Zi_1\lambda_1(E)) = Cr_1 = 0$ and hence $\phi_\sigma(i_1\lambda_1(E)) = 1$. Consider the torsion pair $\langle \mathcal{T}, \mathcal{F} \rangle = \langle \mathcal{P}_2(1), \mathcal{P}(0,1) \rangle = \mathcal{H}_2$ (example 4.7.5) where $\mathcal{P}_2$ is the slicing given by σ_2 on $\mathcal{D}$. By definition 4.7.1 pair we obtain a short exact sequence

$$0 \to T_2 \to E_2 \to F_2 \to 0$$

in $\mathcal{H}_2$ with $T_2 \in \mathcal{T} = \mathcal{P}(1)$ and $F_2 \in \mathcal{T} = \mathcal{P}(0,1)$. Since $i_2(\mathcal{H}) \in \mathcal{H}$, this provides the short exact sequence

$$0 \to i_2(T_2) \to i_2(E_2) \to i_2(F_2) \to 0$$

in $\mathcal{H}$. On the other hand, we have $i_1(E_1) \in i_1(\mathcal{H}_1) \in \mathcal{H}$ as well as $i_2(E_1) \in i_2(\mathcal{H}_2) \in \mathcal{H}$ and so the exact triangle

$$i_2(E_2) \to E \to i_1(E_1) \xrightarrow{+}$$

in $\mathcal{D}^{\uparrow}$ gives rise to the short exact sequence

$$0 \to i_2(E_2) \to E \to i_1(E_1) \to 0 \tag{4.67}$$

in $\mathcal{H}$. Hence, we obtain the chain

$$i_s(T_2) \subset i_2(E_2) \subset E$$

of subobjects in $\mathcal{H}$. The σ-stability of E now provides $\phi_\sigma(i_2(T_2)) < \phi_\sigma(E)$. By [21, Proposition 2.2(3)] we obtain $i_2(\mathcal{P}_2(t)) \subset \mathcal{P}(t)$ for all $t \in \mathbb{R}$. Therefore we obtain $\phi_\sigma(i_2(T_2)) = 1$ providing a contradiction to $\phi_\sigma(E) \in (0,1]$. Hence, we can disregard this case.

3. If $\Im(Zi_2\rho_2(E)) = r_2 = 0$, then $\phi_\sigma(i_2\rho_2(E)) = 1$. But (4.67) combined with the σ-stability of E implies

$$1 = \phi_\sigma(i_2(E_2)) < \phi_\sigma(E) \leq 1,$$

a contradiction. Therefore, we can disregard this case.

4. If $[\varphi] \neq 0, r_1 \neq 0, r_2 \neq 0$ we have $r_1 > 0$ and $r_2 > 0$ and define

$$x := \frac{d_1}{r_1}, y := \frac{d_2}{r_2} \text{ and}$$

$$Q(x,y) := \frac{1}{r_1 r_2} Q(E) = -Ax - Cy - B + C - \delta(Axy + By).$$

Since $r_1, r_2 > 0$ we need to show that $Q(x,y) \geq 0$ and use the key ingredients

$$Cy + Ax + B \leq 0 \text{ (from lemma 4.9.5 and 4.9.20)}, \tag{4.68}$$

$$x \leq y \text{ (since } [\varphi] \neq 0) \text{ and} \tag{4.69}$$

$$-A > 0, C > 0 \text{ (from before)}. \tag{4.70}$$

(a) If $y \geq 0$ then we use (4.70) to obtain $0 \leq Cy$ and obtain $0 \leq Cy \leq -Ax - B$ via (4.68). Since $y \geq 0$ was assumed we obtain $0 \leq -(Axy + By)$ such that $-\delta(Axy + By) \geq 0$ since $\delta > 0$. From (4.68) and (4.70) we obtain $-Ax - Cy - B + C > -Ax - Cy - B \geq 0$ and $Q(x,y) > 0$ follows.

(b) If $y < 0$ and $Ax + B \geq 0$ we obtain $Axy + By \leq 0$ and hence $-\delta(Axy + By) \geq 0$ such that we can conclude as in the previous case.

(c) If $y < 0$ and $Ax + B < 0$ then we get from (4.69) that $x \leq y < 0$. Using (4.70) this gives $-Ax \leq -Ay < 0$. Hence the assumption $Ax + B < 0$ implies $B < Ax + B < 0$ and $B < -Ax \leq -Ay < 0$. Multiplying these inequalities of negative numbers, we obtain

$$B^2 > -Ay(Ax + B).$$

Since $B < 0$ we obtain $B^2 > 0$ and using $C > 0$ we now have

$$C > \frac{-AC}{B^2}(Axy + By).$$

Because $y < 0$ and $Ax + B < 0$ was assumed in this case, we have $Axy + By > 0$. Moreover, $-A > 0$ and $C > 0$ as well as $B \neq 0$ imply $-\frac{AC}{B^2} > 0$. For each δ that satisfies $0 < \delta < -\frac{AC}{B^2}$ we get

$$C > \delta(Axy + By).$$

From (4.68) we finally obtain

$$-Ax - Cy - B + C \geq C > \delta(Axy + By)$$

and hence $Q(x,y) = -Ax - Cy - B + C - \delta(Axy + By) > 0$.

This concludes all cases and – hence – finishes the proof.

□

We will now use the discriminant as a tool to separate different cases and – therefore – have

Notation 4.9.23. For a Matrix $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, denote its trace $w+z$ by $\mathrm{Tr}(A)$ (such that $\mathrm{Discr}(A) = \mathrm{Tr}(A)^2 - 4\det(A)$).

Recall that for a Matrix $M = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$, this implies that we have $\mathrm{Discr}(M) = (A+C)^2 - 4BD$.

We will now investigate the support property for CP-glued pre-stability conditions with negative discriminant.

Lemma 4.9.24. *Let $\sigma = (Z, \mathcal{H}) = (T, f)$ be a pre-stability condition obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from stability conditions σ_1, σ_2 such that* $\mathrm{Discr}(M) < 0$. *For $h \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, we have that σh is a pre-stability condition obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from stability conditions $\sigma_1 h, \sigma_2 h$.*

Proof. Without loss of generality we can assume $h = (K_r, f_r)$, since the feature of two hearts of t-structures on $\mathcal{D}$ to fulfil CP-gluing condition is invariant under anything but rotation. Let $\sigma_i' = \sigma_i h = (Z_i', \mathcal{H}_i')$ for $i \in \{1, 2\}$. Let $\sigma_1 = \sigma_2(T, f)$ and $\mathcal{P}_i$ the slicing of σ_i. Then $\mathcal{P}_1(t) = \mathcal{P}_2(f(t))$. If we let $\mathcal{P}_i'$ be the slicing of σ_i' then $\mathcal{P}_1'(t) = \mathcal{P}_1(f_r(t)) = \mathcal{P}_2(f(f_r(t)))$ and $\mathcal{P}_2'(t) = \mathcal{P}_2(f_r(t))$. Letting $t = 0$ and $t = 1$ respectively and $f_r(0) = r$ we obtain

$$\mathcal{H}_1' = \mathcal{P}_2(f(r), f(r) + 1] \text{ and } \mathcal{H}_2' = \mathcal{P}_2(r, r+1].$$

Since the restrictions of f and T to S^1 agree and via $\mathrm{Discr}(M) < 0$ the eigenvalues of M are not in $\mathbb{R}$, there is no (eigenvalue) $x' \in \mathbb{R}$ with $f(x') = x'$. Since $f(0) \geq 0$ we therefore have $f(r) > r$ for any $r \in \mathbb{R}$. Applying corollary 3.2.31 combined with lemma 3.1.16 proves that σh is a pre-stability condition obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from stability conditions $\sigma_1 h, \sigma_2 h$. □

Lemma 4.9.25. *Let $\sigma = (Z, \mathcal{H})$ be a pre-stability condition obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from stability conditions σ_1, σ_2 and assume that* $0 \leq f(0) < 1$ *such that* $\mathrm{Discr}(M) < 0$. *If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ is a σ-semistable object with $\Im(Z_2(E_1)) < 0$. Then $E_1 \in \mathcal{H}_2[1]$.*

Proof. By the definition of a heart obtained by CP-gluing (see lemma 3.1.5), $E_2 \in \mathcal{H}_2$, since $E \in \mathcal{H}$ was assumed. By lemma 4.9.24, we have that for each $h \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, the object σh is a CP-glued pre-stability condition. As

$\sigma_1 = \sigma_2 g$, if we apply g^{-1}, then $\sigma' = \sigma g^{-1}$ is glued from stability conditions $\sigma_2, \sigma_2 g^{-1} = (Z', \mathcal{H}')$ with respect to $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Since the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action does not change the fact that E is σ'-semistable, this gives $E \in \mathcal{H}'[n]$ for some n. By lemma 3.1.5, we obtain $E_1 \in \mathcal{H}_2[n]$, because $\mathcal{H}_2$ is equal to $\mathcal{P}_1(f^{-1}(0), f^{-1}(1)] = \mathcal{P}_2(0,1]$, where $\mathcal{P}_i$ are the slicings of σ_i, for $i = 1, 2$. Since $0 \leq f(0) < 1$, we obtain that $\mathcal{H}_2 \subset \mathcal{P}_1(-1, 1]$, which implies $n = 0, 1$. We assumed $\Im(Z_2(E_1)) < 0$ and therefore we get that $E_1 \in \mathcal{H}_2[1]$. □

Lemma 4.9.26. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Let $\sigma = (Z, \mathcal{H})$, obtained from CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be a pre-stability condition, where $\sigma_1 = (Z_1, \mathcal{H}_1)$ and $\sigma_2 = (Z_2, \mathcal{H}_2)$, such that there is $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ with $\sigma_1 = \sigma_2 g$ where (T, f) satisfies that $M = T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$ with*

$$\mathrm{Discr}(M) < 0 \text{ and } 0 < f(0) < 1.$$

Then σ is a Bridgeland stability condition.

Proof. By 2.5.44 we need to prove that the support property is satisfied. Firstly, we have $Z_1 = Z \circ i_1$ and $Z_2 = Z \circ i_2$ such that $Z_1 = M Z_2$ and will for an $E = (E_1 \xrightarrow{\varphi} E_2) \in \mathcal{D}^\uparrow$ use the following notation throughout the proof

$$\begin{aligned} d_2 &= -\Re(Z \circ i_2([\rho_2(E)])), d_1 = -\Re(Z \circ i_2([\lambda_1(E)])), \\ r_2 &= \Im(Z \circ i_2([\rho_2(E)])) \text{ and } r_1 = \Im(Z \circ i_2([\lambda_1(E)])). \end{aligned}$$

We have

$$\begin{aligned} Z(v) &= Z i_1 \lambda_1(v) + Z i_2 \rho_2(v) = M Z i_2 \lambda_1(v) + Z i_2 \rho_2(v) \\ &= M \begin{pmatrix} -d_1 \\ r_1 \end{pmatrix} + \begin{pmatrix} -d_2 \\ r_2 \end{pmatrix} = A d_1 + B r_1 - d_2 + i(D d_1 + C r_1 + r_2) \end{aligned}$$

such that $\Re(Z_1 i_1 \lambda_1(v)) = A d_1 + B r_1$ and $\Im(Z_1 i_1 \lambda_1(v)) = D d_1 + C r_1$ for $v \in \mathcal{N}(\mathcal{D}^\uparrow) \otimes \mathbb{R} \cong \mathbb{R}^4$.

We will investigate the quadratic form $Q(v) := d_1 r_2 - r_1 d_2$ and start with a number of facts that will allow us to prove that the support property holds, namely

$$\begin{aligned} 0 < f(0) < 1 \text{ and } C + iD \in \exp(i\pi f(0)) \text{ implies} \\ \Im(C + iD) > 0, \text{ hence } D > 0, \end{aligned} \tag{4.71}$$

$$\begin{aligned} \mathrm{Discr}(M) = (A + C)^2 - 4BD < 0 \text{ implies } 0 \leq (A + C)^2 < 4BD \\ \text{such that (4.71) provides } B > 0, \end{aligned} \tag{4.72}$$

$$\begin{aligned} q_M(d,r) := Dd^2 + (A+C)dr + Br^2 \text{ has discriminante equal to} \\ \mathrm{Discr}(M) = (A+C)^2 - 4BD < 0 \\ \text{such that (4.71) and (4.72) provide} \\ q_M(d,r) > 0 \text{ for all } (d,r) \neq (0,0), \end{aligned} \tag{4.73}$$

$$\begin{aligned} E \in \mathcal{H} \text{ implies } \lambda_1(E) \in \mathcal{H}_1, \rho_2(E) \in \mathcal{H}_2 \text{ by 3.1.5} \\ \text{moreover, } i_1(\mathcal{H}_1) \in \mathcal{H}, i_2(\mathcal{H}_2) \in \mathcal{H} \\ \text{by corollary 3.2.19, such that all objects in the exact triangle} \\ i_2\rho_2(E) \to E \to i_1\lambda_1(E) \xrightarrow{+} \\ \text{are in } \mathcal{H} \text{ and it therefore provides the short exact sequence} \\ 0 \to i_2\rho_2(E) \to E \to i_1\lambda_1(E) \to 0 \in \mathcal{H}, \end{aligned} \tag{4.74}$$

$$\begin{aligned} \text{if } E \in \mathcal{H} \text{ is } \sigma\text{-stable then (4.74) implies} \\ 0 < \phi_\sigma(i_2\rho_2(E)) < \phi_\sigma(E) < \phi_\sigma(i_1\lambda_1(E)) \leq 1, \end{aligned} \tag{4.75}$$

$$\begin{aligned} \text{if } E \in \mathcal{H} \text{ then arguing as in (4.74),} \\ Z(E), Z(i_1\lambda_1(E)), Z(i_2\rho_2(E)) \in \mathbb{H}, \text{ therefore} \\ r_2 = \Im(Z(i_2\rho_2(E))) \geq 0 \text{ and if } r_2 = 0 \\ \text{then } -d_2 = \Im(Z(i_2\rho_2(E))) < 0 \\ \text{analogously } Dd_1 + Cr_1 \geq 0 \text{ and if} \\ Dd_1 + Cr_1 = 0 \text{ then } Ad_1 + Br_1 < 0 \end{aligned} \tag{4.76}$$

and

$$\begin{aligned} E \in \mathcal{H}, r_1 > 0, \text{ if } Dd_1 + Cr_1 = 0 \text{ then, by (4.77),} \\ Ad_1 + Br_1 < 0 \text{ and} \\ q_M(d_1, r_1) = d_1(Dd_1 + Cr_1) + r_1(Ad_1 + Br_1) < 0 \\ \text{contradicting (4.73), such that } Dd_1 + Cr_1 > 0. \end{aligned} \tag{4.77}$$

We will now prove that $Q\big|_{\ker(Z)}$ is negative definite. Let $v \in \ker(Z)$, this is the case if and only if $\Re(Z(v)) = \Im(Z(v)) = 0$, which is equivalent to $d_2 = Ad_1 + Br_1$ and $r_2 = -(Dd_1 + Cr_1)$. Hence $Q(v) = d_1r_2 - r_1d_2 = -d_1(Dd_1 + Cr_1) - r_1(Ad_1 + Br_1) = -q_M(d_1, r_1) < 0$ by (4.73) (if $r_1 = d_1 = 0$ then $r_2 = d_2 = 0$ and hence $v = 0$). Hence $Q\big|_{\ker(Z)}$ is negative definite.

We now have to prove that $E \in \mathcal{H}$ σ-stable then $Q(E) \geq 0$ and distinguish the following cases.

1. If $r_2 = 0$ then $\phi_\sigma(i_2\rho_2(E)) = 1$, contradicting (4.75) and we can disregard this case.

2. If $r_1 = 0$ then (4.76) provides $Dd_1 + Cr_1 \geq 0$ such that $Dd_1 \geq 0$ and therefore, by (4.71), $d_1 \geq 0$. By (4.76) we have $r_2 \geq 0$ such that $Q(E) = d_1r_2 - r_1d_2 = d_1r_2 \geq 0$

3. If $r_2 > 0, r_1 > 0$ then (4.75) provides $\phi_\sigma i_2\rho_2(E) \leq \phi_\sigma i_1\lambda_1(E)$ which implies $\mu_\sigma i_2\rho_2(E) \leq \mu_\sigma i_1\lambda_1(E)$, in other words

$$\frac{-\Re(Zi_2\rho_2(E))}{\Im(Zi_2\rho_2(E))} \leq \frac{-\Re(Zi_1\lambda_1(E))}{\Im(Zi_1\lambda_1(E))}$$

in other words

$$\frac{d_2}{r_2} \leq \frac{-Ad_1 - Br_1}{Dd_1 + Cr_1}.$$

Now, (4.73) provides $q_M(d_1, r_1) > 0$ such that $d_1(Dd_1+Cr_1)+r_1(Ad_1+Br_1) > 0$ and hence

$$\frac{-Ad_1 - Br_1}{Dd_1 + Cr_1} < \frac{d_1}{r_1}$$

since $r_1 > 0$ by assumption and, by (4.77), $Dd_1 + Cr_1 > 0$ as well. Hence,

$$\frac{d_2}{r_2} < \frac{d_1}{r_1}$$

and since $r_1, r_2 > 0$ this implies $Q(E) = d_1r_2 - r_1d_2 > 0$.

If $r_2 > 0, r_1 < 0$, we obtain by lemma 4.9.25, which is applicable because $r_1 < 0$, that $\lambda_1(E) \in \mathcal{H}_2[1]$. Since $\rho_2(E) \in \mathcal{H}_2$ we obtain $[\varphi] \in \mathrm{Hom}^{<0}_{\mathcal{D}^\uparrow}(\mathcal{H}_2, \mathcal{H}_2) = 0$ which implies $E \cong i_1(E_1) \oplus i_2(E_2)$ by corollary 4.5.6. Since E stable either $E_1 = 0$ or $E_2 = 0$ which contradicts $r_1, r_2 > 0$ such that we can disregard this case.

□

Lemma 4.9.27. *Let* $\sigma \in \Theta_1$ *normalised with* $\mathrm{Discr}(M) \geq 0$ *for the associated matrix* M *and negative eigenvalues. Then there is an* $h \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ *such that* σh *is obtained from CP-gluing stability conditions* $\sigma_1 h, \sigma_2 h$ *via the semiorthogonal decomposition* $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ *with* $\sigma_i h = (Z'_i, \mathcal{H}'_i), i \in \{1, 2\}$ *and*

$$\mathrm{Hom}^{\leq 1}(i_1(\mathcal{H}'_1), i_2(\mathcal{H}'_2)) = 0.$$

Proof. The existence of real eigenvalues of T is provided from that of T^{-1} since $\mathrm{Discr}(M) \geq 0$. Let $\beta < 0$ be an eigenvalue of T with corresponding eigenvector v, in other words $Tv = \beta v$. Written in polar coordinates we have $v = m(\cos(\phi), \sin(\phi))$ with $\phi \in (-\pi, \pi]$ and $m \in \mathbb{R}_{>0}$. However, by linearity we can, without loss of generality assume $m = 1$.

We investigate σh where $h = (K_\phi, f_\phi) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. Firstly we consider $gh = (TK_\phi, f \circ f_\phi)$. By the correspondence between $f \circ f_\phi$ and TK_ϕ over S^1 it is sufficient for the computation of $f \circ f_\phi$ to compute $TK_\phi v_0$ where $v_0 = (1, 0)$. We have $(TK_\phi)v_0 = Tv = \beta v$. Therefore, if we study the induced map $f : S^1 \to S^1$, where $S^1 = (-1, 1]$, as $\beta < 0$, we show $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1$. We distinguish between two cases

1. If $-1 < \phi/\pi \leq 0$ we have $f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1$ on S^1, such that $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1 + 2k, k \in \mathbb{Z}$. We use that f is an increasing function to see that $-1 < \phi/\pi \leq 0$ implies
$$-1 < f(-1) < \phi/\pi + 1 + 2k \leq f(0) < 1$$
which forces $k = 0$.

2. If $0 < \phi/\pi \leq 1$ we have $f_\phi(0) = f(\phi/\pi) = \phi/\pi - 1$ on S^1, such that $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi - 1 + 2k, k \in \mathbb{Z}$. We use that f is an increasing function to see that $0 < \phi/\pi \leq 1$ implies
$$0 < f(0) < \phi/\pi - 1 + 2k \leq f(1) < 2$$
which forces $k = 1$.

In essence, the consideration of both cases reveals $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1$. Now, let $\mathcal{P}_2'$ be the slicing of σ_2, then

$$\mathcal{H}_1' = \mathcal{P}_2(f \circ f_\phi(0), f \circ f_\phi(1)] = \mathcal{P}(\phi/\pi + 1, \phi/\pi + 2]$$
$$\text{and } \mathcal{H}_2' = \mathcal{P}_2(f_\phi(0), f_\phi(1)] = \mathcal{P}(\phi/\pi, \phi/\pi + 1]$$

Hence,
$$\mathrm{Hom}^{\leq 1}(i_1(\mathcal{H}_1'), i_2(\mathcal{H}_2')) = 0$$
holds and σh is indeed obtained from CP-gluing by corollary 3.2.36. □

We are finally ready to give the following important proposition.

Proposition 4.9.28. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is s smooth projective curve. If*
$$\sigma \in \Theta_1 \cup \Theta_2 \cup \Theta_3$$
then σ satisfies the support property and is therefore a Bridgeland stability condition.

Proof. We will prove the statement for $i = 1$, where $i = 2$ and $i = 3$ follow by applying the Serre functor. This is seen from

$$S_{\mathcal{D}^{\uparrow}}(\Theta_1) \in \Theta_2, S_{\mathcal{D}^{\uparrow}}(\Theta_2) \in \Theta_3, S_{\mathcal{D}^{\uparrow}}(\Theta_1) \in \Theta_1$$

(see (4.57)), and lemma 2.5.55.

Let σg be normalised and note that $f(0) = 0$ and $\mathrm{Discr}(M) < 0$ is not possible since $f(0) = 0$ forces $D = 0$ such that $\mathrm{Discr}(M) = (A+C)^2 - 4BD = (A + C)^2 \geq 0$. The proof falls into the following cases:

1. If $f(0) \geq 1$, we obtain the result from lemma 4.9.1.

2. If $0 \leq f(0) < 1$ and $\mathrm{Discr}(M) \geq 0$, we have the existence of real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. If $\lambda_1, \lambda_2 > 0$, then, by lemma 4.8.24 we can find $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that σg is obtained via CP-gluing of two copies of the same heart – therefore $D = 0$. By lemma 4.9.22 and the fact that the support property is stable under the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action, we have that σ fulfils the support property. If $\lambda_1, \lambda_2 < 0$, then lemma 4.9.27 there is $h \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, such that σh is obtained by CP-gluing from $\sigma_1 h, \sigma_2 h$ via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ with $\sigma_i h = (Z'_i, \mathcal{H}'_i)$ and $\mathrm{Hom}^{\leq 1}(i_1\mathcal{H}'_1, i_2\mathcal{H}'_2) = 0$. We hence obtain the result from lemma 4.9.1.

3. If $0 < f(0) < 1$ and $\mathrm{Discr}(M) < 0$, the result is obtained from lemma 4.9.26.

□

We will now study the support property for pre-stability conditions in Γ. We aim at proving proposition 4.9.36 broken down into a Series of lemmas, which will – in turn – provide a key ingredient for establishing theorem 4.9.37.

Lemma 4.9.29. *Let $\sigma = (Z, \mathcal{H}(C_1, D_1))$ be a pre-stability condition in Γ. If $F \in \mathcal{A} = \mathrm{Coh}(C)$, C an elliptic curve, is μ-semistable, then $i_1(F)$, $i_2(F)$ and $\Delta(F)$ are σ-semistable.*

Proof. We consider a JHF of F with respect to μ. All the μ-stable factors A_i, for $i = 0, \dots, n$, have the same slope $\mu(F)$ and by lemma 4.8.35, we additionally obtain that $i_2(A_i)$ is σ-stable and, by lemma 4.7.39 in $\mathcal{H}$, for all $i \in \{1, \dots, n\}$. As $Z\big|_{i_2(\mathcal{A})} = Z_\mu$, we obtain from the equality of the slopes that $\phi(i_2(A_i)) = \phi(i_2(F)) = \lambda$, with $\lambda \in \mathbb{R}$. Since the category $\mathcal{P}(\lambda)$ is closed under extensions, we obtain that $i_2(F)$ is σ-semistable. Now since F is μ-semistable, we can use lemma 4.7.39 to see that $i_1(F)$ is in $\mathcal{H}$ or in $\mathcal{H}[1]$ and $\Delta(F)$ is in $\mathcal{H}$ or in $\mathcal{H}[1]$. Since the slope determines the phase up to addition of $2n, n \in \mathbb{Z}$, the same conclusion can be drawn for $i_1(F)$ and $\Delta(F)$. □

Lemma 4.9.30. *Let $\mathcal{A} = \mathrm{Coh}(C)$, C an elliptic curve and $\mathcal{H}$ be the heart of a normalised pre-stability condition in Γ. We have $\mathcal{H} \cap \mathcal{D}_2 = i_2(\mathcal{A})$.*

Proof. Let $E \in \mathcal{H}, E = E_1 \xrightarrow{\varphi} E_2$. Using the description of E's cohomology provided by lemma 4.8.5, assuming $E \in \mathcal{D}_2$ at the same time, we obtain $H^0(E_1) = H^1(E_1) = 0$. By considering the long exact cohomology-sequence induced by the exact triangle $E_1 \xrightarrow{\varphi} E_2 \to \mathrm{Cone}(\varphi) \xrightarrow{+}$, in $\mathcal{D}$, we obtain $H^{-1}(\mathrm{Cone}(\varphi)) = H^1(E_2) = 0$, which implies $E \in i_2(\mathcal{A})$.

On the other hand, lemma 4.7.39 provides $i_2(\mathcal{A}) \subset \mathcal{H}$. Since obviously $i_2(\mathcal{A}) \subset \mathcal{D}_2$, the proof is finished. □

Lemma 4.9.31. *Let $\mathcal{A} = \mathrm{Coh}(C)$ and C be an elliptic curve. If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}$ is σ-semistable and $\sigma = (Z, \mathcal{H})$ where σ is a normalised pre-stability condition in Γ then $E \in \mathcal{H}_{12}$ or $E \in \mathcal{H}_{12}[-1]$ or $E \in \mathcal{H}_{23}[-1]$ or $E \in \mathcal{H}_{31}$.*

Proof. By lemma 4.8.35, we have that $i_1(\mathbb{C}(x)), i_2(\mathbb{C}(x))$ and $\Delta(\mathbb{C}(x))$ are σ-stable. There are elements $g_1, g_2 \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ with $\delta_i = \sigma g_i = (W_i, \mathcal{B}_i)$ for $i = 1, 2$, such that δ_1 satisfies that $i_1(\mathbb{C}(x))$ is δ_1-stable of phase one, $i_2(\mathbb{C}(x)), \Delta(\mathbb{C}(x))$ are δ_1-stable and δ_2 satisfies that $\Delta(\mathbb{C}(x))$ is δ_2-stable of phase one and $i_2(\mathbb{C}(x)), i_1(\mathbb{C}(x))$ are δ_2-stable. We can apply lemma 4.8.3 to describes the cohomology of objects in the respective hearts.

If

- $E \in \mathcal{H}$, then the cohomology of E_1, E_2 and $\mathbb{K}(E)[1]$ vanishes except for
$$H^0(E_1), H^1(E_1), H^0(E_2), H^1(E_2), H^{-1}(\mathbb{K}(E)[1]) \text{ and } H^0(\mathbb{K}(E)[1]),$$
- $E \in \mathcal{B}_1$, then the cohomology of E_1, E_2 and $\mathbb{K}(E)[1]$ vanishes except for
$$H^{-1}(E_1), H^0(E_1), H^{-1}(E_2), H^0(E_2), H^{-1}(\mathbb{K}(E)[1]) \text{ and } H^0(\mathbb{K}(E)[1]),$$
- $E \in \mathcal{B}_2$, then all its cohomology vanishes except for
$$H^0(E_1), H^1(E_1), H^{-1}(E_2), H^0(E_2), H^{-1}(\mathbb{K}(E)[1]) \text{ and } H^0(\mathbb{K}(E)[1]).$$

Additionally we see from the exact triangle $E_1 \to E_2 \to \mathbb{K}(E)[1] \xrightarrow{+}$ that $H^i = 0$ for all $i \in \mathbb{Z}$ for two of the objects, then $H^i = 0$ for all $i \in \mathbb{Z}$ for the third one as well.

Since E is δ_i-semistable we have that $E \in \mathcal{B}_i[n]$, for $i = 1, 2$ and $n \in \mathbb{Z}$, where the only possible cases are $n = 1, 0, -1, -2$. We study the non-trivial cases.

1. if $E \in \mathcal{H}\cap\mathcal{B}_1[-2]$ then the only non-vanishing cohomology is $H^1(E_1) = H^0(E_1[1])$ and $H^1(E_2) = H^0(E_2[1])$ such that $E_1[1] \in \mathcal{A}$ and $E_2[1] \in \mathcal{A}$ and hence $E \in \mathcal{H}_{12}[-1]$
2. if $E \in \mathcal{H} \cap \mathcal{B}_1[-1] \cap \mathcal{B}_2[-1]$ then the only non-vanishing cohomology is $H^0(E_1[1]), H^0(\mathbb{K}(E)[1])$, such that $E \in \mathcal{H}_{23}[-1]$
3. if $E \in \mathcal{H} \cap \mathcal{B}_1[-1] \cap \mathcal{B}_2$ then, by similar arguments as in the previous case $E \in \mathcal{H}_{13}$.
4. If $E \in \mathcal{H} \cap \mathcal{B}_1$ then $H^i(E_1)$ and $H^i(E_2)$ vanishes if $i \neq 0$ such that $E_1, E_2 \in \mathcal{A}$ and therefore $E \in \mathcal{H}_{12}$.

□

Lemma 4.9.32. *If $\mathcal{D}$ has a Serre functor, then*

$$S_{\mathcal{D}^\uparrow}(\mathcal{H}_{12}) = \mathcal{H}_{23}[1], S_{\mathcal{D}^\uparrow}(\mathcal{H}_{23}) = \mathcal{H}_{31}[2], S_{\mathcal{D}^\uparrow}(\mathcal{H}_{31}) = \mathcal{H}_{12}[1]$$

Proof. We obtain this from combining (4.35) with definition 4.8.1. □

Lemma 4.9.33. *Let $\mathcal{A} = \mathrm{Coh}(C)$ and C be an elliptic curve and $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}_{ij}$ be σ-semistable in $\mathcal{H}$, where (ij) is an element of the set of ordered pairs $\{(12),(31),(23)\}$ and $\sigma = (Z,\mathcal{H})$ and σ a pre-stability condition in Γ. Then we have $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[2]) = 0$.*

Proof. Assume that $E \in \mathcal{H}_{12} = \mathcal{A}^\uparrow$. We have $E \in \mathcal{A}^\uparrow \cap \mathcal{H} = \mathcal{F}$, where $\mathcal{H} = \langle \mathcal{F}, \mathcal{T}[-1]\rangle$ as in lemma 4.8.5. Additionally,

$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[2]) = \mathrm{Hom}_{\mathcal{D}^\uparrow}(E[2], S_{\mathcal{D}^\uparrow}(E))^* = \mathrm{Hom}(E[1], E_2 \to \mathrm{Cone}(\varphi))^*$$

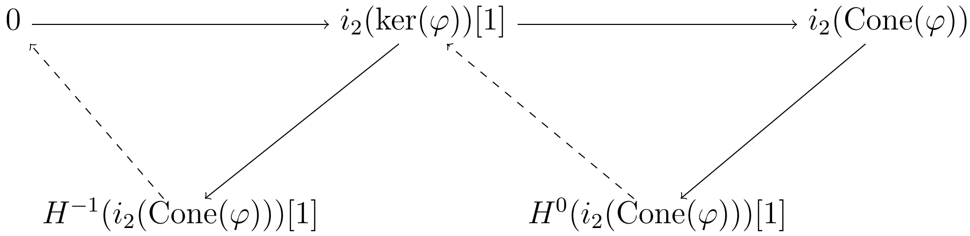
and it therefore suffices to prove that $\mathrm{Hom}(E[1], E_2 \to \mathrm{Cone}(\varphi)) = 0$.

Consider the exact triangle $i_2(\mathrm{Cone}(\varphi)) \to S_{\mathcal{D}^\uparrow}(E)[-1] \to i_1(E_2) \xrightarrow{+}$ which, via applying the Hom-functor, induces a long exact sequence which implies that it is enough to prove that $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E[1], i_1(E_2)) = 0$ and that $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E[1], i_2(\mathrm{Cone}(\varphi))) = 0$. We obtain the first vanishing from the fact that we have $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E[1], i_1(E_2)) = \mathrm{Hom}_{D^b(C)}(E_1[1], E_2) = 0$ provided by $E_1, E_2 \in \mathcal{A}$ combined with the fact that $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}$. We must therefore prove that $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E[1], i_2(\mathrm{Cone}(\varphi))) = 0$.

1. If $\ker(\varphi) = 0$, we obtain that $\mathrm{Cone}(\varphi) = \mathrm{coker}(\varphi)$ and via $\mathbb{K}[1] \dashv i_2$ this implies

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^\uparrow}(E[1], i_2(\mathrm{Cone}(\varphi))) &= \mathrm{Hom}_{\mathcal{D}^\uparrow}(E[1], i_2(\mathrm{coker}(\varphi))) \\ &= \mathrm{Hom}_{\mathcal{D}^\uparrow}(\mathrm{coker}(\varphi)[1], \mathrm{coker}(\varphi)) = 0.\end{aligned}$$

2. If $\ker(\varphi) \neq 0$, we will obtain the Hom-vanishing by showing that $\phi(E)+1 > \phi^+(i_2(\mathrm{Cone}(\varphi)))$, to which end we now compute $\phi^+(i_2(\mathrm{Cone}(\varphi)))$. By lemma 4.9.30, we have that $\mathcal{H} \cap \mathcal{D}_2 = i_2(\mathcal{A})$ and hence $i_2(\mathrm{Cone}(\varphi)) \notin \mathcal{H}$. Therefore we need to consider its filtration in the t-structure induced by $\mathcal{H}$, given by



with $H^{-1}(i_2(\mathrm{Cone}(\varphi)))[1] \in i_2(\mathcal{A})[1] \subset \mathcal{H}[1], H^0(i_2(\mathrm{Cone}(\varphi)))[1] \in i_2(\mathcal{A}) \subset \mathcal{H}$. By definition, $\phi^+(i_2(\mathrm{Cone}(\varphi))) = \phi^+(i_2(\ker(\varphi))) + 1$. We examine the HN-filtration $0 \subset H_1 \cdots \subset H_{n-1} \subset H_n = i_2(\ker(\varphi))$ of $i_2(\ker(\varphi))$ in $\mathcal{H}$ with respect to σ. Lemma 4.9.29 provides that if $F \in \mathcal{A}$ is μ-semistable then $i_2(F)$ is σ-semistable. Therefore, we consider the HN-filtration of $\ker(\varphi)$ with respect to μ. Since $i_2(\mathcal{A}) \subset \mathcal{H}$ and $Z\big|_{i_2(\mathcal{A})} = Z_\mu$, we obtain a filtration that fulfils all the conditions of a HN-filtration of $i_2(\ker(\varphi))$ in $\mathcal{H}$ with respect to σ. By the uniqueness of the HN-filtration, we deduce that $H_i \in i_2(\mathcal{A})$, for all $i = 0, \dots, n$. Moreover, we have that $H_1 \neq 0$ is σ-semistable and $\phi(H_1) = \phi^+(i_2(\ker(\varphi))) = \phi^+(i_2(\mathrm{Cone}(\varphi))) - 1$.

   Let $H_1 = i_2(F_1)$, with $F_1 \in \mathcal{A}$. By definition we have $F_1 \subset \ker(\varphi)$. As $\ker(\varphi) \to 0$ is a subobject of E in $\mathcal{A}^\uparrow$ and $\mathcal{F}$ is closed under subobjects we obtain $F_1 \to 0 \in \mathcal{F} \subset \mathcal{H}$. Since F_1 is μ-semistable, we use lemma 4.9.29 to see that $i_1(F_1)$ too is σ-semistable. Moreover, we have a non-zero morphism $i_1(F_1) \to E$. As both $i_1(F_1)$ and E are σ-semistable this implies $\phi_\sigma(i_1(F_1)) \leq \phi_\sigma(E)$.

   Let $d = \deg(F_1)$ and $r = \mathrm{rank}(F_1)$. By the definition of $\mathcal{F}$, using lemma 4.8.16, we get that $\ker(\varphi)$ and therefore F_1 is torsion-free, implying $r > 0$. As $i_1(F_1) \in \mathcal{F}$, we also have that $Cr + Dd \geq 0$. Moreover, $D < 0$ and $\mathrm{Discr}(M) < 0$ implies that $\phi(H_1) < \phi(i_1(F_1))$ holds true. Therefore, we obtain $\phi^+(i_2(\mathrm{Cone}(\varphi))) - 1 = \phi(i_2(F_1)) < \phi(E)$, which is what we wanted to prove.

We obtain the statement when $E \in \mathcal{H}_{31}$ or $E \in \mathcal{H}_{23}$ by applying the Serre functor – lemma 4.9.32 provides, in particular,

$$S_{\mathcal{D}^\uparrow}(\mathcal{H}_{23}) \subset \mathcal{H}_{31}[2], S_{\mathcal{D}^\uparrow}(\mathcal{H}_{31}) \subset \mathcal{H}_{12}[1]$$

such that $E \in \mathcal{H}_{23}$ σ-semistable provides

$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[2]) = \mathrm{Hom}_{\mathcal{D}^\uparrow}(S^2_{\mathcal{D}^\uparrow}(E), S^2_{\mathcal{D}^\uparrow}(E[2]))$$

by the previous and since E is $S^2_{\mathcal{D}^\uparrow}(\sigma)$-semistable by lemma 2.5.54 and $S^2_{\mathcal{D}^\uparrow}(\sigma) \in \Gamma$ by 4.8.38, we have $\mathrm{Hom}_{\mathcal{D}^\uparrow}(S^2_{\mathcal{D}^\uparrow}(E), S^2_{\mathcal{D}^\uparrow}(E[2])) = 0$. Similarly, $E \in \mathcal{H}_{31}$ σ-semistable provides

$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[2]) = \mathrm{Hom}_{\mathcal{D}^\uparrow}(S_{\mathcal{D}^\uparrow}(E), S_{\mathcal{D}^\uparrow}(E[2])) = 0.$$

□

Corollary 4.9.34. *Let $\mathcal{A} = \mathrm{Coh}(C)$ and C be an elliptic curve. If E is σ-stable where σ is a pre-stability condition in Γ, then $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[2]) = 0$.*

Proof. Because of lemma 4.9.31 and since stable implies semistable, lemma 4.9.33 applies – the shift does not change the vanishing of the Hom. □

Lemma 4.9.35. *Let $\mathcal{A} = \mathrm{Coh}(C)$ and C be an elliptic curve. If $E = (E_1 \xrightarrow{\varphi} E_2) \in \mathcal{D}^\uparrow$ is a σ-stable object, where σ is a pre-stability condition in Γ, then $-\chi(E, E) = d_2 r_1 - d_1 r_2 \geq 0$.*

Proof. The proof falls into two cases:

1. $[\varphi] = 0$. This implies that either $E_1 = 0$ or $E_2 = 0$, as otherwise, by corollary 4.5.6, E would be a direct sum of non-zero $i_1(E_1)$ and $i_2(E_2)$ and we would obtain a contradiction to the assumption that E is σ-stable and therefore cannot be written as a direct sum of non-zero subobjects. It follows that $-\chi(E, E) = 0$.

2. $[\varphi] \neq 0$. As $\chi(E[n], E[n]) = \chi(E, E)$ for all $n \in \mathbb{Z}$ we may assume $E \in \mathcal{H}$ and since $\mathcal{H}$ is the heart of a bounded t-structure, we have that $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[n]) = 0$ for all $n < 0$. By corollary 4.9.34, we have that $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[2]) = 0$ and so $\mathcal{A}^\uparrow$ has homological dimension 2, which implies that $\mathcal{H}_{23}[-1]$ and $\mathcal{H}_{31}$ also have homological dimension 2. Therefore, it follows that $\mathrm{Hom}_{\mathcal{D}^\uparrow}(E, E[n]) = 0$ for $n \geq 2$. As a consequence, we obtain $-\chi(E, E) = -\hom_{\mathcal{D}^\uparrow}(E, E) + \hom_{\mathcal{D}^\uparrow}(E, E[1])$.

   Since E was assumed to be σ-stable, it follows that $-\hom_{\mathcal{D}^\uparrow}(E, E) = -1$. To prove our claim, it therefore suffices to show that the strict inequality $\hom_{\mathcal{D}^\uparrow}(E, E[1]) > 0$ holds. We have $\hom_{\mathcal{D}^\uparrow}(E, E[1]) = \hom(E[1], S_{\mathcal{D}^\uparrow}(E))$ where $S_{\mathcal{D}^\uparrow}(E) = E_2[1] \xrightarrow{i_E[1]} \mathrm{Cone}(\varphi)[1]$. We also have $\hom_{\mathcal{D}^\uparrow}(E[1], S_{\mathcal{D}^\uparrow}(E)) > 0$ because there is a non-zero morphism $E \to S_{\mathcal{D}^\uparrow}(E)[-1]$. Therefore, $-\chi(E, E) = d_2 r_1 - d_1 r_2 > 0$.

□

Proposition 4.9.36. *Let $\mathcal{A} = \mathrm{Coh}(C)$ and C be an elliptic curve. Let $\sigma = (Z, \mathcal{H})$ on $\mathcal{D}^{\uparrow}$ be a pre-stability condition in Γ, then σ satisfies the support property and therefore it is a Bridgeland stability condition.*

Proof. We will prove that σ satisfies the support property with respect to the quadratic form $Q(r_1, d_1, r_2, d_2) = d_2r_1 - d_1r_2$. We – using a normalised σ combined with the fact that the group action preserves the support property – have

$$\ker(Z) = \{(r_1, d_1, r_2, d_2) \mid d_2 = Ad_1 + Br_1 \text{ and } r_2 = -Cr_1 - Dd_1\}.$$

Let $(r_1, d_1, r_2, d_2) \in \ker(Z)$, then $Q(r_1, d_1, r_2, d_2) = Dd_1^2 + (A+C)d_1r_1 + Br_1^2 < 0$. Since $-1 < f(0) = r < 0$, we have that $D < 0$. Moreover, since we have $\mathrm{Discr}(M) = (A+C)^2 - 4BD < 0$, we additionally obtain $B < 0$ and $Dd_1^2 + (A+C)d_1r_1 + Br_1^2 < 0$ for all $(r_1, d_1) \in \mathbb{R}^2$. Hence, Q is negative definite in $\ker(Z)$.

Now, let $E = E_1 \xrightarrow{\varphi} E_2$ be a σ-semistable object. By [8, Lemma A.6] it is enough to show that $Q(E) \geq 0$ for σ-stable objects. By lemma 4.9.35 we have that $d_2r_1 - d_1r_2 \geq 0$ and the proof is finished. □

Theorem 4.9.37. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is an elliptic curve. Then*

$$\mathrm{pre}\,\mathrm{Stab}(\mathcal{D}^{\uparrow}) = \mathrm{Stab}(\mathcal{D}^{\uparrow}).$$

Proof. Let $\sigma \in \mathrm{pre}\,\mathrm{Stab}\,\mathcal{D}^{\uparrow}$ then σ satisfies the support property and, therefore, is a Bridgeland stability condition. To see this apply proposition 4.9.28 for $\sigma \in \Theta_i$, and proposition 4.9.36 for $\sigma \in \Gamma$. By theorem 4.8.36 these are the only cases we need to consider. □

4.10 Topological description of $\mathrm{Stab}(\mathcal{D}^{\uparrow})$

It is now our purpose to study the topology of $\mathrm{Stab}(\mathcal{D}^{\uparrow})$ by investigating one of it's characterising subsets. The aim of this section is to prove that the set S_{12} to be defined in 4.10.1 is an open, connected four dimensional complex manifold. We start by introducing new language.

Definition 4.10.1. Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. Define the sets

$$S_{12} = \{\sigma \in \mathrm{Stab}(\mathcal{D}^{\uparrow}) \mid i_1(\mathbb{C}(x)), i_2(\mathbb{C}(x)), i_1(\mathcal{O}_C), i_2(\mathcal{O}_C) \ \sigma\text{-stable}\},$$

$$S_{23} = \{\sigma \in \mathrm{Stab}(\mathcal{D}^{\uparrow}) \mid i_2(\mathbb{C}(x)), \Delta(\mathbb{C}(x)), i_2(\mathcal{O}_C), \Delta(\mathcal{O}_C) \ \sigma\text{-stable}\},$$

and

$$S_{31} = \{\sigma \in \mathrm{Stab}(\mathcal{D}^{\uparrow}) \mid i_1(\mathbb{C}(x)), \Delta(\mathbb{C}(x)), i_1(\mathcal{O}_C), \Delta(\mathcal{O}_C) \ \sigma\text{-stable}\},$$

for all closed points $x \in C$.

The following lemma is a very useful – and intriguing – fact on $\mathrm{Stab}(\mathcal{D})$. We will subsequently be able to use it with regard to $\mathrm{Stab}(\mathcal{D}^{\uparrow})$ also.

Lemma 4.10.2. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. Let*

$$\mathcal{M} := \{(m_0, m_1, \phi_0, \phi_1) \in \mathbb{R}^4 \mid \phi_1 < \phi_0 < \phi_1 + 1 \text{ and } m_0, m_1 > 0\}$$

There is a homeomorphism

$$\rho : \mathrm{Stab}(\mathcal{D}) \to \mathcal{M}$$
$$\sigma = (Z, \mathcal{H}) \mapsto (m_0, m_1, \phi_0, \phi_1)$$

where $m_0 = |Z(\mathbb{C}(x))|$ and $m_1 = |Z(\mathcal{O}_C)|$, $\phi_0 = \phi_\sigma(\mathbb{C}(x))$ and $\phi_1 = \phi_\sigma(\mathcal{O}_C)$.

Proof. The stability of $\mathbb{C}(x)$ and $\mathcal{O}_C$ in combination with $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_C, \mathbb{C}(X)) \neq 0 \neq \mathrm{Hom}_{\mathcal{A}}(\mathbb{C}(X), \mathcal{O}_C[1])$ provides $\phi_1 < \phi_0 < \phi_1 + 1$. Moreover, $m_0, m_1 > 0$ by definition.

We obtain that ρ is continuous from two fact. Firstly from that, that the operators $\phi^-_{\mathcal{P}}(E) = \phi^-_\sigma(E)$ and $\phi^-_{\mathcal{P}}(E) = \phi^-_\sigma(E)$ of [18, Section 3], considered as functions on the set of slicings/stability conditions are equal whenever E semistable (which is the case for $E = \mathbb{C}(x)$ and for $E = \mathcal{O}_C$) and – that is the point – are continuous. Secondly that, that the same holds for the mass-function m of [18, Section 5].

Next, we will prove that ρ is bijective. Identifying σ with an element $(T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ via $\sigma = \sigma_\mu(T, f)$, which is possible by theorem 2.5.51, we must prove that T and f can be reconstructed from the vector $(m_0, m_1, \phi_0, \phi_1)$. We obtain T^{-1} as

$$T^{-1} = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$$

where $m_0 \exp(\pi\phi_0 i) = A + Di$ and $m_1 \exp(\pi\phi_1 i) = B + Ci$. On the other hand, to reconstruct f from $(m_0, m_1, \phi_0, \phi_1)$, we need to provide $f : \mathbb{R}/2\mathbb{Z} \to \mathbb{R}/2\mathbb{Z}$, which is given by the formula

$$f(t) =$$
$$\frac{1}{\pi}(\arg(m_1(\sin(\phi_1\pi)\cos(t\pi) - \cos(\phi_1\pi)\sin(t\pi)$$
$$+ i m_0(m_1(\sin(\phi_1\pi)\cos(t\pi) - \cos(\phi_1\pi)\sin(t\pi)))),$$

as well as $n = \lfloor f(0) \rfloor$. To obtain the latter, note that applying f to

$$-n < \phi_0 = f^{-1}(1) \leq -n + 1$$

gives

$$f(-n) < 1 \leq f(-n+1)$$

which – using the fact that f is increasing – provides

$$f(0) - n < 1 \leq f(0) - n + 1$$

and hence $n \leq f(0) < n+1$ such that n is provided by ϕ_0, given as $n = \lfloor \phi_0 \rfloor$.

Finally, to see that ρ is a local homeomorphism, consider

$$\begin{aligned} \nu : \{(m_0, m_1, \phi_0, \phi_1) \in \mathbb{R}^4 \mid \phi_1 < \phi_0 < \phi_1 + 1 \text{ and } m_0, m_1 > 0\} &\to \mathrm{GL}_2^+(\mathbb{R}) \\ (m_0, m_1, \phi_0, \phi_1) &\mapsto \begin{pmatrix} -m_0 \cos(\phi_0 \pi) & m_1 \cos(\phi_1 \pi) \\ -m_0 \sin(\phi_0 \pi) & m_1 \sin(\phi_1 \pi) \end{pmatrix}^{-1} \end{aligned}$$

which makes sense because the matrix above has det > 0.The map ν is a covering and we have $\nu \circ \rho = \mathcal{Z}$, where $\mathcal{Z} : \mathrm{Stab}(\mathcal{D}) \cong \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \to \mathrm{GL}_2^+(\mathbb{R})$ is the universal covering. Hence, ρ is a local homeomorphism. Combining all this, we conclude that ρ is indeed a homeomorphism. □

From now on we will, throughout this chapter, freely identify the following three topological spaces:

$$\widetilde{\mathrm{GL}}_2^+(\mathbb{R}), \mathrm{Stab}(\mathcal{D}) \text{ and } \mathcal{M}.$$

These homeomorphisms are explicitly given by the identities:

$$\begin{aligned} \sigma = \sigma_\mu(T, f), T^{-1} = M = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}, \\ m_0 = |Z_\sigma(\mathbb{C}(x))|, m_1 = |Z_\sigma(\mathcal{O}_C)|, \phi_0 = \phi_\sigma(\mathbb{C}(x)), \phi_1 = \phi_\sigma(\mathcal{O}_C), \\ A + Di = m_0 \exp(i\pi\phi_0), B + Ci = m_1 \exp(i\pi\phi_1), f(\phi_0) = 1, \\ m_0 = |A + Di|, m_1 = |B + Ci|, \phi_0 = f^{-1}(1) \text{ and } \phi_1 = f^{-1}(1/2). \end{aligned} \tag{4.78}$$

Definition 4.10.3. Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. Define

$$\begin{aligned} \mathcal{P}_{12} = \{(\sigma_1, \sigma_2) \in (\widetilde{\mathrm{GL}}_2^+(\mathbb{R}))^2 \mid \phi_0 < \phi_2 + 1, \phi_1 < \phi_3 + 1 \\ \text{and if } \phi_0 > \phi_2, \text{ then } \det(M_1 + M_2) > 0\} \end{aligned}$$

with $\phi_i, i \in \{0, \dots, 3\}$ as in lemma 4.10.2.

Definition 4.10.4. Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. For $\sigma \in S_{12}$ define

$$\phi_0 = \phi_\sigma(i_1(\mathbb{C}(x))), \phi_1 = \phi_\sigma(i_1(\mathcal{O}_C)), \phi_2 = \phi_\sigma(i_2(\mathbb{C}(x))) \text{ and } \phi_3 = \phi_\sigma(i_2(\mathcal{O}_C)).$$

We immediately obtain the following.

Lemma 4.10.5. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. For every $\sigma \in S_{12}$, we have $\phi_1 < \phi_0 < \phi_1 + 1$ and moreover $\phi_3 < \phi_2 < \phi_3 + 1$ with $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4.*

Proof. This follows from the fact that $i_1(\mathbb{C}(x)), i_1(\mathcal{O}_C), i_2(\mathbb{C}(x))$ and $i_2(\mathcal{O}_C)$ are stable and the existence of non-vanishing morphisms between the respective pairs. □

Lemma 4.10.6. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. There is a map*

$$\pi : S_{12} \to \mathcal{P}_{12}$$
$$\sigma \mapsto ((m_0, m_1, \phi_0, \phi_1)(m_2, m_3, \phi_2, \phi_3)) = (\sigma_1, \sigma_2),$$

where we have $\sigma = (Z, \mathcal{H})$ with $m_0 = |Z(i_1(\mathbb{C}(x)))|, m_1 = |Z(i_1(\mathcal{O}_C))|$, $m_2 = |Z(i_2(\mathbb{C}(x)))|$, $m_3 = |Z(i_2(\mathcal{O}_C))|$ and $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4.

Proof. Since every $\sigma \in S_{12}$ satisfies $\phi_1 < \phi_0 < \phi_1 + 1$ and $\phi_3 < \phi_2 < \phi_3 + 1$, then by lemma 4.10.2 we have a unique stability condition σ_1 given by $(m_0, m_1, \phi_0, \phi_1)$, where $m_0 = |Z(i_1(\mathbb{C}(x)))|$ and $m_1 = |Z(i_1(\mathcal{O}_C))|$ and a unique stability condition σ_2 given by $(m_2, m_3, \phi_2, \phi_3)$, where we have $m_2 = |Z(i_2(\mathbb{C}(x)))|$ and $m_3 = |Z(i_2(\mathcal{O}_C))|$. Therefore, we obtain two stability conditions $\sigma_1 = (Z_1, \mathcal{H}_1) = (T_1, f_1)$ and $\sigma_2 = (Z_2, \mathcal{H}_2) = (T_2, f_2)$ in $\mathrm{Stab}(\mathcal{D})$ and define π as $\sigma \mapsto (\sigma_1, \sigma_2)$. □

Lemma 4.10.7. *The map π is well-defined, continuous, open and $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-equivariant.*

Proof. We have that $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acts freely on S_{12}. As π is defined in terms of the slicing, we obtain a $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-equivariant continuous map from S_{12} to $\widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \times \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.

We now show that $(\sigma_1, \sigma_2) \in \mathcal{P}_{12}$. First we prove that $m - n \geq -1$. Let with $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4. Since $i_1(\mathbb{C}(x)), i_2(\mathbb{C}(x))$ are stable and we have a non-zero morphism $i_1(\mathbb{C}(x)) \to i_2(\mathbb{C}(x))[1]$, it follows that $\phi_0 - \phi_2 < 1$.

If $\phi_0 > \phi_2$, then by lemma 4.5.31 we get that $\Delta(\mathbb{C}(x))$ is stable. We show now that in this case $\det(M_1 + M_2) > 0$. By the equivariance of π we obtain that there is $g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ such that by acting by g we obtain a stability condition $\sigma' = \sigma g$ such that $\pi(\sigma') = (\sigma_1 g, \sigma_\mu)$. Let $\sigma' = \sigma_1 g = (T', f')$ and $M' = T'^{-1}$. By lemma 4.8.15, we have $\det(M' + I) > 0$. Note that $M' = M_2^{-1} M_1$, therefore

$$\begin{aligned} 0 &< \det(M_2^{-1} M_1 + I) \\ &= \det(M_2^{-1} M_1) + \det(I) + \det(M_2^{-1} M_1) \operatorname{Tr}(M_2^{-1} M_1) \\ &= \det(M_2)^{-1} \det(M_1) + 1 + \det(M_2)^{-1} \det(M_1) \operatorname{Tr}(M_2^{-1} M_1) \\ &= \det(M_2)^{-1} (\det(M_1) + \det(M_2) + \det(M_1) \operatorname{Tr}(M_2^{-1} M_1)) \\ &= \det(M_2) \det(M_1 + M_2). \end{aligned}$$

As $\phi_3 < \phi_2 < \phi_3 + 1$, we obtain $\det(M_2) > 0$. This implies $\det(M_1 + M_2) > 0$. Moreover, as $i_1(\mathcal{O}_C)$ and $i_2(\mathcal{O}_C)$ are σ-stable and there is a non-zero morphism $i_1(\mathcal{O}_C) \to i_2(\mathcal{O}_C)[1]$, we obtain $\phi_1 < \phi_3 + 1$.

Since $\pi' : S_{12} \to \mathrm{GL}^+(2, \mathbb{R})^2$ is a local homeomorphism, where π' maps a stability condition to it's stability function, the fact that π is a local homeomorphism follows from the fact that $p \circ \pi = \pi'$, where $p : \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \to \mathrm{GL}^+(2, \mathbb{R})$ is the universal covering. □

We require some technical language in order to prepare the proof of proposition 4.10.27.

Definition 4.10.8. Define

- the set
$$\mathcal{V}_{12} = \{\sigma \in S_{12} \mid \pi(\sigma) = (\sigma_1, \sigma_2) \text{ such that } \sigma_2 = \sigma_\mu\},$$
- furthermore
$$\begin{aligned} \mathcal{L}_{12} = \{(\sigma, \sigma_\mu) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})^2 \mid f(0) > -1, 3/2 > f^{-1}(1/2) \\ \text{and if } f(0) < 0 \text{ then } \det(M + I) > 0\}, \end{aligned}$$
- moreover we define
$$\begin{aligned} Y \text{ to be the set of all } (m_0, m_1, \phi_0, \phi_1) \text{ where} \\ m_i > 0, \phi_0 < 2, \phi_1 < \frac{3}{2}, \phi_1 < \phi_0 < \phi_1 + 1 \\ \text{and if } 1 \leq \phi_0 < 2 \text{ and } 0 < \phi_1 < \frac{3}{2}, \text{ then } \delta(m_0, m_1, \phi_0, \phi_1) > -1, \text{ where} \\ \delta : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times (1, 2) \times \left(0, \frac{3}{2}\right) \to \mathbb{R} \text{ is given by} \\ (m_0, m_1, \phi_0, \phi_1) \mapsto m_0 m_1 \sin((\phi_0 - \phi_1)\pi) - m_0 \cos(\phi_0 \pi) + m_1 \sin(\phi_1 \pi) \end{aligned}$$

- and finally

$$\pi_0 = \pi\big|_{\mathcal{V}_{12}}.$$

We prove the following lemmas for the newly-defined sets.

Lemma 4.10.9. *The set Y is connected.*

Proof. We denote M by $M = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$, and have $A + Di = m_0 \exp(i\pi\phi)$ as well as $B + Ci = m_1 \exp(i\pi\phi)$ and set

$$a := \cos(\phi_0\pi), d := \sin(\phi_0\pi), b := \cos(\phi_1\pi), c := \sin(\phi_1\pi), \Delta := bd - ac$$

such that

$$\det(M) = BD - AC = m_0m_1(bd - ac) = m_0m_1 \sin((\phi_0 - \phi_1)\pi) > 0$$
$$\text{and } \operatorname{Tr}(M) = C - A = m_1c - m_0a.$$

Moerover,

$$\delta = \det(M) + \operatorname{Tr}(M) = \det(M + I) - 1 > -1 \iff \det(M + I) > 0.$$

Now $\Delta > 0$ provides $\delta = m_0m_1\Delta + m_1c - m_0a > -1 \iff m_0m_1 + m_1\frac{c}{\Delta} - m_0\frac{a}{\Delta} > -\frac{1}{\Delta}$ which in turn is true if and only if $m_1(m_0 + \frac{c}{\Delta}) - \frac{a}{\Delta}(m_0 + \frac{c}{\Delta}) > -\frac{1}{\Delta} - \frac{ac}{\Delta^2}$ which holds true if and only if $(m_0 + \frac{c}{\Delta})(m_1 - \frac{a}{\Delta}) > -\frac{\Delta + ac}{\Delta^2} = \frac{bd}{\Delta^2} =: R$. For fixed ϕ_0, ϕ_1 we now consider a set $(m_0, m_1, \phi_0, \phi_1) \in Y$. Its connectedness is obvious except in the case where $\phi_1 > 0$ and $\phi_0 \geq 1$. If in this case $R \leq 0$ then we see that the set is connected. On the other hand, this is not clear if in this case $R > 0$. The connectedness then is in question if simultaneously with $R > 0$ we had $m_0 < -\frac{c}{\Delta}$ and at the same time $m_1 < \frac{a}{\Delta}$ and hence, using $m_0, m_1 > 0$ as well as $\Delta > 0$, that $a > 0$ and $c < 0$. However, using $R = \frac{bd}{\Delta^2}$ reveals that the very case of $a > 0$ and $c < 0$ cannot occur simultaneously with $R > 0$. Hence, all fibres of the projection

$$Y \xrightarrow{P} X \in \mathbb{R}^2,$$
$$(m_0, m_1, \phi_0, \phi_1) \mapsto (\phi_0, \phi_1)$$

where $X = \{(\phi_0, \phi_1) \mid \exists\, m_0, m_1 \text{ such that } (m_0, m_1, \phi_0, \phi_1) \in Y\}$ are connected and non-empty.

Therefore, consider $S(\frac{1}{10}, \frac{1}{10}, \phi_0, \phi_1)$ for all $(\phi_0, \phi_1) \in X$. Since $\delta = m_0m_1\Delta + m_1c - m_0a$ we obtain

$$|\delta| \leq |m_0m_1\Delta| + |m_1c| + |m_0a| \leq \frac{1}{10}(|\Delta| + |c| + |a|) \leq \frac{3}{10} < 1$$

such that $\delta > -1$. If we now take $Q, Q' \in Y$ then we obtain that $S(P(Q))$ and Q are in a path-connected component of $P^{-1}(\phi_0, \phi_1)$ and that $S(P(Q'))$ and Q' are in a path-connected component of $P^{-1}(\phi_0', \phi_1')$. Since X is connected, there is a path from $P(Q)$ to $P(Q')$ and therefore from $S(P(Q))$ to $S(P(Q'))$ (note that we use that connectedness and path-connectedness is the same in a metric space). With this, the proof is finished. □

Lemma 4.10.10. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. We have $\pi_0(\mathcal{V}_{12}) \subset \mathcal{L}_{12}$.*

Proof. We consider $\pi(\sigma) = (\sigma_1, \sigma_\mu)$ where we need to prove that the restrictions on $\sigma_1 = \sigma_\mu(T_1, f_1)$ and σ_μ are fulfilled. This is seen by using the definition of $\mathcal{P}_{12}$ in 4.10.3, where we simply have $M_1 = (T_1)^{-1}, M_2 = I, \phi_2 = 1$ and $\phi_3 = \frac{1}{2}$ with $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4. Subbing these into the definition of $\mathcal{P}_{12}$ now provides the result – for instance, consider the condition $\phi_1 < \phi_3 + 1 = \frac{1}{2} + 1 = \frac{3}{2}$. Since $f_1^{-1}(\frac{1}{2}) = \phi_1$ (see (4.78)) we obtain $f_1^{-1}(\frac{1}{2}) < \frac{3}{2}$. □

Lemma 4.10.11. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for a smooth projective curve C. If $\sigma \in \mathcal{V}_{12}$ and $\pi_0(\sigma) = (\sigma_\mu g, \sigma_\mu)$ with $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $0 \leq f(0)$ then $\sigma \in \Theta_{12}$.*

Proof. Let $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4. If $\Delta(\mathbb{C}(x))$ is σ-stable, its stability provides $\phi_2 < \phi(\Delta(\mathbb{C}(x))) < \phi_0$. Let $\lfloor f(0) \rfloor = n$. Since $\phi_2 = 1$ we obtain $1 < \phi_0 \leq -n+1$ such that $-n > 0$. We obtain $\lfloor f(0) \rfloor = n \leq -1$ which provides $f(0) < 0$ and that is a contradiction. Hence, letting $X = \mathbb{C}(x)$ we have $X \in \mathcal{D}$ stable for which $\Delta(X)$ is not σ-stable such that the result follows by theorem 4.5.29. □

Lemma 4.10.12. *Let $\mathcal{A} = \mathrm{Coh}(C)$, for a smooth projective curve C. If $\sigma \in \mathcal{V}_{12}$ and $\pi_0(\sigma) = (\sigma_\mu g, \sigma_\mu)$ with $g = (T, f) \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $0 > f(0) > -1$ then $\Delta(\mathbb{C}(x))$ is σ-stable.*

Proof. Let $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4. Let $\lfloor f(0) \rfloor = n$. If we assume $\Delta(\mathbb{C}(x))$ to be not σ-stable, then we obtain by corollary 4.5.21 and lemma 4.5.31 that $\phi_0 \leq \phi_2$. Since $\phi_2 = 1$ we obtain $-n < \phi_0 \leq 1$ such that $n > -1$. We obtain $f(0) \geq \lfloor f(0) \rfloor = n \geq 0$ which is a contradiction. □

Lemma 4.10.13. *The map*

$$\rho : \mathcal{L}_{12} \to Y \subset \mathbb{R}^4$$
$$(T, f) \mapsto (m_0, m_1, f^{-1}(1), f^{-1}(\frac{1}{2}))$$

with $m_0 = |A + Di|, m_1 = |B + Ci|$ is a homeomorphism.

Proof. Let $\phi_i, i \in \{0, \dots, 3\}$ as in definition 4.10.4. It suffices to prove that Y is the image of $\mathcal{L}_{12}$ under the map defined in lemma 4.10.2. Let $\sigma_1 \in \mathcal{L}_{12}$. We have $m_0, m_1 > 0$ for start. We will cumulatively use that f is increasing. First of all this implies that we obtain $0 > f^{-1}(-1)$ from $f(0) > -1$. Hence we obtain

$$2 > f^{-1}(-1) + 2 = f^{-1}(-1+2) = f^{-1}(1) = \phi_0,$$

as well as

$$\begin{aligned} \phi_1 = f^{-1}(\frac{1}{2}) < f^{-1}(1) \text{ and} \\ \phi_1 = f^{-1}(\frac{1}{2}) < f^{-1}(1) = \phi_0 = f^{-1}(1) < f^{-1}(\frac{3}{2}) \\ = f^{-1}(\frac{1}{2}+1) = f^{-1}(\frac{1}{2}) + 1 = \phi_1 + 1. \end{aligned}$$

Additionally, we have $\phi_1 = f^{-1}(\frac{1}{2}) < \frac{3}{2}$ by assumption and if $f(0) \geq 0$, and hence $f(1) \geq 1$ we obtain $1 \geq f^{-1}(1) = \phi_0$. If, on the other hand, $f(0) < 0$, we obtain

$$1 < f{-}1(1) = \phi_0 = f{-}1(1) < 2 \text{ and } 0 < f^{-1}(\frac{1}{2}) = \phi_1 = f^{-1}(\frac{1}{2}) < 2.$$

We now use $\det(M+I) > 0$ together with

$$\begin{aligned} A + Di = m_0(\cos(f^{-1}(1)\pi) + i\sin(f^{-1}(1)\pi)) \\ B + Ci = m_1(\cos(f^{-1}(\frac{1}{2})\pi) + i\sin(f^{-1}(\frac{1}{2})\pi)) \end{aligned}$$

provided by the correspondence between M and f^{-1}, which gives

$$-1 < \det(M+I) - 1 = \delta(m_0, m_1, f^{-1}, f^{-1}(\frac{1}{2})).$$

□

Lemma 4.10.14. *The set $\mathcal{L}_{12} \subset \mathcal{P}_{12}$ is open and connected.*

Proof. Define

$$\begin{aligned} U_1 = \{g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \mid f(0) > 0\} \text{ and} \\ U_2 = \{g \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \mid \frac{1}{2} > f(0) > -1, \det(M+I) > 0 \text{ and } f^{-1}(\frac{1}{2}) < \frac{3}{2}\}. \end{aligned}$$

The sets U_1 and U_2 are open in $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and $U_1 \cup U_2 \subset \mathcal{L}_{12}$.

Let $g \in \mathcal{L}_{12}$, if $f(0) \neq 0$ we have $g \in U_1 \cup U_2$. If $f(0) = 0$ we have $-A, C \in \mathbb{R}_{>0}$ which gives $\mathrm{Tr}(M) > 0$, hence $\det(M + I) > 0$ and therefore $g \in U_2$ which provides $\mathcal{L}_{12} \subset U_1 \cup U_2$ and hence $\mathcal{L}_{12} = U_1 \cup U_2$ such that $\mathcal{L}_{12}$ is open.

Since Y from definition 4.10.8 is connected, by lemma 4.10.9 we obtain by lemma 4.10.13 that $\mathcal{L}_{12}$ too is connected. □

In preparation of case 1 of the proof of proposition 4.10.27, we have the following lemmas. We start by considering the situation where both σ_1 and σ_2 have the abelian category $\mathcal{A}$ as their heart.

Lemma 4.10.15. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for an elliptic curve C. Let Z_1 be a stability function on $\mathcal{A}$. There is a stability condition obtained by CP-gluing from $\sigma_1 = (Z_1, \mathcal{A})$ and $\sigma_2 = \sigma_\mu$ via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$.*

Proof. We must prove the HN-property – the result then follows from 4.9.28. In this situation CP-gluing conditions are fulfilled and the resulting heart is $\mathcal{A}^\uparrow$. Since under the conditions of this lemma we have $\phi_0 = 1$ and so $A + Di = m_0 \exp(i\pi) = -m_0$. In particular, $D = 0$ and $A = -m_0 < 0$.

We use [4, Corollary 3.6] to prove the HN-property. Let $E = E_1 \to E_2 \in \mathcal{A}^\uparrow$ and consider subobjects $F = (F_1 \to F_2)$ of E. Let $r_i^E, d_i^E, r_i^F, d_i^F$ be rank and degree of E_i and F_i. The class of E in the numerical Grothendieck group is given by $(r_1^E, d_1^E, r_2^E, d_2^E) \in \mathbb{Z}^4$ and we have $Z(E) = Ad_1^E - d_2^E + Br_1^E + i(Cr_1^E + r_2^E)$. We need to prove that there are only finitely many classes $(r_1^F, d_1^F, r_2^F, d_2^F)$ coming from sub-objects $F \subset E$ that satisfy $Ad_1^F - d_2^F + Br_1^F < \max\{0, Ad_1^E - d_2^E + Br_1^E\}$. Because degrees of subsheaves are bounded above by by lemma 4.7.18, there exist integers M_1, M_2 that may depend on E but not on F such that $d_1^F \leq M_1$ and $d_2^F \leq M_2$. Because $A < 0$ we need only concern ourselves with such $F \subset E$ for which

$$AM_1 \leq Ad_1^F < \max\{0, Ad_1^E - d_2^E + Br_1^E\} + d_2^F - Br_1^F, \tag{4.79}$$

in particular $AM_1 < \max\{0, Ad_1^E - d_2^E + Br_1^E\} + d_2^F - Br_1^F$. Because $F \subset E$ provides $0 \leq r_1^F \leq r_1^E$, this leads to the inequality

$$AM_1 - \max\{0, Ad_1^E - d_2^E + Br_1^E\} + \min\{0, Br_1^E\} < d_2^F < M_2.$$

Now, since r_1^F and d_2^F are integers we obtain that there are only finitely many pairs (r_1^F, d_2^F) arising from subobjects $F \subset E$ that satisfy the required condition. We see from (4.79) that there is only a finite number of d_1^F occurring for each such pair such that the number of classes of subobjects $F \subset E$ for which $\Re(Z(F)) < \max\{0, \Re(Z(E))\}$ is finite and therefore the conditions for [4, Corollary 3.6] are fulfilled and the proof finished. □

Proposition 4.10.16. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for an elliptic curve C. If $\sigma_2 \in \mathrm{Stab}(\mathcal{D})$ and $\sigma_1 = \sigma_2(T, f) \in \mathrm{Stab}(\mathcal{D})$ with $f(0) \geq 0$, then there is a stability condition obtained by CP-gluing σ_1 and σ_2 via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$.*

Proof. It is sufficient to prove that the HN-property is fulfilled – because the support property then follows from 4.9.28. Assume $\mathcal{P}_i$ to be the slicing of σ_i. We use

$$\begin{aligned} &\mathrm{Hom}^{\leq 0}_{\mathcal{D}^\uparrow}(i_1\mathcal{P}_1(\theta, \theta+1], i_2\mathcal{P}_2(\theta, \theta+1]) \\ = &\mathrm{Hom}^{\leq 0}_{\mathcal{D}^\uparrow}(i_1\mathcal{P}_2(f(\theta), f(\theta)+1], i_2\mathcal{P}_2(\theta, \theta+1]) = 0 \end{aligned}$$

if and only if $f(\theta) \geq \theta$ applying lemma 3.2.30. Via [21, Theorem 3.6], this reduces the problem to proving that there is a $\theta \in (0, 1)$ with $f(\theta) \geq \theta$. Now we consider different cases.

1. If $f(0) \geq 1$ we use $\theta \in (0, 1)$ to see that $f(\theta) \geq f(0) \geq 1 \geq \theta$.

2. If $1 > f(0) > 0$ we obtain from the correspondence between f and T (definition of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$) that for $T^{-1} = \begin{pmatrix} -A & B \\ -D & C \end{pmatrix}$ we obtain

   $$f(\theta) = \frac{1}{\pi}\arg(C\cos(\theta\pi) - B\sin(\theta\pi) + (-A\sin(\theta\pi) + D\cos(\theta\pi)i).$$

   From $1 > f(0) > 0$ we obtain $D > 0$ which implies that there is a $\theta \in (0, 1)$ such that $\cot(\theta\pi) = \dfrac{A}{D}$ and hence $-A\sin(\theta\pi) + D\cos(\theta\pi) = 0$. Then $D(C\cos(\theta\pi) - B\sin(\theta\pi)) = DC\cos(\theta\pi) - DB\sin(\theta\pi) = AC\sin(\theta\pi) - DB\sin(\theta\pi) = -\det(T^{-1})\sin(\theta\pi)$ such that

   $$f(\theta) = \frac{1}{\pi}\arg(-\det(T^{-1})\sin(\theta\pi)) = 1 > \theta \text{ provided by } \det(T^{-1}) \geq 0.$$

3. If $f(0) = 0$ then σ_1 and σ_2 have the same heart $\mathcal{P}_\mu(s, s+1]$ where $s \in \mathbb{R}$. Apply a shift $[-\lceil s \rceil]$ to reduce the problem to a situation where the (common) heart is simply $\mathcal{P}_\mu(r, r+1], r \in (-1, 0]$. If $r = 0$ apply lemma 4.10.15. Otherwise we adapt the strategy of lemma 4.10.15. Glue two copies of $\mathcal{P}_\mu(r, r+1]$ (the hearts of σ_1, σ_2) and obtain the heart obtained by CP-gluing $\mathcal{P}_l(r, r+1] = (\mathcal{A}^\uparrow)^l$ (see definition 4.8.23). Now consider a torsion pair $\langle \mathcal{P}_l(0, r+1], \mathcal{P}_l(r, 0] \rangle$ for this (abelian) category and obtain for any $E \in \mathcal{P}_l(r, r+1]$ an exact sequence

   $$0 \to E' \to E \to E''[-1] \to 0,$$

where $E' \in \mathcal{P}_l(0, r+1]$ and $E'' \in \mathcal{P}_l(r+1, 1]$ and therefore $E', E'' \in \mathcal{A}^{\uparrow}$. For any subobject $F \subset E$ we obtain an analogous sequence such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F''[-1] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''[-1]} & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E''[-1] & \longrightarrow & 0 \end{array} \tag{4.80}$$

where we obtain f' from $\mathrm{Hom}(F', E''[-1]) = 0$ provided by the induced torsion pair $\langle \mathcal{P}_\mu(0, r+1], \mathcal{P}_\mu(r, 0] \rangle$ and hence f'' by [32, Section 1.1, (TR3)]. Moreover f' is a monomorphism by the snake lemma. We use the usual notation $\lambda_1(E) = E_1, \lambda_1(f) = f_1$ and $\rho_2(E) = E_2, \rho_2(f) = f_2$ for objects E and morphisms f and denote $\deg(F_i)$ by d_i^F and $\mathrm{rank}(F_i)$ by r_i^F. In order to emulate the proof of lemma 4.10.15 we must prove that there are only finitely many values possible for r_i' and that the d_i' are bounded above. From

$$0 \to E' \to E \to E''[-1] \to 0,$$

we obtain the equations $d_i^F = \deg(F_i') - \deg(F_i'')$ and $r_i^F = \mathrm{rank}(F_i') - \mathrm{rank}(F_i'')$. Since f' is a monomorphism we have $0 \leq \mathrm{rank}(F_i') \leq \mathrm{rank}(E_i')$. We must now prove that there are only finitely many values for $\mathrm{rank}(F_i'')$. From the application of λ_1 or ρ_2 respectively to (4.80) we firstly obtain the exact sequence

$$0 \to K_i''[-1] \to Q_i' \to Q_i \to Q_i''[1] \to 0 \tag{4.81}$$

via the application of the snake lemma and secondly have the canonical exact sequence

$$0 \to K_i'' \to F_i'' \to E_i'' \to Q_i'' \to 0 \tag{4.82}$$

where Q_i, Q_i' and Q_i'' are the cokernels of f_i, f_i' and f_i'' and K_i'' the kernel of f_i''. In the Grothendieck group we obtain

$$[K_i''[-1]] - [Q_i'] + [Q_i] - [Q_i''[1]] = 0$$

from (4.81) and

$$[K_i''] - [F_i''] + [E_i''] - [Q_i''] = 0$$

from (4.82). Adding these two equations and rearranging now provides $F'' = E'' - Q' + Q$ and therefore $\mathrm{rank}(F'') = \mathrm{rank}(E'') - \mathrm{rank}(Q') + \mathrm{rank}(Q)$. Since E'' and hence $\mathrm{rank}(E'')$ is fixed and there are only finitely many choices for $\operatorname{rank}(Q')$ and $\operatorname{rank}(Q)$ there are only finitely many choices for $\operatorname{rank}(F'')$.

To see that the d_i^F are bounded, we first recall that the degrees of subsheaves of E_i' are bounded above by lemma 4.7.18. Because $d_i^F = \deg(F_i') - \deg(F_i'')$ it is then sufficient to show that $\deg(F_i'')$ is bounded below. Because there are only finitely many possible values for $\operatorname{rank}(F_i'')$, this is equivalent to the slope of F_i'' being bounded below. That this is indeed true follows from $F_i'' \in \mathcal{P}_\mu(r+1, 1]$ and $r > -1$, using $\lambda_1(\mathcal{P}_l(r+1,1]) = \mathcal{P}_\mu(r+1,1] = \rho_2(\mathcal{P}_l(r+1,1])$. This finished the proof for $f(0) = 0$ and hence the proof of the proposition too is finished.

□

Lemma 4.10.17. *Let* $\mathcal{A} = \operatorname{Coh}(C)$ *for an elliptic curve* C*. Let* $\sigma_1, \sigma_2, \sigma_3 \in \operatorname{Stab}(\mathcal{D})$*.*

1. *If* $\sigma_2 = \sigma_\mu(T_2, f_2), \sigma_3 = \sigma_\mu(T_3, f_3)$ *and* $f_2(0) - f_3(0) \geq 1$*, then*

$$\sigma_{23} \in \Theta_2 \subset \Theta_{23} \subset S_{23} \subset \operatorname{Stab}(\mathcal{D}^\uparrow).$$

   where σ_{23} *is obtained by CP-gluing* σ_2 *and* σ_3 *via the semiorthogonal decomposition* $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$*.*

2. *If* $\sigma_3 = \sigma_\mu(T_3, f_3), \sigma_1 = \sigma_\mu(T_1, f_1)$ *and* $f_2(0) - f_3(0) \geq 1$*, then*

$$\sigma_{31} \in \Theta_3 \subset \Theta_{31} \subset S_{31} \subset \operatorname{Stab}(\mathcal{D}^\uparrow).$$

   where σ_{31} *is obtained by CP-gluing* σ_3 *and* σ_1 *via the semiorthogonal decomposition* $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$*.*

Proof. We prove the first statement. We know from corollary 4.2.24 that $f_2(0) - f_3(0) \geq 1$ is equivalent to the CP-gluing condition for $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$. We define $\sigma_1' = \sigma_2[1] = \sigma_\mu(-T_2, f_2 - 1)$ and $\sigma_2' = \sigma_3$. Then $\mathcal{A}_1' = \mathcal{A}_2[-1]$ and $\mathcal{A}_2' = \mathcal{A}_3$. The CP-gluing condition for σ_1', σ_2' that produces the heart $\mathcal{A}_{12}'$ then is $f_2(0) - 1 \geq f_3(0)$, which is what we were given. We now obtain

$$E \in \mathcal{A}_{12}' \text{ if and only if } S_{\mathcal{D}^\uparrow}[-1](E) \in \mathcal{A}_{23}.$$

Therefore, $S_{\mathcal{D}^\uparrow}[-1](\sigma_{12}')$ and σ_{23} have the same heart where σ_{12}' is the stability condition obtained by CP-gluing σ_1' and σ_2' via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and σ_{23} the stability condition obtained by CP-gluing σ_2 and σ_3 via the semiorthogonal decomposition $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$. Moreover, the stability functions agree as well. We therefore obtain $S_{\mathcal{D}^{\uparrow}}(\sigma'_{12}) = \sigma_{23}$ by applying [21, Proposition 2.2(2)]. Proposition 4.10.16 implies now that σ_{23} has the HN-property and the result follows.

For the proof of the second statement we define $\sigma'_2 = \sigma_3, \sigma'_3 = \sigma_1$ and show that $E \in \mathcal{A}'_{23}$ if and only $S_{\mathcal{D}^{\uparrow}}[-1](E) \in \mathcal{A}_{31}$. □

Definition 4.10.18. Define

$$S = \{\mathcal{O}_C, \mathbb{C}(x) \mid x \in C\}.$$

In preparation of case 2 of the proof of proposition 4.10.27, we have the following series of lemmas.

Lemma 4.10.19. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for an elliptic curve C. Let $(T, f) \in \mathcal{L}_{12}$ satisfy $f(0) < 0$ and $\mathrm{Discr}(M) \geq 0$. Let λ_1, λ_2 be the eigenvalues of the matrix M and let $v = (\cos(\pi\theta), \sin(\pi\theta))$ with $\theta \in [-1, 1)$ be an eigenvector of M. Let $g = (K_\theta, f_\theta)$ be the rotation by $\pi\theta$ and let $\sigma_1 = \sigma_\mu(T, f), \sigma_2 = \sigma_\mu$ and $\sigma_3 = \sigma_\mu(T_3, f_3)$, where $T_3 = (M + I)^{-1}$ and $f_3(0) \in [-1, 1)$.*

- *If $\lambda_i < -1$, then there is a stability condition obtained by CP-gluing $\sigma_2 g$ and $\sigma_3 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ in $\mathrm{Stab}(\mathcal{D}^{\uparrow})$.*
- *If $-1 < \lambda_i < 0$, then there is a stability condition obtained by CP-gluing $\sigma_3 g$ and $\sigma_1 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ in $\mathrm{Stab}(\mathcal{D}^{\uparrow})$.*
- *If $0 < \lambda_i$, then there is a stability condition obtained by CP-gluing $\sigma_1 g$ and $\sigma_2 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ in $\mathrm{Stab}(\mathcal{D}^{\uparrow})$.*

Proof. The matrix $T_1 := T = M^{-1}$ has eigenvalues $\alpha_i = 1/\lambda_i$ and the same eigenvectors as M and the matrix $T_3 = (M + I)^{-1}$ has eigenvalues $\beta_i = 1/(1+\lambda_i)$ and the same eigenvectors as M ($i \in \{1, 2\}$). To maintain consistent notation let $f_1(t) := f(t)$ and $f_2(t) = t$.

Recall that $-1 < f(0) < 0$ is equivalent to $1 < \phi_0 < 2$ implying that $A + Di = m_0 \exp(i\pi\phi_0)$ has negative imaginary part – in other words $D < 0$. We obtain from the compatibility of f_3 and $M + I$ that $\exp(i\pi f_3(0)) = \frac{C+1+Di}{|C+1+Di|}$. From $D < 0$ we see that $-1 < f_3(0) < 0$ because we have assumed $f_3(0) \in [-1, 1)$.

Assume λ to be the eigenvalue corresponding to the eigenvector $v, \alpha = 1/\lambda, \beta = 1/(1 + \lambda)$. then

$$Mv = \lambda v, T_1 v = \alpha v, T_3 v = \beta v.$$

We therefore have $f_1(\theta) = \theta + m$ where $m = 2k$ is an even integer when $\alpha > 0$ and $m = 2k - 1$ is an odd integer if $\alpha < 0$ ($k \in \mathbb{Z}$). A similar statement holds for $f_3(\theta)$, but now the sign of β is determining the parity of the added integer.

We will now prove $k = 0$ in all cases, that is, the added integer is in the set $\{-1, 0\}$. We distinguish two cases. First we assume $-1 \leq \theta < 0$. When h is one of the functions f_1 or f_3, we know that h is strictly increasing, which means $h(t+1) = h(t) + 1$ and $-1 < h(0) < 0$. We therefore obtain

$$-2 < h(-1) \leq h(\theta) < h(0) < 0.$$

Using $h(\theta) = \theta + m$, we obtain

$$m - 1 \leq h(\theta) < m.$$

Combining these inequalities we get $-2 < m$ and $m - 1 < 0$ such that for the integer m we now have $m \in \{-1, 0\}$. Now we assume $0 \leq \theta < 1$ and obtain

$$-1 < h(0) \leq h(\theta) < h(1) < 1 \text{ as well as } m \leq h(\theta) < m + 1.$$

The combination of both equalities now gives $-1 < m + 1$ and $m < 1$, such that $m \in \{-1, 0\}$. Considering the three different cases of the statement of this lemma we obtain

- if $\lambda < -1$, then $\alpha, \beta < 0$ such that $f_2(\theta) = \theta, f_3(\theta) = \theta - 1$ and hence $f_2(\theta) - f(\theta) = 1$,
- if $-1 < \lambda < 0$, then $\alpha < 0 < \beta < 1$ such that $f_3(\theta) = \theta, f_1(\theta) = \theta - 1$ and hence $f_3(\theta) - f_1(\theta) = 1$ and
- if $0 < \lambda$ then $0 < \alpha$ such that $f_1(\theta) = \theta, f_2(\theta) = \theta$ such that $f_1(\theta) - f_2(\theta) = 0$.

Writing $\sigma_i g = \sigma_\mu(T_i', f_i')$ gives $f_i'(t) = f_i(f_\theta(t)) = f_i(t + \theta)$ and so $f_i'(0) = f_i(\theta)$. Thus, in each case the corresponding CP-gluing condition is satisfied for two of the three stability conditions $\sigma_i g$ which by proposition 4.10.16 and lemma 4.10.17 finishes the proof. □

Corollary 4.10.20. *With the same assumptions and notation as in lemma 4.10.19, we have*

- *if $\lambda_i < -1$, then the stability condition obtained by CP-gluing $\sigma_2 g$ and $\sigma_3 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ is in S_{12},*
- *if $-1 < \lambda_i < 0$, then the stability condition obtained by CP-gluing $\sigma_3 g$ and $\sigma_1 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$ is in S_{12} and*

- *if* $0 < \lambda_i$, *then the stability condition obtained by CP-gluing* $\sigma_1 g$ *and* $\sigma_2 g$ *via the semiorthogonal decomposition* $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ *is in* S_{12}.

Proof. As above we write $\sigma_i = \sigma_\mu(T_i, f_i) \in \mathrm{Stab}(\mathcal{D})$ and let Z_i denote the stability function of σ_i. For $i = 1, 3$ we have $-1 < f_i(0) < 0$. We obtain $1 < \phi_{\sigma_i}(\mathbb{C}(x)) < 2$. Moreover $\mathrm{Hom}(\mathcal{O}_C, \mathbb{C}(x)) \neq 0$ implies $\phi_{\sigma_i}(\mathbb{C}(x)) - 1 < \phi_{\sigma_i}(\mathcal{O}_C) < \phi_{\sigma_i}(\mathbb{C}(x))$ and therefore $0 < \phi_{\sigma_i}(\mathcal{O}_C) < 2$. This means that $\phi_{\sigma_i}(\mathcal{O}_C)$ is determined by $Z_i(\mathcal{O}_C)$. Because the definition of $\mathcal{L}_{12}$ provides $\phi_{\sigma_1}(\mathcal{O}_C) = \phi_1 < 3/2$, the complex number $Z_1(\mathcal{O}_C)$ is not in the fourth quadrant. As

$$Z_3(\mathcal{O}_C) = Z_1(\mathcal{O}_C) + Z_2(\mathcal{O}_C) = Z_1(\mathcal{O}_C) + i,$$

$Z_3(\mathcal{O}_C)$ cannot be in the fourth quadrant either, in other words, this shows $0 < \phi_{\sigma_3}(\mathcal{O}_C) < 3/2$.

We now study the three cases. First consider σ to be the stability condition obtained by CP-gluing $\sigma_2 g$ and $\sigma_3 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$. Then $i_2(X)$ is σ-stable for all $X \in S$,

$$\begin{aligned} \phi_\sigma(i_2(X)) &= \phi_{\sigma_2 g}(X) = f_\theta^{-1}(\phi_{\sigma_2}(X)) = \phi_{\sigma_2}(X) - \theta \\ \text{and } \phi_\sigma(\Delta(X)) &= \phi_{\sigma_3 g}(X) = f_\theta^{-1}(\phi_{\sigma_3}(X)) = \phi_{\sigma_3}(X) - \theta \end{aligned}$$

also. When $i_1(X)$ is not σ-stable we obtain $\phi_\sigma(\Delta(X)) \geq \phi_\sigma(i_2(X)) + 1$ from the HNF/JHF, see corollary 4.5.21 and lemma 4.5.31. To prove stability of $i_1(X)$ it hence suffices to show $\phi_\sigma(\Delta(X)) < \phi_\sigma(i_2(X)) + 1$ which is equivalent to $\phi_{\sigma_3}(X) < \phi_{\sigma_2}(X) + 1$.

When $X = \mathbb{C}(x)$ we have $\phi_{\sigma_2}(X) + 1 = \phi_{\sigma_\mu}(\mathbb{C}(x)) + 1 = 2$. We have seen above that $\phi_{\sigma_3}(\mathbb{C}(x)) < 2$ and therefore $i_1(\mathbb{C}(x))$ must be σ-stable. For $X = \mathcal{O}_C$ we obtain the stability of $i_1(X)$ from $\phi_{\sigma_3}(\mathcal{O}_C) < \phi_{\sigma_2}(\mathcal{O}_C) + 1 = 3/2$ which we have seen before. This proves $\sigma \in S_{12}$.

Now consider σ to be the stability condition obtained by CP-gluing $\sigma_3 g$ and $\sigma_1 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_3, \mathcal{D}_1 \rangle$. Then $i_1(X)$ is σ-stable for all $X \in S$,

$$\begin{aligned} \phi_\sigma(\Delta(X)) &= \phi_{\sigma_3 g}(X) = f_\theta^{-1}(\phi_{\sigma_3}(X)) = \phi_{\sigma_3}(X) - \theta \\ \text{and } \phi_\sigma(i_1(X)) &= \phi_{\sigma_1 g}(X) = f_\theta^{-1}(\phi_{\sigma_1}(X)) = \phi_{\sigma_1}(X) - \theta \end{aligned}$$

also. When $i_2(X)$ is not σ-stable we – similar as above – obtain $\phi_\sigma(i_2(X)) \geq \phi_\sigma(\Delta(X)) + 1$ from the HNF/JHF, again see corollary 4.5.21 and lemma 4.5.31. To prove stability of $i_1(X)$ it hence suffices to show $\phi_\sigma(i_2(X)) < \phi_\sigma(\Delta(X)) + 1$ which is equivalent to $\phi_{\sigma_1}(X) < \phi_{\sigma_3}(X) + 1$. For $X = \mathbb{C}(x)$ we have seen above $\phi_{\sigma_1}(\mathbb{C}(x)) < 2 < \phi_{\sigma_3}(\mathbb{C}(x)) + 1$ such that $i_2(\mathbb{C}(x))$ must be σ-stable. For $X = \mathcal{O}_C$ we have to show $\phi_{\sigma_1}(\mathcal{O}_C) < \phi_{\sigma_1}(\mathcal{O}_C) + 1$. Recall $Z_1(\mathcal{O}_C) = B + iC$ and $Z_1(\mathcal{O}_C) = B + i(C+1)$. When $B \geq 0$, then $Z_1(\mathcal{O}_C)$ is in the first quadrant, because $0 < \phi_{\sigma_1}(\mathcal{O}_C) = \phi_1 < 3/2$. This puts $Z_3(\mathcal{O}_C)$ into the first quadrant as well and hence $\phi_{\sigma_1}(\mathcal{O}_C) < \phi_{\sigma_3}(\mathcal{O}_C) < \phi_{\sigma_1}(\mathcal{O}_C) + 1$ implying that $i_2(\mathcal{O}_C)$ is σ-stable in this case. When $B < 0, Z_1(\mathcal{O}_C)$ and $Z_3(\mathcal{O}_C)$ are both on the left half-plane such that $1/2 < \phi_{\sigma_3}(\mathcal{O}_C) < \phi_{\sigma_1}(\mathcal{O}_C) < 3/2$. This provides us with

$$\phi_{\sigma_1}(\mathcal{O}_C) < 3/2 < \phi_{\sigma_3}(\mathcal{O}_C) + 1$$

and $i_2(\mathcal{O}_C)$ is σ-stable in this case as well. Hence we obtain $\sigma \in S_{12}$.

Since $\Theta_1 \subset \Theta_{12} \subset S_{12}$, there is nothing to prove in the remaining case. □

Corollary 4.10.21. *With the same assumptions and notation as in lemma 4.10.19 we have* $\pi_0(\sigma g^{-1}) = (T, f)$.

Proof. In corollary 4.10.20 we saw that each of the intervals $(-\infty, -1), (-1, 0)$ and $(0, \infty)$ for the eigenvalues provides a different CP-gluing situation which we will use to prove this corollary. For $X \in S$ consider the three equations

$$\begin{aligned} Z_{\sigma g^{-1}}(i_1(X)) &= K_\theta Z_\sigma(i_1(X)) = K_\theta Z_{\sigma_1 g}(X) = Z_1(X), \\ Z_{\sigma g^{-1}}(i_2(X)) &= K_\theta Z_\sigma(i_2(X)) = K_\theta Z_{\sigma_2 g}(X) = Z_2(X) \\ \text{and } Z_{\sigma g^{-1}}(\Delta(X)) &= K_\theta Z_\sigma(\Delta(X)) = K_\theta Z_{\sigma_3 g}(X) = Z_3(X) \end{aligned}$$

and note that in each of the three CP-gluing situations, two of these three equations hold true. In fact however we see from $Z_3 = Z_1 + Z_2$ and $[\Delta(X)] = [i_1(X)] + [i_2(X)]$ that the third one also holds in all cases. It remains to show that $\phi_{\sigma g^{-1}}(i_1(X)) = \phi_{\sigma_1}(X)$ as well as $\phi_{\sigma g^{-1}}(i_2(X)) = \phi_{\sigma_1}(X)$ for all $X \in S$.

If $0 < \lambda_i$ and we therefore obtain from corollary 4.10.20 that the stability condition obtained by CP-gluing from $\sigma_1 g$ and $\sigma_2 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is in S_{12}, from the definition of gluing and of the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action we obtain

$$\phi_{\sigma g^{-1}}(i_1(X)) = f_\theta(\phi_\sigma(i_1(X))) = f_\theta(\phi_{\sigma_1 g}(X)) = f_\theta f_\theta^{-1} \phi_{\sigma_1}(X) = \phi_{\sigma_1}(X)$$

and

$$\phi_{\sigma g^{-1}}(i_2(X)) = f_\theta(\phi_\sigma(i_2(X))) = f_\theta(\phi_{\sigma_2 g}(X)) = f_\theta f_\theta^{-1} \phi_{\sigma_2}(X) = \phi_{\sigma_2}(X)$$

for all $X \in S$.

If $0 < \lambda_i$ and we therefore obtain from corollary 4.10.20 that the stability condition obtained by CP-gluing from $\sigma_2 g$ and $\sigma_3 g$ via the semiorthogonal decomposition $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ is in S_{12}, from the definition of gluing and of the action $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ we obtain $\phi_{\sigma g^{-1}}(i_2(X)) = \phi_{\sigma_2}(X)$ and $\phi_{\sigma g^{-1}}(\Delta(X)) = \phi_{\sigma_3}(X)$

for all $X \in S$. We obtain via $\sigma \in S_{12}$ that – in addition to $i_2(X)$ and $\Delta(X)$ stable – we too have $i_1(X)$ σ-stable for $X \in S$. Hence $i_1(X), i_2(X)$ and $\Delta(X)$ are also σg^{-1}-stable. This implies

$$\phi_{\sigma g^{-1}}(i_2(X)) < \phi_{\sigma g^{-1}}(\Delta(X)) < \phi_{\sigma g^{-1}}(i_1(X)) < \phi_{\sigma g^{-1}}(i_2(X)) + 1$$

arguing in the usual way via the exact triangle $i_2(X) \to \Delta(X) \to i_1(X) \xrightarrow{+}$. Since $\phi_{\sigma g^{-1}}(i_2(X)) = \phi_{\sigma_2}(X) \in \{1, \frac{1}{2}\}$ we obtain $\phi_{\sigma g^{-1}}(i_2(X)) \in (0,2)$. Because $\phi_{\sigma_1}(X) \in (0,2)$ and $Z_{\sigma g^{-1}}(i_1(X)) = Z_1(X)$ we obtain $\phi_{\sigma g^{-1}}(i_1(X)) = \phi_{\sigma_1}(X)$.

In the remaining case from corollary 4.10.20 we have $\phi_{\sigma g^{-1}}(\Delta(X)) = \phi_3(X)$ and $\phi_{\sigma g^{-1}}(i_1(X)) = \phi_1(X)$ for all $X \in S$. From

$$\begin{aligned} 0 < \phi_{\sigma_1}(X) - 1 &= \phi_{\sigma g^{-1}}(i_1(X)) - 1 \\ < \phi_{\sigma g^{-1}}(i_2(X)) &< \phi_{\sigma g^{-1}}(\Delta(X)) = \phi_{\sigma_3}(X) < 2 \end{aligned}$$

we obtain, as above, $\phi_{\sigma g^{-1}}(i_2(X)) = \phi_{\sigma_2}(X)$. □

In preparation of case 3 of the proof of proposition 4.10.27, we have the following lemmas. We start with the following elementary one, that Bridgeland uses in [17, Page 264].

Lemma 4.10.22. *For $\eta \in (0, \frac{1}{2}] \subset \mathbb{R}$ and $w, w' \in \mathbb{C}$ with $|w - w'| < \eta|w|$, we have*

$$|w - w'| < 2\eta|w'|.$$

Proof. Via the triangle inequality in $\mathbb{C}$ and $|w - w'| < \eta|w|$ we obtain

$$|w| = |w| - |w'| + |w'| \leq |w - w'| + |w'| \leq \eta|w| + |w'|,$$

such that $(1-\eta)|w| < |w'|$. Since $\eta \in (0, \frac{1}{2}] \subset \mathbb{R}$, we have $\eta > 0, 1 - \eta > 0$ and $0 < 1/(1-\eta) \leq 2$ which implies

$$\eta|w| < \frac{\eta}{1-\eta}|w'| \leq 2\eta|w'|.$$

Hence,

$$|w - w'| < \eta|w| < 2\eta|w'|.$$

□

Lemma 4.10.23. *Let Q be a quadratic form on $\Lambda_{\mathbb{R}}$ and $W \in \operatorname{Hom}(\Lambda, \mathbb{C})$. The following are equivalent*

1. *There exists a real number $C_{Q,W} > 0$ such that $C_{Q,W}||v|| \leq |W(v)|$ for all $v \in \Lambda_{\mathbb{R}} \setminus \{0\}$ that satisfy $Q(v) \geq 0$.*

2. Q is negative definite on $\ker(W)$.

Proof. To see that 1 implies 2 consider $Q(v) \geq 0$ and $v \neq 0$, we have $0 < C_{Q,W}||v|| \leq |W(v)|$ such that $W(v) \neq 0$, hence $Q(v) < 0$ for all non-zero $v \in \ker(W)$.

To see that 2 implies 1 we use the fact that $\operatorname{rank}(\Lambda)$ is finite which gives us that the unit sphere, and therefore the set $K = \{v \mid Q(v) \geq 0, ||v|| = 1\} \subset \Lambda_{\mathbb{R}}$ is compact. Since we assumed Q to be negative definite on $\ker(W)$, K is also disjoint to $\ker(W)$, hence the function $f(v) := |W(v)|$ is positive and continuous on K and therefore attains its (positive) minimum – which we now denote by $C_{Q,W}$. We obtain $C_{Q,W}||v|| \leq |W(v)|$ for $v \in K$ and hence, by the linearity of W, for all $v \in \Lambda_{\mathbb{R}} \setminus \{0\}$ that satisfy $Q(v) \geq 0$. Since $C_{Q,W} > 0$ we are – indeed – finished. □

Definition 4.10.24. Let $W \in \operatorname{Hom}(\Lambda, \mathbb{C})$. Define

$$||W||_\infty := \max\{|W(v)| \mid v \in \Lambda_{\mathbb{R}}, ||v|| = 1\}.$$

The topology on $\operatorname{Hom}(\Lambda, \mathbb{C})$ is given by this norm. We now use lemma 4.10.22 to prove the following.

Lemma 4.10.25. *Given a quadratic form Q on $\Lambda_{\mathbb{R}}, W \in \operatorname{Hom}(\Lambda, \mathbb{C})$ such that $Q\big|_{\ker(W)} < 0$ and $C_{Q,W} > 0$ a corresponding constant as in lemma 4.10.23. Then, for all $\epsilon \in (0,1)$ and all $W' \in \operatorname{Hom}(\Lambda, \mathbb{C})$ that satisfy the inequality $||W - W'||_\infty < \sin(\pi\epsilon)\frac{C_{Q,W}}{2}$ and all non-zero $v \in \Lambda_{\mathbb{R}}$ that satisfy $Q(v) \geq 0$ we have*

$$|W(v) - W'(v)| < \sin(\pi\epsilon)|W'(v)|.$$

Proof. The inequality $||W - W'||_\infty < \sin(\pi\epsilon)\frac{C_{Q,W}}{2}$ implies the inequality

$$|W(v) - W'(v)| < \sin(\pi\epsilon)\frac{C_{Q,W}}{2}||v|| \tag{4.83}$$

for all $v \in \Lambda_{\mathbb{R}} \setminus \{0\}$. Lemma 4.10.23 provides us with $C_{Q,W}||v|| \leq |W(v)|$ for all $v \in \Lambda_{\mathbb{R}} \setminus \{0\}$ that satisfy $Q(v) \geq 0$, such that – since the quantities in question are positive – the inequality (4.83) extends to

$$|W(v) - W'(v)| < \sin(\pi\epsilon)\frac{C_{Q,W}}{2}||v|| \leq \sin(\pi\epsilon)\frac{|W(v)|}{2}.$$

To this, we apply lemma 4.10.22 where we let $w = W(v), w' = W'(v)$ and $\eta = \frac{\sin(\pi\epsilon)}{2}$. We obtain $|W(v) - W'(v)| < \sin(\pi\epsilon)|W'(v)|$ for all $v \in \Lambda_{\mathbb{R}} \setminus \{0\}$ that satisfy $Q(v) \geq 0$. □

Corollary 4.10.26. *Given a quadratic form Q on $\Lambda_{\mathbb{R}}, W \in \mathrm{Hom}(\Lambda, \mathbb{C})$ such that $Q\big|_{\ker(W)} < 0$ and $C_{Q,W} > 0$ a corresponding constant as in lemma 4.10.23. Suppose $\sigma' \in \mathrm{Stab}(\mathcal{D}^{\uparrow})$ satisfies the support property with respect to Q, that $\epsilon \in (0,1)$ and that $W' := \mathcal{Z}(\sigma') \in \mathrm{Hom}(\Lambda, \mathbb{C})$ satisfies the inequality $||W - W'||_{\infty} < \sin(\pi\epsilon)\frac{C_{Q,W}}{2}$, then*

$$||W - W'||_{\sigma'} \leq \sin(\pi\epsilon).$$

Proof. If $E \in \mathcal{D}^{\uparrow}$ is σ'-semistable then $Q(E) \geq 0$ since σ' satisfies the support property with respect to Q. Therefore we can apply lemma 4.10.25 to obtain

$$|W(v) - W'(v)| < \sin(\pi\epsilon)|W'(v)|$$

for all σ'-semistable objects E. This implies that $||W - W'||_{\sigma'} \leq \sin(\pi\epsilon)$. □

We are now ready to provide the crucial proposition 4.10.27 following.

Proposition 4.10.27. *Let $\mathcal{A} = \mathrm{Coh}(C)$, for an elliptic curve C. The map*

$$\begin{aligned} \pi_0 : \mathcal{V}_{12} &\to \mathcal{L}_{12} \\ \sigma &\mapsto \sigma_1 \end{aligned}$$

is a homeomorphism.

Proof. Since we have seen in lemma 4.10.7, that π is a local homeomorphism, it remains to prove that π_0 is bijective. To prove injectivity, consider $\sigma = (Z_1, \mathcal{H}_1), \tau = (Z_2, \mathcal{H}_2) \in \mathcal{V}_{12}$ and $\sigma_1 \in \mathrm{Stab}(\mathcal{D})$ with $\pi(\sigma) = \sigma_1 = \pi(\tau)$. If $0 \leq f(0)$, lemma 4.10.11 provides $\sigma, \tau \in \Theta_{12}$ and, by lemma 4.8.13, we now obtain $\sigma = \tau$. If $-1 < f(0) < 0$, then $\Delta(\mathbb{C}(x))$ is σ-stable/τ-stable by lemma 4.10.12. We now obtain $\sigma = \tau$ by lemma 4.8.16 via the assumed equality of their stability functions combined with $d(\mathcal{P}, \mathcal{Q}) < 1$, with $\mathcal{P}, \mathcal{Q}$ being their respective slicings.

We will now prove surjectivity. We distinguish between three cases:

1. $f(0) \geq 0$
2. $f(0) < 0$ and $\mathrm{Discr}(M) \geq 0$
3. $f(0) < 0$ and $\mathrm{Discr}(M) < 0$.

We obtain case 1 from corollary 4.10.16 (giving us that (T, f) is in the image of π_0 for $(T, f) \in \mathcal{L}_{12}, f(0) \geq 0$).

To study case 2 assume that (T, f) satisfies $f(0) < 0$ and $\mathrm{Discr}(M) \geq 0$ resulting in it having real eigenvalues λ_1, λ_2. By definition of $\mathcal{L}_{12}$ we have

$\det(M) > 0$ and $\det(M+I) > 0$ which implies $\lambda_1 \notin \{-1, 0\}$. There are only three possibilities, these are

$$\lambda_1, \lambda_2 < -1,$$
$$-1 < \lambda_1, \lambda_2 < 0$$
$$\text{and } 0 < \lambda_1, \lambda_2.$$

We can hence apply corollary 4.10.21 which concludes the proof of case 2.

To prove the result for case 3 we will use the deformation theorem of Bridgeland. Define

$$\mathcal{L}_{12}^{\Gamma} := \{(T, f) \in \mathcal{L}_{12} \mid f(0) < 0, \mathrm{Discr}(M) < 0\} \subset \mathcal{L}_{12},$$

which is an open subset of $\mathcal{L}_{12}$. Suppose $(T, f) \in \mathcal{L}_{12}^{\Gamma}$ corresponds to $(Z, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$ and define $W = Z \circ \lambda_1 + Z_\mu \circ \rho_2$. Because $f(0) < 0$ gives $1 < \phi_0 < 2$ and $\phi_1 < \phi_0 < \phi_1 + 1$ now implies $0 < \phi_1 < 2$, any $(T, f) \in \mathcal{L}_{12}^{\Gamma}$ is determined by T and also by W.

For $X \in S$ let $\theta_X = \frac{\arg(W(i_1(X)))}{\pi}, \theta'_X = \frac{\arg(W(\Delta(X)))}{\pi}$ implying $\theta_X, \theta'_X \in [0, 2)$. Since $-1 < f(0) < 0$ is equivalent to $1 < \phi_0 = f^{-1}(1) < 2$ and $A + Di = m_0 \exp(i\pi\phi_0)$, we obtain $D < 0$ and from $\mathrm{Discr}(M) = (A+C)^2 - 4AB < 0$, we get $B < 0$ also. Now, let $\xi_X := \phi_\mu(X)$, in other words $\xi_{\mathbb{C}(x)} = 1$ and $\xi_{\mathcal{O}_C} = \frac{1}{2}$ and using $[\Delta(X)] = [i_1(X)] + [i_2(X)]$ we obtain

$$\xi_X < \theta'_X < \theta_X < \xi_X + 1 \tag{4.84}$$

for all $X \in S$.

Let $\epsilon > 0$ satisfy

$$\epsilon < \min\{\frac{1}{8}, \xi_X + 1 - \theta_X, \theta_X - \theta'_X, \theta'_X - \xi_X \mid X \in S\}.$$

This implies that the inequalities 4.84 remain invariant under adding or subtracting ϵ to either θ_X or θ'_X. We have seen in the proof of proposition 4.9.36, where we only needed $B, D < 0$, that the quadratic form $Q := d_2 r_1 - d_1 r_2$ is negative definite on $\ker(W)$ for any W that comes from an element $(T, f) \in \mathcal{L}_{12}^{\Gamma}$. Now we assume $C_{Q,W}$ to be a constant as in lemma 4.10.23 that corresponds to this quadratic form Q and the fixed W. There exists $(T', f') \in \mathcal{L}_{12}^{\Gamma}$ such that the entries of the matrix T' are rational and

$$||W - W'||_\infty \le \sin(\pi\epsilon)\frac{C_{Q,W}}{2},$$

where W' is obtained from (T', f') in the same manner in which W was from (T, f). By lemma 4.7.31 in combination with proposition 4.9.36, which applies because $\mathrm{Discr}((T')^{-1}) < 0$ and supplies the support property, there is a $\sigma' \in \mathcal{V}_{12} \subset \mathrm{Stab}(\mathcal{D}^{\uparrow})$ such that $\pi_0(\sigma') = (T', f')$ and σ' satisfies the support property with respect to Q. Since we also have $\mathcal{Z}(\sigma') = W'$, we obtain by corollary 4.10.26 that

$$||W - W'||_{\sigma'} \leq \sin(\pi\epsilon) \tag{4.85}$$

and from Bridgeland's deformation theorem, see [7, Theorem B3], we obtain that there is a $\sigma \in \mathrm{Stab}(\mathcal{D}^{\uparrow})$ such that $\mathcal{Z}(\sigma) = W$ and $d(\sigma, \sigma') < \epsilon$.

We now apply lemma 4.8.35 to σ'. It provides us with $i_1(X), i_2(X)$ and $\Delta(X)$ being σ'-stable for $X \in S$. Using this we will now prove that in this situation $i_1(X), i_2(X), \Delta(X)$ are σ-stable for $X \in S$. By our choice of ξ_X, θ_X and θ'_X, there exist integers k, m, n such that $\phi_\sigma(i_1(X)) = \theta_X + 2k, \phi_\sigma(i_2(X)) = \xi_X + 2m$ and $\phi_\sigma(\Delta(X)) = \theta'_X + 2n$ whenever these objects are σ-stable.

(a) Suppose $\Delta(X)$ is not σ-stable. We now let $\theta'_X = \frac{\arg(W'(\Delta(X)))}{\pi} \in [0, 2)$ (without changing θ_X and ξ_X). Hence, $\phi_{\sigma'}(\Delta(X)) = \theta'_X + 2n$ and moreover, by (4.85), θ'_X differs from the argument of $W(\Delta(X))$ by no more then ϵ such that the inequality (4.84) holds true. Via the HNF/JHF (corollary 4.5.21 and lemma 4.5.31) we get $\phi_+ = \phi_\sigma(i_2(X)) \geq \phi_\sigma(i_1(X)) = \phi_-$, which translates into $\phi_+ = \xi_X + 2m \geq \phi_- = \theta_X + 2k > \xi_X + 2k$, using (4.84), such that we obtain $m > k$. On the other hand, we have

$$\epsilon > d(\sigma, \sigma') \geq |\phi_{\sigma'}(\Delta(X)) - \phi_+| = |2(n - m) + \theta'_X - \xi_X|.$$

Since we have $\theta'_X - \xi_X \in (0, 1)$ by (4.84), this yields $m = n$. Similarly, we obtain from

$$\epsilon > d(\sigma, \sigma') \geq |\phi_- - \phi_{\sigma'}(\Delta(X))| = |2(k - n) + \theta_X - \theta'_X|,$$

in combination with $\theta_X - \theta'_X \in (0, 1)$ by (4.84), that $k = n$. Hence $k = n = m$, which contradicts $m > k$.

(b) Suppose $i_1(X)$ is not σ-stable. We proceed similar as before, now letting $\theta_X = \frac{\arg(W'(i_1(X)))}{\pi} \in [0, 2)$ and obtain, using HNF/JHF that $n > m$ on one hand and using $(\theta_X - \theta'_X), (\xi_X + 1 - \theta_X) \in (0, 1)$, that $m = k = n$ on the other.

(c) Suppose $i_2(X)$ is not σ-stable. We – again – proceed as before showing that $k > n$. However, in this case we use $\phi_{\sigma'}(i_2(X)) = \phi_\mu(X) = \xi_X$ and

$$\phi_{\sigma'}(i_2(X)) - \phi_+ = \xi_X - \theta_X - 2k + 1 \in (-2k, -2k + 1)$$
$$\text{as well as } \phi_- - \phi_{\sigma'}(i_2(X)) = \theta'_X - \xi_X + 2n \in (2n, 2n + 1)$$

such that $k = 0 = n$.

This concludes our investigation regarding the stability of the three embeddings under the given particular circumstances. We have proved that in this situation $i_1(X), i_2(X), \Delta(X)$ are σ-stable for $X \in S$.

Since $\sigma \in S_{12}$ and $\Delta(X)$ is σ-stable for $X \in S$, we have

$$\phi_\sigma(i_2(X)) < \phi_\sigma(\Delta(X)) < \phi_\sigma(i_1(X)) < \phi_\sigma(i_2(X)) + 1 \tag{4.86}$$

for all $X \in S$. Since $\mathcal{Z}(\sigma) = W$ for any $X \in S$, the difference of $\phi_\sigma(X)$ and the corresponding parameter value ϕ_i of (T, f) is always an even integer. Without loss of generality we can assume $\phi_\sigma(i_2(\mathbb{C}(x))) = 1$ and $\phi_\sigma(i_2(\mathcal{O}_C)) = 1/2$. This is possible because $W(i_2(\mathbb{C}(x))) = -1$ and $W(i_2(\mathcal{O}_C)) = i$ and $\phi_\sigma(i_2(\mathcal{O}_C)) < \phi_\sigma(i_2(\mathbb{C}(x))) < \phi_\sigma(i_2(\mathcal{O}_C)) + 1$ which is why we have $\phi_\sigma(i_2(\mathbb{C}(x))) = n$ and $\phi_\sigma(i_2(\mathcal{O}_C)) = n - 1/2$ and can apply a suitable shift – corresponding to an element of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ acting on the stability condition – if necessary. By (4.86) we obtain

$$\begin{aligned} 1 = \phi_\sigma(i_2(\mathbb{C}(x))) < \phi_\sigma(\Delta(\mathbb{C}(x))) < \phi_\sigma(i_1(\mathbb{C}(x))) < \phi_\sigma(i_2(\mathbb{C}(x))) + 1 = 2 \\ \frac{1}{2} = \phi_\sigma(i_2(\mathcal{O}_C)) < \phi_\sigma(\Delta(\mathcal{O}_C)) < \phi_\sigma(i_1(\mathcal{O}_C)) < \phi_\sigma(i_2(\mathcal{O}_C)) + 1 = \frac{3}{2}, \end{aligned}$$

so in particular $\phi_\sigma(i_1(\mathbb{C}(x))), \phi_\sigma(i_2(\mathbb{C}(x))), \phi_\sigma(i_1(\mathcal{O}_C)), \phi_\sigma(i_2(\mathcal{O}_C)) \in (0, 2)$. Therefore they have to agree with $\phi_0, \phi_1, \phi_2 = 1$ and $\phi_3 = 1/2$ respectively, which shows $\sigma \in \mathcal{V}_{12}$ and $\pi_0(\sigma) = (T, f)$. This proves that π_0 is surjective and therefore the proof is finished. □

Corollary 4.10.28. *Let $\mathcal{A} = \mathrm{Coh}(C)$, for an elliptic curve C. The map*

$$\begin{aligned} \pi : S_{12} &\to \mathcal{P}_{12} \\ \sigma &\mapsto (\sigma_1, \sigma_2). \end{aligned}$$

is a homeomorphism.

Proof. This is an implication of proposition 4.10.27. □

Lemma 4.10.29. *The pairs $\sigma = (Z', \mathcal{H})$ constructed by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ and the pairs $\sigma = (Z, \mathcal{H}(C_1, D_1))$ given in lemma 4.7.31 with $f^{-1}(\frac{1}{2}) < \frac{3}{2}$ are Bridgeland stability conditions.*

Proof. We have seen that whenever $\sigma_1 = (T_1, f_1) = (Z_1, \mathcal{H}_1) \in \mathrm{Stab}(\mathcal{D})$ and σ_2 are discrete, we obtain a Bridgeland stability condition by applying proposition 4.9.28 after [21, Proposition 3.5]. Without loss of generality we assume $\sigma_2 = \sigma_\mu$. Let σ be a pair obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from σ_1 and $\sigma_2 = \sigma_\mu$. As $f_1(0) \geq 0$, we get $(\sigma_1, \sigma_\mu) \in \mathcal{L}_{12}$. Therefore, there is a stability condition $\tau \in \mathcal{V}_{12}$ such that $\pi(\tau) = (\sigma_1, \sigma_\mu)$. By lemma 4.10.11, we obtain $\tau \in \Theta_{12}$ and by Lemma 4.8.13, we have that τ is a pair obtained by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ from σ_1 and σ_2. As a consequence the pair $(Z, \mathcal{H})$ gives a Bridgeland stability condition.

Let $\sigma = (Z_r, \mathcal{A}_r)$. If we consider σ_1 as in lemma 4.7.31 with $f^{-1}(\frac{1}{2}) < \frac{3}{2}$, we obtain $-1 < f_1(0) < 0$ and since we assumed $\det(M_1 + I) > 0$, we have that $(\sigma_1, \sigma_\mu) \in \mathcal{L}_{12}$. Therefore, a stability condition $\tau \in \mathcal{V}_{12}$ such that $\pi(\tau) = (\sigma_1, \sigma_2)$ exists. By lemma 4.10.12, we have that $\Delta(\mathbb{C}(x))$ is τ-stable. Hence, lemma 4.8.16 gives that τ is given by the construction of lemma 4.7.31. Then $\tau = (Z_r, \mathcal{A}_r)$ and as a consequence is a Bridgeland stability condition. □

Lemma 4.10.30. *Let $\mathcal{A} = \mathrm{Coh}(C)$ for an elliptic curve C. Let $\sigma \in \mathrm{pre\,Stab}$ and assume that there is a $g \in \mathrm{GL}_2^+(\mathbb{R})$ such that σg is constructed by CP-gluing via $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ or one of the kind of lemma 4.7.31, then σ is a Bridgeland stability condition.*

Proof. Let $\sigma \in \mathrm{pre\,Stab}$ such that there is a $g \in \mathrm{GL}_2^+(\mathbb{R})$ of the kind of lemma 4.7.31 such that $f^{-1}(\frac{1}{2}) \geq \frac{3}{2}$. In this situation either $i_1(\mathcal{O}_C)$ or $i_2(\mathcal{O}_C)$ is not σ-stable, in other words, $\sigma \in S_{23}$ or $\sigma \in S_{31}$. If $\phi_\sigma(\Delta(\mathcal{O}_C)) > \frac{3}{2}$ we obtain that $i_2(\mathcal{O}_C)$ and $\Delta(\mathcal{O}_C)$ are σ-stable. On the other hand, $\phi_\sigma(\Delta(\mathcal{O}_C)) < \frac{1}{2}$ implies that $i_1(\mathcal{O}_C)$ and $\Delta(\mathcal{O}_C)$ are σ-stable. All of this is due to non-vanishing morphisms and HNFs/JHFs connected to the canonical exact triangle

$$i_2(\mathcal{O}_C) \to \Delta(\mathcal{O}_C) \to i_2(\mathcal{O}_C) \xrightarrow{+} i_2(\mathcal{O}_C)[1]$$

in the usual way. □

Theorem 4.10.31. *Let $\mathcal{A} = \mathrm{Coh}(C)$, for an elliptic curve C. The space of stability conditions $\mathrm{Stab}(\mathcal{D}^\uparrow) = S_{12} \cup S_{23} \cup S_{31}$ is a connected, four dimensional complex manifold.*

Proof. Note that $\Theta'_{ij} \subset S_{ij}$, where Θ'_{ij} is as in corollary 4.5.30 (it should also be observed that $\Theta'_{ij} = \Theta_{ij}$ by theorem 4.9.37). Since $\mathcal{V}_{12}$ is connected, S_{12} is connected too. As, moreover, $S_{12} \cap S_{23} = S_{23} \cap S_{31} = S_{12} \cap S_{31}$ is not empty, $\mathrm{Stab}(\mathcal{D}^\uparrow)$ is indeed connected. □

5 More on $\mathcal{D}^\uparrow$

This section aims at the investigation of questions that have to do with the category $\mathcal{D}^\uparrow$ and are all – to some degree – related to the discussion of the space $\mathrm{Stab}(\mathcal{D}^\uparrow)$ in the previous chapter.

5.1 Shape of recollement t-structures

Subsection 4.3 saw the introduction of new t-structures via the technique of recollement – the question remains however, how the resulting t-structures in fact "look like", by which is meant that the question arises, if a more intuitive understanding of them could be obtained that exceeds their purely abstract functorial description. This – however – can only mean that we must try to understand how the involved objects look like in terms of their usual cohomology which will hence be the objective of this subsection.

In order to investigate the t-structures, we hence introduce the following.

Definition 5.1.1. By $\tau^{\leq 0}_{1,\alpha,\beta}$ and $\tau^{\geq 1}_{1,\alpha,\beta}$ denote the truncation functors associated to the t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ defined in 4.3.19.

In order to conduct our investigation we also need to introduce the following.

Definition 5.1.2. We define an "isomorphic arrow" in $\mathcal{D}^{\uparrow}$ (which will be denoted by $A \overset{\cong}{\to} B$) to be an object $X \in \mathcal{D}^{\uparrow}$ for which $X \cong \Delta(B)$ for a $B \in \mathcal{D}$.

We will now – broken down into a series of lemmas – provide the explanation for the fact that the definition of an object to be an isomorphic arrow in 5.1.2 matches the intuition (that is being an isomorphism in $\mathcal{D}$) and that therefore the name is well chosen.

Lemma 5.1.3. *Let $E \in \mathcal{D}^{\uparrow}$ such that E, viewed as a morphism in $\mathcal{C}(\mathcal{A})$ is a quasi-isomorphism. Then and only then, E is an isomorphic arrow.*

Proof. First, if we have $E \in \mathcal{D}^{\uparrow}$ such that $E \in \operatorname{im}(i_1)$, then $E \cong (E_1 \to 0)$ and hence $\rho_2(E) = \rho_2(E_1 \to 0) = 0$ which implies $E \in \ker(\rho_2)$. If – on the other hand – we have $E \in \ker(\rho_2)$ then $\rho_2(E) = 0$ and hence $E \cong (E_1 \to 0)$ for an $E_1 \in \mathcal{D}$. In other words $E \in \operatorname{im}(i_1)$, which means $\operatorname{im}(i_1) = \ker(\rho_2)$. Since $\operatorname{im}(\Delta)$ is left admissible by lemma 4.2.2, we obtain $\operatorname{im}(\Delta) = ({}^{\perp}\operatorname{im}(\Delta))^{\perp}$ by lemma 4.2.4. Furthermore, we obtain ${}^{\perp}\operatorname{im}(\Delta) = \ker(\rho_2)$ by lemma 4.2.10 via the fact that $\rho_2 \dashv \Delta$. Similarly we obtain $(\operatorname{im}(i_1))^{\perp} = \ker(\mathbb{K})$ from lemma 4.2.11. Hence, we have

$$\operatorname{im}(\Delta) = ({}^{\perp}\operatorname{im}(\Delta))^{\perp} = (\ker(\rho_2))^{\perp} = (\operatorname{im}(i_1))^{\perp} = \ker(\mathbb{K}).$$

Using the exact triangle

$$\mathbb{K}(E) \to \lambda_1(E) \overset{\mu_E}{\to} \rho_2(E) \overset{+}{\to}$$

we obtain that $\mathbb{K}(E) = 0$ if and only if μ_E is an isomorphism in $\mathcal{C}(\mathcal{A})$. The equality $\mathbb{K}(E) = 0$ is just another way of saying that $E \in \ker(\mathbb{K})$ and via $\ker(\mathbb{K}) = \operatorname{im}(\Delta)$ we hence obtain $E \in \operatorname{im}(\Delta)$ if and only if μ_E is an isomorphism in $\mathcal{C}(\mathcal{A})$. The application of corollary 4.3.4 finishes the proof. □

Remark 5.1.4. Note that, as stated at the end of the proof of 5.1.3, an important consequence is that μ_Z, viewed as an arrow in $\mathcal{C}(\mathcal{A})$ is an isomorphism if and only if $\mathbb{K}(Z)$ vanishes.

We will finally require the following crucial fact on isomorphic arrows.

Corollary 5.1.5. *Assume that $A, F \in \mathcal{D}$, then*

$$\operatorname{Hom}(i_1(A), \Delta(F)) = 0.$$

Proof. This is a special case of lemma 4.1.4. □

Definition 5.1.6. We define a "monomorphic arrow" in the abelian category $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$ to be an object $X \in \mathcal{D}^{\uparrow}$ such that we have $X \cong \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$ and $\tau^{\leq 0}_{1,\alpha,\alpha}(i_1(\mathbb{K}(X))) = 0$ where $\alpha \in \mathbb{R}$.

The following definition will only be required at a later stage. However we will provide it at this point to ensure completion.

Definition 5.1.7. We define an "epimorphic arrow" in the abelian category $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$ to be an object $X \in \mathcal{D}^{\uparrow}$ such that we have $X \cong \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$ and $\tau^{\geq 1}_{1,\alpha,\alpha}(i_1(\mathbb{K}(X)[1])) = 0$ where $\alpha \in \mathbb{R}$.

Remark 5.1.8. Note that the term "isomorphic arrow" appears in the context of the derived category $\mathcal{D}^{\uparrow}$, in contrast to both the terms "monomorphic/epimorphic arrow" that, even when ordinarily defined for objects in $\mathcal{D}^{\uparrow}$, belong to the abelian situation of the category $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$. Note that for $\alpha \in \mathbb{Z}$, the above definition is coherent with the vanishing of kernel and cokernel respectively and therefore generalises the concept from $\mathcal{A}$.

The term "isomorphic arrow" is particularly well-behaved and turns out to work well under the transition to the abelian case without the need to change language. To illustrate this – and to hint at the difference between the terminology – we will include the following corollary.

Corollary 5.1.9. *If we assume $X \in \mathcal{D}^{\uparrow}$ to be an isomorphic arrow in $\mathcal{D}^{\uparrow}$, then $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$, considered as a morphism $A \xrightarrow{f} B$ is an isomorphism in $\mathcal{D}$ and hence in $\tau^{\leq 0}_{\alpha}(\tau^{\geq 1}_{\alpha+1}(\mathcal{D}))$ for any $\alpha \in \mathbb{R}$.*

Proof. As part of the proof of theorem 2.5.33, we proved, that $\tau_{\geq 0}(\mathrm{Cone}(f))$ is the cokernel, and $(\tau_{\leq -1}(\mathrm{Cone}(f)))[-1]$ the kernel of a morphism $A \xrightarrow{f} B$. Hence $\mathrm{Cone}(f) = 0$ implies

$$\begin{aligned} \mathrm{coker}(f) = \tau_{\geq 0}(\mathrm{Cone}(f)) = \tau_{\geq 0}(0) = 0 \\ = (\tau_{\leq -1}(0))[-1] = (\tau_{\leq -1}(\mathrm{Cone}(f)))[-1] = \ker(f). \end{aligned}$$

Since $X = A \xrightarrow{f} B$ being an isomorphic arrow in $\mathcal{D}^{\uparrow}$ implies $\mathrm{Cone}(f) = 0$, this finishes the proof. □

We will now, broken down into the following series of lemmas prove the theorem that fully characterises the $\mathcal{D}^{\geq 1}$ (and hence – since the $\mathcal{D}^{\leq 0}$ is known – the t-structure) for type-1-recollement-data.

Lemma 5.1.10. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha < \beta$. For an object $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ we have $\tau^{\leq 0}_{1,\beta,\beta}(X) = 0$.*

Proof. Since $\alpha < \beta$, we have $\mathcal{D}^{\leq 0}_{1,\beta,\beta} \subset \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$. Using $i^{\leq 0}_{1,\beta,\beta} \dashv \tau^{\leq 0}_{1,\beta,\beta}$, we obtain

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i^{\leq 0}_{1,\beta,\beta}(\tau^{\leq 0}_{1,\beta,\beta}(X)), X) \cong \mathrm{Hom}_{\mathcal{D}^{\leq 0}_{1,\beta,\beta}}(\tau^{\leq 0}_{1,\beta,\beta}(X), \tau^{\leq 0}_{1,\beta,\beta}(X)).$$

Again, we have $\alpha < \beta$, the inclusion $\mathcal{D}^{\leq 0}_{1,\beta,\beta} \subset \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$, introduced above provides us with $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta})^{\perp} \subset (\mathcal{D}^{\leq 0}_{1,\beta,\beta})^{\perp}$ and on the other hand we have both $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta})^{\perp} = \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ and $(\mathcal{D}^{\leq 0}_{1,\beta,\beta})^{\perp} = \mathcal{D}^{\geq 1}_{1,\beta,\beta}$ which means $\mathcal{D}^{\geq 1}_{1,\alpha,\beta} \subset \mathcal{D}^{\geq 1}_{1,\beta,\beta}$. Since $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$, we therefore also have $X \in \mathcal{D}^{\geq 1}_{1,\beta,\beta}$. Hence, we obtain the equation

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_{1,\beta,\beta}(\tau^{\leq 0}_{1,\beta,\beta}(X)), X) \subset \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\mathcal{D}^{\leq 0}_{1,\beta,\beta}, \mathcal{D}^{\geq 1}_{1,\beta,\beta}) = 0.$$

Hence $\mathrm{Hom}_{\mathcal{D}^{\leq 0}_{1,\beta,\beta}}(\tau^{\leq 0}_{1,\beta,\beta}(X), \tau^{\leq 0}_{1,\beta,\beta}(X)) = 0$ which means that $\tau^{\leq 0}_{1,\beta,\beta}(X) = 0$. □

Lemma 5.1.11. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha < \beta - 1$. For an object $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ we have $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X)$ is an isomorphic arrow.*

Proof. For $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ let $Z = \tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X)$ and consider the exact triangle

$$i_1\mathbb{K}(Z) \to Z \to \Delta\rho_2(Z) \xrightarrow{+} . \tag{5.1}$$

It is our aim to prove that $i_1\mathbb{K}(Z) = 0$, as this would imply $Z \cong \Delta\rho_2(Z)$, which – by definition – would make Z an isomorphic arrow . Since $i_1 \dashv \mathbb{K}$, and $\mathbb{K} \circ \Delta = 0$ we obtain

$$\mathrm{Hom}(i_1\mathbb{K}(Z), \Delta\rho_2(W)) = \mathrm{Hom}(\mathbb{K}(Z), \mathbb{K}\Delta\rho_2(W)) = \mathrm{Hom}(\mathbb{K}(Z), 0) = 0$$

for any $W \in \mathcal{D}^{\uparrow}$. Letting $W = Z$ and applying $\mathrm{Hom}(i_1\mathbb{K}(Z), -)$ to the exact triangle (5.1), we obtain

$$\mathrm{Hom}(i_1\mathbb{K}(Z), i_1\mathbb{K}(Z)) \cong \mathrm{Hom}(i_1\mathbb{K}(Z), Z).$$

Hence, we need to prove that $\mathrm{Hom}(i_1\mathbb{K}(Z), Z) = 0$. Consider the exact triangle

$$\tau^{\leq -1}_{1,\alpha,\alpha}(X) \to X \to \tau^{\geq 0}_{1,\alpha,\alpha}(X) \xrightarrow{+} \tag{5.2}$$

and note that $Z = \tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X) = \tau^{\leq -1}_{1,\alpha,\alpha}(X)$. Note additionally that from $\tau^{\geq 0}_{1,\alpha,\alpha}(X) \in \mathcal{D}^{\geq 0}_{1,\alpha,\alpha}$ we obtain

$$\tau^{\geq 0}_{1,\alpha,\alpha}(X)[-1] \in \mathcal{D}^{\geq 0}_{1,\alpha,\alpha}[-1] = \mathcal{D}^{\geq 1}_{1,\alpha,\alpha}.$$

Applying $\mathrm{Hom}(i_1\mathbb{K}(Z), -)$ to the exact triangle (5.2), we obtain the exact sequence

$$\mathrm{Hom}(i_1\mathbb{K}(Z), \tau^{\geq 0}_{1,\alpha,\alpha}(X)[-1]) \to \mathrm{Hom}(i_1\mathbb{K}(Z), Z) \to \mathrm{Hom}(i_1\mathbb{K}(Z), X).$$

Hence, in order for $\mathrm{Hom}(i_1\mathbb{K}(Z), Z)$ to vanish, we require

$$\mathrm{Hom}(i_1\mathbb{K}(Z), \tau^{\geq 0}_{1,\alpha,\alpha}(X)[-1]) = 0 = \mathrm{Hom}(i_1\mathbb{K}(Z), X).$$

Since $\tau^{\geq 0}_{1,\alpha,\alpha}(X)[-1] \in \mathcal{D}^{\geq 1}_{1,\alpha,\alpha}$ and $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$, this will be true if we had both $i_1\mathbb{K}(Z) \in \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$ and $i_1\mathbb{K}(Z) \in \mathcal{D}^{\leq 0}_{1,\alpha,\alpha}$. But, since $\alpha < \beta$, we have $\mathcal{D}^{\leq 0}_{1,\alpha,\beta} \subset \mathcal{D}^{\leq 0}_{1,\alpha,\alpha}$ and – hence – it suffices to prove that $i_1\mathbb{K}(Z) \in \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$. Recall that

$$\mathcal{D}^{\leq 0}_{1,\alpha,\beta} = \{Y \in \mathcal{D}^{\uparrow} | \lambda_1(Y) \in \mathcal{D}^{\leq 0}_{\alpha}, \rho_2(Y) \in \mathcal{D}^{\leq 0}_{\beta}\}.$$

We hence need to examine the image of $i_1\mathbb{K}(Z)$ under the functors ρ_2 and λ_1. Since $\rho_2 \circ i_1 = 0$, we have $\rho_2 i_1\mathbb{K}(Z) = 0 \in \mathcal{D}^{\leq 0}_{\beta}$. It remains to prove that $\lambda_1 i_1\mathbb{K}(Z) \in \mathcal{D}^{\leq 0}_{\alpha}$. First note, that by lemma 3.2.18 we have i_1 fully faithful. This, combined with the fact that $\lambda_1 \dashv i_1$, implies $\lambda_1 \circ i_1 \xrightarrow{\cong} \mathrm{id}_{\mathcal{D}}$. This means that $\lambda_1 i_1\mathbb{K}(Z) \cong \mathbb{K}(Z)$. Consider the exact triangle

$$\Delta\rho_2(Z)[-1] \to i_1\mathbb{K}(Z) \to Z \xrightarrow{+}.$$

Applying the (exact) functor λ_1 now gives the exact triangle

$$\lambda_1\Delta\rho_2(Z)[-1] \to \mathbb{K}(Z) \to \lambda_1(Z) \xrightarrow{+}.$$

Since $Z = \tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X)$ we have $\lambda_1(Z) \in \mathcal{D}^{\leq 0}_{\alpha+1} \subset \mathcal{D}^{\leq 0}_{\alpha}$. By lemma 3.2.28 we have Δ fully faithful. This, combined with the fact that $\Delta \dashv \lambda_1$, implies

$\mathrm{id}_{\mathcal{D}} \xrightarrow{\cong} \lambda_1 \circ \Delta$. This means that $\lambda_1 \Delta \rho_2(Z)[-1] \cong \rho_2(Z)[-1]$. Therefore we have

$$\begin{aligned}\lambda_1 \Delta \rho_2(Z)[-1] \cong \rho_2(Z)[-1] = \rho_2(\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X))[-1] \in \rho_2(\mathcal{D}^{\leq 0}_{1,\alpha+1,\alpha+1})[-1] \\ = \mathcal{D}^{\leq 0}_{\alpha+1}[-1] = \mathcal{D}^{\leq 0}_{\alpha}.\end{aligned}$$

Since $\mathcal{D}^{\leq 0}_{\alpha}$ is extension closed, we obtain $\mathbb{K}(Z) \in \mathcal{D}^{\leq 0}_{\alpha}$. With this, our proof is finished. □

Lemma 5.1.12. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha < \beta$. For an object $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ we have that $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$ is a monomorphic arrow in $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$.*

Proof. Let $Y \in \mathcal{D}^{\uparrow}$ fulfil $Y = \tau^{\leq 0}_{1,\alpha,\alpha}(i_1(\mathbb{K}(X)))$. Hence, we have chosen Y such that $Y \in \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$. By lemma 5.1.11, there is an $F \in \mathcal{D}^{\uparrow}$ such that

$$\Delta(\rho_2(F)) \to X \to \tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X) \xrightarrow{+} \tag{5.3}$$

is an exact triangle. Since $\rho_2(Y) = 0$ we have $Y \cong i_1(\lambda_1(Y))$ and hence, using corollary 5.1.5, we obtain

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, \Delta(\rho_2(F))) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(\lambda_1(Y)), \Delta(\rho_2(F))) = 0.$$

Hence, applying $\mathrm{Hom}(Y, -)$ to the exact triangle (5.3), provides us with

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, \tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)) \cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, X) \subset \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta}) = 0.$$

We know from $\alpha < \beta$ that $\mathcal{D}^{\leq 0}_{1,\alpha,\beta} \subset \mathcal{D}^{\leq 0}_{1,\alpha,\alpha}$ and hence $Y \in \mathcal{D}^{\leq 0}_{1,\alpha,\alpha}$. Hence, we can use the adjunction $(i^{\leq 0}_{1,\alpha,\alpha}, \tau^{\leq 0}_{1,\alpha,\alpha})$ to obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) &\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i^{\leq 0}_{1,\alpha,\alpha}(Y), \tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)) \\ &\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, \tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)) = 0.\end{aligned}$$

If we let $X = A \xrightarrow{f} B$, we have $Y = \tau^{\leq 0}_{1,\alpha,\alpha}(i_1(\mathbb{K}(X)))$ and $\mathbb{K}(X)$ is given by the complex $\mathrm{Cone}(f)[-1] = B[-1] \oplus A$ with p the canonical projection associated to $\mathrm{Cone}(f)$, we obtain that $p[-1]\big|_{\lambda_1(Y)} = p[-1]\big|_{\lambda_1(\tau^{\leq 0}_{1,\alpha,\alpha}(i_1(\mathbb{K}(X))))} = p[-1]\big|_A = 0$ implies $Y = 0$. Since there is a morphism $(p[-1], 0)$ from $i_1(\mathbb{K}(X))$ to X, we obtain $(p[-1]\big|_{\lambda_1(Y)}, 0) \in \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)))$. This implies $p[-1]\big|_{\lambda_1(Y)} = 0$ and, therefore, $Y = 0$ which finishes the proof. □

Lemma 5.1.13. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\beta - 1 \leq \alpha < \beta$. For an object $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ we have $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\beta,\beta}(X))$ is a monomorphic arrow in $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$.*

Proof. The arguments that were used in the proof of lemma 5.1.12 can be completely adapted to prove lemma 5.1.13, replacing $\tau^{\geq 1}_{1,\alpha+1,\alpha+1}$ with $\tau^{\geq 1}_{1,\beta,\beta}$ everywhere throughout the proof where it is applied to an object (X or Y). □

Lemma 5.1.14. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha < \beta$. For an object $X \cong \tau^{\leq 0}_{1,\beta,\beta}(X) \in \mathcal{D}^{\uparrow}$ and $X = 0$ we have $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$.*

Proof. This is obvious. □

Lemma 5.1.15. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha < \beta$. For an object $X \cong \tau^{\geq 1}_{1,\beta,\beta}(\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X)) \in \mathcal{D}^{\uparrow}$ and X an isomorphic arrow, we have $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$.*

Proof. Let $Y \in \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$. Hence, regarding Y as an arrow $Y_1 \xrightarrow{f} Y_2$ we obtain

$$\begin{aligned} \rho_2(\tau^{\geq 1}_{1,\beta,\beta}(Y)) &= \rho_2(\tau^{\geq 1}_{\beta}(Y_1) \overset{\tau^{\geq 1}_{\beta}(f)}{\to} \tau^{\geq 1}_{\beta}(Y_2)) \\ &= \tau^{\geq 1}_{\beta}(Y_2) = \tau^{\geq 1}_{\beta}(\rho_2(Y)). \end{aligned} \tag{5.4}$$

Since $Y \in \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$ we have $\rho_2(Y) \in \mathcal{D}^{\leq 0}_{\beta}$. Therefore $\tau^{\geq 1}_{\beta}(\rho_2(Y)) = 0$. This, combined with (5.4), implies that $\rho_2(\tau^{\geq 1}_{1,\beta,\beta}(Y)) = 0$ and hence we obtain $(\tau^{\geq 1}_{1,\beta,\beta}(Y) \cong i_1(\lambda_1(\tau^{\geq 1}_{1,\beta,\beta}(Y))$. Using $\tau^{\geq 1}_{1,\beta,\beta} \dashv i^{\geq 1}_{1,\beta,\beta}$ and the fact that X is an isomorphic arrow, we obtain for an $F \in \mathcal{D}^{\uparrow}$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, X) &= \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, i^{\geq 1}_{1,\beta,\beta}(X)) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\geq 1}_{1,\beta,\beta}(Y), X) \\ = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(i_1(\tau^{\geq 1}_{1,\beta,\beta}(Y)), \Delta(F)) &= \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\geq 1}_{1,\beta,\beta}(Y), \mathbb{K}(\Delta(F))) \\ &= \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\geq 1}_{1,\beta,\beta}(Y), 0) = 0. \end{aligned}$$

Hence, we finally obtain $X \in (\mathcal{D}^{\leq 0}_{1,\alpha,\beta})^{\perp} = \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$. □

Lemma 5.1.16. *For a t-structure $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ on $\mathcal{D}^{\uparrow}$ assume that $\alpha < \beta - 1$. For an object $X \cong \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)) \in \mathcal{D}^{\uparrow}$ and X a monomorphic arrow in $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$, we have $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$.*

Proof. Let $Y \in \mathcal{D}^{\leq 0}_{1,\alpha,\beta}$. Since $\alpha < \beta$ this implies $Y \cong \tau^{\leq 0}_{1,\alpha,\alpha}(Y)$. We use the adjoint pair of functors $\tau^{\geq 1}_{1,\alpha+1,\alpha+1} \dashv i^{\geq 1}_{1,\alpha+1,\alpha+1}$ to obtain

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y,X) &\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(Y, \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\leq 0}_{1,\alpha,\alpha}(Y), \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\leq 0}_{1,\alpha,\alpha}(Y), i^{\geq 1}_{1,\alpha+1,\alpha+1}(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)))) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\tau^{\leq 0}_{1,\alpha,\alpha}(Y)), \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(Y)), \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))).
\end{aligned}$$

Since $\alpha < \beta - 1$, we have $\rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(Y)) = 0$. Therefore $g \in \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(Y)), \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)))$ implies that g factors via $i_1(\mathrm{Cone}(f)[-1]) = \mathrm{Cone}(f)[-1] \to 0$ given by the fact that

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(Y)), \Delta(\rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))))) \\
&= \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(Y))), \rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)))) \\
&= \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(0, \rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)))) = 0
\end{aligned}$$

(via $\rho \dashv \Delta$) and the canonical exact triangle

$$\begin{array}{ccccc}
\mathrm{Cone}(f)[-1] & \xrightarrow{p[-1]} & A & \longrightarrow & B \\
\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}_B}\downarrow \quad \xrightarrow{+} \\
0 & \longrightarrow & B & \longrightarrow & B
\end{array}$$

where $(A \xrightarrow{f} B) = \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$ and p the canonical projection associated with $\mathrm{Cone}(f)$. Since X was assumed to be a monomorphic arrow in $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$, we have $\tau^{\leq 0}_{1,\alpha,\alpha}(i_1(\mathbb{K}(X))) = 0$ or, in other words, $\tau^{\leq 0}_{1,\alpha,\alpha}(\mathrm{Cone}(f)[-1]) = 0$ and therefore $g = 0$. This implies that we obtain

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(X,Y) \cong \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(Y)), \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) = 0.$$

Hence, we finally see that $X \in (\mathcal{D}^{\leq 0}_{1,\alpha,\beta})^{\perp} = \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$. □

Lemma 5.1.17. *For a t-structure* $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ *on* $\mathcal{D}^{\uparrow}$ *assume that* $\beta - 1 \leq \alpha < \beta$. *For an object* $X \cong \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\beta,\beta}(X)) \in \mathcal{D}^{\uparrow}$ *and* X *a monomorphic arrow in* $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$, *we have* $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$.

Proof. The arguments that were used in the proof of lemma 5.1.16 can be completely adapted to prove lemma 5.1.17, replacing $\tau^{\geq 1}_{1,\alpha+1,\alpha+1}$ with $\tau^{\geq 1}_{1,\beta,\beta}$ everywhere throughout the proof where it is applied to an object (X or Y). □

Theorem 5.1.18. *For a t-structure* $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ *on* $\mathcal{D}^{\uparrow}$ *assume that* $\alpha < \beta$. *For an object* $X \in \mathcal{D}^{\uparrow}$, *we have* $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ *if and only if*

1. $\tau^{\leq 0}_{1,\beta,\beta}(X) = 0$,

2. $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X)$ *is an isomorphic arrow,*

3. - *if* $\alpha < \beta - 1$, *then* $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$ *is a monomorphic arrow in* $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$,
   - *if* $\beta - 1 \leq \alpha < \beta$, *then* $\tau^{\leq 0}_{1,\alpha,\alpha}(X)$ *is a monomorphic arrow in* $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$,

Proof. For the "if"-part break up X into three objects, using the truncation-functors $\tau^{\leq 0}_{1,\beta,\beta}, \tau^{\leq 0}_{1,\alpha+1,\alpha+1}$ and $\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)$. Apply lemmas 5.1.14, 5.1.15 and 5.1.16/5.1.17, respectively, to these objects and deduce from the fact that $\mathcal{D}^{\geq 1}$ is extension closed that $X \in \mathcal{D}^{\geq 1}$.

For the "only if"-part combine lemmas 5.1.10, 5.1.11 and 5.1.12/5.1.13. □

Remark 5.1.19. Note that the second bullet point of part 3 of theorem 5.1.18 makes sense, because part 1 ensures that $\tau^{\leq 0}_{1,\alpha,\alpha}(X)$ is indeed an object in the abelian category $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$.

We can now say, what hearts we obtain from a t-structure of the kind we have just investigated.

Corollary 5.1.20. *For a t-structure* $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ *on* $\mathcal{D}^{\uparrow}$ *assume that* $\alpha < \beta$. *The heart* $H_{1,\alpha,\beta}$ *of* $(\mathcal{D}^{\leq 0}_{1,\alpha,\beta}, \mathcal{D}^{\geq 1}_{1,\alpha,\beta})$ *is given by the following.*

1. *If* $\alpha < \beta - 2$ *then* $X \in H_{1,\alpha,\beta}$, *if and only if*

   - $\tau^{\leq 0}_{1,\beta+1,\beta+1}(X) = 0$,
   - $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(X))$ *is an isomorphic arrow*
   - $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(\tau^{\geq 1}_{1,\beta,\beta}(X)) = 0$,
   - $\rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)) = 0$ *and*
   - $\tau^{\geq 1}_{1,\alpha,\alpha}(X)) = 0$.

2. *If* $\beta - 2 \leq \alpha < \beta - 1$ *then*

   - $\tau^{\leq 0}_{1,\beta+1,\beta+1}(X) = 0$,

- $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(X))$ *is an isomorphic arrow,*
- $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))$ *is a monomorphic arrow in the (abelian) category* $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(\mathcal{D}^{\uparrow}))$,
- $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(\tau^{\geq 1}_{1,\beta,\beta}(X)) = 0$,
- $\rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X)) = 0$ *and*
- $\tau^{\geq 1}_{1,\alpha,\alpha}(X)) = 0$.

3. *If* $\beta - 1 \leq \alpha$ *then*

- $\tau^{\leq 0}_{1,\beta+1,\beta+1}(X) = 0$,
- $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(X))$ *is a monomorphic arrow in the (abelian) category* $\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$,
- $\rho_2(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\beta,\beta}(X)) = 0$ *and*
- $\tau^{\geq 1}_{1,\alpha,\alpha}(X)) = 0$.

Proof. Combine theorem 5.1.18 with definition 2.5.27. □

It will now be our task to extend our findings on t-structures that come from type-1-recollement-data to t-structures that come from type-2-recollement-data or type-3-recollement-data. We can rephrase corollary 4.3.17 using the particular properties of the functors $\mathbb{K}$ and $\mathbb{C}$.

Corollary 5.1.21. *There are t-structures on* $\mathcal{D}^{\uparrow}$ *given by*

1.

$$\mathcal{D}^{\leq 0} = \{Z \in \mathcal{D}^{\uparrow} \mid \rho_2(Z) \in \mathcal{D}_2^{\leq 0}, \lambda_1(Z) \in \mathcal{D}_1^{\leq 0}\}$$
$$\mathcal{D}^{\geq 1} = \{Z \in \mathcal{D}^{\uparrow} \mid \rho_2(Z) \in \mathcal{D}_2^{\geq 1}, \mathrm{Cone}(\mu_Z)[-1] \in \mathcal{D}_1^{\geq 1}\}$$

2.

$$\mathcal{D}^{\leq 0} = \{Z \in \mathcal{D}^{\uparrow} \mid \mathrm{Cone}(\mu_Z)[-1] \in \mathcal{D}_2^{\leq 0}, \rho_2(Z) \in \mathcal{D}_1^{\leq 0}\}$$
$$\mathcal{D}^{\geq 1} = \{Z \in \mathcal{D}^{\uparrow} \mid \mathrm{Cone}(\mu_Z)[-1] \in \mathcal{D}_2^{\geq 1}, \lambda_1(Z) \in \mathcal{D}_1^{\geq 1}\}$$

3.

$$\mathcal{D}^{\leq 0} = \{Z \in \mathcal{D}^{\uparrow} \mid \lambda_1(Z) \in \mathcal{D}_2^{\leq 0}, \mathrm{Cone}(\mu_Z) \in \mathcal{D}_1^{\leq 0}\}$$
$$\mathcal{D}^{\geq 1} = \{Z \in \mathcal{D}^{\uparrow} \mid \lambda_1(Z) \in \mathcal{D}_2^{\geq 1}, \rho_2(Z) \in \mathcal{D}_1^{\geq 1}\}$$

for t-structure $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 1})$ *and* $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 1})$ *on* $\mathcal{D}$ *and* μ_Z *as in corollary 4.3.4.*

Proof. Using $\mathrm{Cone}(\mu_Z)[-1] = \mathbb{K} = \mathbb{C}[-1]$ by corollary 4.3.4 and by lemma 4.2.28, we replace $\mathbb{K}$ and $\mathbb{C}$ in corollary 4.3.17. □

We will now provide analogies to theorem 5.1.18 and theorem 5.1.20 by describing the t-structures provided by type-3-recollement-data. We have the following observation (recall the definition of an "epimorphic arrow" made in 5.1.7).

Corollary 5.1.22. *For a t-structure* $(\mathcal{D}_{3,\alpha,\beta}^{\leq 0}, \mathcal{D}_{3,\alpha,\beta}^{\geq 1})$ *on* $\mathcal{D}^{\uparrow}$ *assume that* $\alpha \leq \beta$. *Then the t-structure has a description as*

$$\mathcal{D}^{\leq 0} = \{Z \in \mathcal{D}^{\uparrow} \mid \lambda_1(Z) \in \mathcal{D}_2^{\leq 0}, \rho_2(Z) \in \mathcal{D}_1^{\leq 0}\}$$
$$\mathcal{D}^{\geq 1} = \{Z \in \mathcal{D}^{\uparrow} \mid \lambda_1(Z) \in \mathcal{D}_2^{\geq 1}, \rho_2(Z) \in \mathcal{D}_1^{\geq 1}\}.$$

Proof. The $\mathcal{D}^{\leq 0}$ is unique for a given $\mathcal{D}^{\geq 1}$ by lemma 4.2.6. □

In other words, $\alpha > \beta$ is now the interesting case.

Theorem 5.1.23. *For a t-structure* $(\mathcal{D}_{3,\alpha,\beta}^{\leq 0}, \mathcal{D}_{3,\alpha,\beta}^{\geq 1})$ *on* $\mathcal{D}^{\uparrow}$ *assume that* $\alpha > \beta$ *For an object* $X \in \mathcal{D}^{\uparrow}$, *we have* $X \in \mathcal{D}_{3,\alpha,\beta}^{\leq 0}$ *if and only if*

1. - *if* $\alpha > \beta + 1$, *then* $\tau_{1,\alpha-1,\alpha-1}^{\leq 0}(\tau_{1,\alpha,\alpha}^{\geq 1}(X))$ *is an epimorphic arrow in* $\tau_{1,\alpha-1,\alpha-1}^{\leq 0}(\tau_{1,\alpha,\alpha}^{\geq 1}(\mathcal{D}^{\uparrow}))$,
   - *if* $\beta < \alpha \leq \beta + 1$, *then* $\tau_{1,\alpha,\alpha}^{\geq 1}(X)$ *is an epimorphic arrow in* $\tau_{1,\alpha-1,\alpha-1}^{\leq 0}(\tau_{1,\alpha,\alpha}^{\geq 1}(\mathcal{D}^{\uparrow}))$,
2. $\tau_{1,\alpha-1,\alpha-1}^{\geq 1}(\tau_{1,\beta,\beta}^{\leq 0}(X))$ *is an isomorphic arrow,*
3. $\tau_{1,\beta,\beta}^{\geq 1}(X) = 0$.

Proof. This is simply the dual of theorem 5.1.18 and all parts of the proof can hence be obtained dually. □

In analogy to the previous procedure, following theorem 5.1.18, we will now explain how the hearts of the t-structures that we have just described, look like.

Corollary 5.1.24. *For a t-structure* $(\mathcal{D}_{3,\alpha,\beta}^{\leq 0}, \mathcal{D}_{3,\alpha,\beta}^{\geq 1})$ *on* $\mathcal{D}^{\uparrow}$ *assume that* $\alpha > \beta$. *The heart* $H_{3,\alpha,\beta}$ *of* $(\mathcal{D}_{3,\alpha,\beta}^{\leq 0}, \mathcal{D}_{3,\alpha,\beta}^{\geq 1})$ *is given by the following.*

1. *If* $\alpha > \beta + 2$ *then* $X \in H_{3,\alpha,\beta}$, *if and only if*

- $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X) = 0$,
- $\lambda_1(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha,\alpha}(X))) = 0$,
- $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\alpha,\alpha}(X)) = 0$,
- $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(X))$ *is an isomorphic arrow and*
- $\tau^{\geq 1}_{1,\beta,\beta}(X) = 0$.

2. If $\beta + 1 \leq \alpha < \beta + 2$ *then*

- $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X) = 0$,
- $\lambda_1(\tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) = 0$,
- $\tau^{\leq 0}_{1,\alpha-1,\alpha-1}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(X))$ *is an epimorphic arrow in the (abelian) category* $\tau^{\leq 0}_{1,\alpha-1,\alpha-1}(\tau^{\geq 1}_{1,\alpha,\alpha})(\mathcal{D}^{\uparrow})$,
- $\tau^{\leq 0}_{1,\beta,\beta}(\tau_{1,\alpha-1,\alpha-1}(X))$ *is an isomorphic arrow and*
- $\tau^{\geq 1}_{1,\beta,\beta}(X) = 0$.

3. If $\beta + 1 \geq \alpha$ *then*

- $\tau^{\leq 0}_{1,\alpha+1,\alpha+1}(X) = 0$ *and*
- $\lambda_1(\tau^{\leq 0}_{1,\beta+1,\beta+1}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(X))) = 0$
- $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\alpha,\alpha}(X))$ *is an epimorphic arrow in the (abelian) category* $\tau^{\leq 0}_{1,\alpha-1,\alpha-1}(\tau^{\geq 1}_{1,\alpha,\alpha})(\mathcal{D}^{\uparrow})$ *and*
- $\tau^{\geq 1}_{1,\beta,\beta}(X) = 0$.

Proof. Combine theorem 5.1.23 with definition 2.5.27. □

Remark 5.1.25. Note that by proposition 4.3.21 we have now given a cohomological description of the t-structures that we obtain via recollement from the three semiorthogonal decompositions that we are working with. This is with the exception of the small interval from remark 4.3.22, where the situation remains unclear.

Remark 5.1.26. Like we did previously, in the description of t-structures coming from type-1-recollement-data in theorem 5.1.18 and their hearts in corollary 5.1.20, we are using the functors $\tau^{\leq 0}_{1,x,x}$ and $\tau^{\geq 1}_{1,y,y}$ to describe the t-structures. It might seem odd, that we should use these functors again. The point is, that they are special cases of the functors $\tau^{\leq 0}_{1,x_1,x_2}$ and $\tau^{\geq 1}_{1,y_1,y_2}$ where we let $x_1 = x_2$ and $y_1 = y_2$, belong to t-structures that simply are "real shifts" of the standard t-structure on $\mathcal{D}^{\uparrow}$ which makes them particularly nice and hence a good basis to understand the more advanced t-structures we were describing in 5.1.18, 5.1.20, 5.1.23 and 5.1.24.

We conclude this subsection with a few examples to illustrate our findings. In order to focus on the essence of this subchapter we will illustrate whichever of both categories the t-structure is composed of, that does not have the "easy" CP-gluing description of section 3.

Example 5.1.27. *Let $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ where $\alpha, \beta \in \mathbb{Z}$ and $\alpha + 3 = \beta$. Then the cohomology of X has the form*

$$
\begin{array}{rccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & \vdots & & \vdots \\
 & \downarrow & & \downarrow \\
\beta : & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \\
\alpha + 2 : & X_{\alpha+2} & \xrightarrow{\cong} & Y_{\alpha+2} \\
 & \downarrow & & \downarrow \\
\alpha + 1 : & X_{\alpha+1} & \xrightarrow{\cong} & Y_{\alpha+1} \\
 & \downarrow & & \downarrow \\
\alpha : & X_{\alpha} & \hookrightarrow & Y_{\alpha} \\
 & \downarrow & & \downarrow \\
\alpha - 1 : & X_{\alpha-1} & \longrightarrow & Y_{\alpha-1} \\
 & \downarrow & & \downarrow \\
 & \vdots & & \vdots
\end{array}
$$

Note: We are giving a description of the cohomology-complex of X, therefore the vertical arrows in the above are 0*.*

Remark 5.1.28. Example 5.1.27 has a particularly nice shape because α and β are whole numbers, and hence the abelian categories obtained by the truncations in theorem 5.1.18 correspond to embeddings of $\mathcal{A}^{\uparrow}$ into $\mathcal{D}^{\uparrow}$. However,

$\alpha, \beta \notin \mathbb{Z}$ yields a much more complicated situation as we will see in example 5.1.29.

Example 5.1.29. *Let $X \in \mathcal{D}^{\geq 1}_{1,\alpha,\beta}$ where $\alpha, \beta \in \mathbb{R}, \alpha, \beta \notin \mathbb{Z}$ and $\alpha + 3 = \beta$. Then the cohomology of X has the form*

$$
\begin{array}{rccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
 & \vdots & & \vdots \\
 & \downarrow & & \downarrow \\
\lfloor \beta \rfloor + 1 : & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \\
\lfloor \beta \rfloor : & X_{\lfloor \beta \rfloor} & \xrightarrow{c} & Y_{\lfloor \beta \rfloor} \\
 & \downarrow & & \downarrow \\
\lfloor \alpha \rfloor + 2 : & X_{\lfloor \alpha \rfloor + 2} & \xrightarrow{\cong} & Y_{\lfloor \alpha \rfloor + 2} \\
 & \downarrow & & \downarrow \\
\lfloor \alpha \rfloor + 1 : & X_{\lfloor \alpha \rfloor + 1} & \xrightarrow{b} & Y_{\lfloor \alpha \rfloor + 1} \\
 & \downarrow & & \downarrow \\
\lfloor \alpha \rfloor : & X_{\lfloor \alpha \rfloor} & \xrightarrow{a} & Y_{\lfloor \alpha \rfloor} \\
 & \downarrow & & \downarrow \\
\lfloor \alpha \rfloor - 1 : & X_{\lfloor \alpha \rfloor - 1} & \longrightarrow & Y_{\lfloor \alpha \rfloor - 1} \\
 & \downarrow & & \downarrow \\
 & \vdots & & \vdots
\end{array}
$$

and the objects given by a, b and c can be described by short exact sequences in $\mathcal{A}$ as follows.

1. *For the object given by the morphism "a" we obtain the short exact*

sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^1_{\lfloor\alpha\rfloor} & \longrightarrow & X_{\lfloor\alpha\rfloor} & \longrightarrow & X^2_{\lfloor\alpha\rfloor} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_2} & & \\ 0 & \longrightarrow & Y^1_{\lfloor\alpha\rfloor} & \longrightarrow & Y_{\lfloor\alpha\rfloor} & \longrightarrow & Y^2_{\lfloor\alpha\rfloor} & \longrightarrow & 0 \end{array}$$

where $X^1_{\lfloor\alpha\rfloor} \xrightarrow{a_1} Y^1_{\lfloor\alpha\rfloor} \in \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\lfloor\alpha\rfloor+1,\lfloor\alpha\rfloor+1}(\mathcal{D}^{\uparrow}))$ *is a monomorphic arrow in the abelian category* $\tau^{\leq 0}_{1,\lfloor\alpha\rfloor,\lfloor\alpha\rfloor}(\tau^{\geq 1}_{1,\lfloor\alpha\rfloor+1,\lfloor\alpha\rfloor+1}(\mathcal{D}^{\uparrow}))$ *and* $X^2_{\lfloor\alpha\rfloor} \xrightarrow{a_2} Y^2_{\lfloor\alpha\rfloor} \in \tau^{\leq 0}_{1,\lfloor\alpha\rfloor+1,\lfloor\alpha\rfloor+1}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$.

2. *For the object given by the morphism "b" we obtain the short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^1_{\lfloor\alpha\rfloor+1} & \longrightarrow & X_{\lfloor\alpha\rfloor+1} & \longrightarrow & X^2_{\lfloor\alpha\rfloor+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b_2} & & \\ 0 & \longrightarrow & Y^1_{\lfloor\alpha\rfloor+1} & \longrightarrow & Y_{\lfloor\alpha\rfloor+1} & \longrightarrow & Y^2_{\lfloor\alpha\rfloor+1} & \longrightarrow & 0 \end{array}$$

where $X^1_{\lfloor\alpha\rfloor+1} \xrightarrow{b_1} Y^1_{\lfloor\alpha\rfloor+1} \in \tau^{\leq 0}_{1,\alpha+1,\alpha+1}(\tau^{\geq 1}_{1,\lfloor\alpha\rfloor+2,\lfloor\alpha\rfloor+2}(\mathcal{D}^{\uparrow}))$ *is an isomorphic arrow and* $X^2_{\lfloor\alpha\rfloor+1} \xrightarrow{b_2} Y^2_{\lfloor\alpha\rfloor+1} \in \tau^{\leq 0}_{1,\lfloor\alpha\rfloor+1,\lfloor\alpha\rfloor+1}(\tau^{\geq 1}_{1,\alpha+1,\alpha+1}(\mathcal{D}^{\uparrow}))$ *is a monomorphic arrow in the category* $\tau^{\leq 0}_{1,\lfloor\alpha\rfloor+1,\lfloor\alpha\rfloor+1}(\tau^{\geq 1}_{1,\lfloor\alpha\rfloor+2,\lfloor\alpha\rfloor+2}(\mathcal{D}^{\uparrow}))$.

3. *For the object given by the morphism "c" we obtain the short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^1_{\lfloor\beta\rfloor} & \longrightarrow & X_{\lfloor\beta\rfloor} & \longrightarrow & X^2_{\lfloor\beta\rfloor} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c_2} & & \\ 0 & \longrightarrow & Y^1_{\lfloor\beta\rfloor} & \longrightarrow & Y_{\lfloor\beta\rfloor} & \longrightarrow & Y^2_{\lfloor\beta\rfloor} & \longrightarrow & 0 \end{array}$$

where $X^1_{\lfloor\beta\rfloor} \xrightarrow{c_1} Y^1_{\lfloor\beta\rfloor} = 0 \in \mathcal{D}^{\uparrow}$ *and moreover we have* $X^2_{\lfloor\beta\rfloor} \xrightarrow{c_2} Y^2_{\lfloor\beta\rfloor} \in \tau^{\leq 0}_{1,\lfloor\beta\rfloor,\lfloor\beta\rfloor}(\tau^{\geq 1}_{1,\beta,\beta}(\mathcal{D}^{\uparrow}))$ *is an isomorphic arrow.*

Remark 5.1.30. We choose the description via exact sequences in 5.1.29 in order to demonstrate how the respective cohomology objects split up along the boundaries provided by the real numbers α and β, obviously

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^1_{\lfloor\beta\rfloor} & \longrightarrow & X_{\lfloor\beta\rfloor} & \longrightarrow & X^2_{\lfloor\beta\rfloor} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c_2} & & \\ 0 & \longrightarrow & Y^1_{\lfloor\beta\rfloor} & \longrightarrow & Y_{\lfloor\beta\rfloor} & \longrightarrow & Y^2_{\lfloor\beta\rfloor} & \longrightarrow & 0 \end{array}$$

is simply $(X_{\lfloor\beta\rfloor+1} \xrightarrow{c} Y_{\lfloor\beta\rfloor+1}) \cong (X^2_{\lfloor\beta\rfloor+1} \xrightarrow{c_2} Y^2_{\lfloor\beta\rfloor+1})$. One should think about the three objects a, b and c as composed of the factors in the short exact sequences with the subobject sitting on top of the quotient within the ladder displayed at the start of example 5.1.29.

We will now provide examples for the better understanding of the situation of $(\mathcal{D}^{\leq 0}_{3,\alpha,\beta}, \mathcal{D}^{\geq 1}_{3,\alpha,\beta})$. Since this is dual to the $(\mathcal{D}^{\leq 0}_{3,\alpha,\beta}, \mathcal{D}^{\geq 1}_{3,\alpha,\beta})$-case as we pointed out before, the interesting category is now $\mathcal{D}^{\leq 0}_{3,\alpha,\beta}$. The following example is the dual of 5.1.27.

Example 5.1.31. *Let $X \in \mathcal{D}^{\leq 0}_{3,\alpha,\beta}$ where $\alpha, \beta \in \mathbb{Z}$ and $\alpha - 3 = \beta$. Then the cohomology of X has the form*

$$
\begin{array}{lccc}
 & \vdots & & \vdots \\
 & \downarrow & & \downarrow \\
\alpha : & X_\alpha & \longrightarrow & Y_\alpha \\
 & \downarrow & & \downarrow \\
\beta+2 : & X_{\beta+2} & \twoheadrightarrow & Y_{\beta+2} \\
 & \downarrow & & \downarrow \\
\beta+1 : & X_{\beta+1} & \xrightarrow{\cong} & Y_{\beta+1} \\
 & \downarrow & & \downarrow \\
\beta : & X_\beta & \xrightarrow{\cong} & Y_\beta \\
 & \downarrow & & \downarrow \\
\beta-1 : & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \\
 & \vdots & & \vdots \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}
$$

And finally we demonstrate the dual to example 5.1.32 with the following.

Example 5.1.32. *Let $X \in \mathcal{D}^{\leq 0}_{3,\alpha,\beta}$ where $\alpha, \beta \in \mathbb{Z}$ and $\alpha - 3 = \beta$. Then the* *cohomology of X has the form*

$$
\begin{array}{lccc}
 & \vdots & & \vdots \\
 & \downarrow & & \downarrow \\
\lfloor\alpha\rfloor+1: & X_{\lfloor\alpha\rfloor+1} & \longrightarrow & Y_{\lfloor\alpha\rfloor+1} \\
 & \downarrow & & \downarrow \\
\lfloor\alpha\rfloor: & X_{\lfloor\alpha\rfloor} & \xrightarrow{c} & Y_{\lfloor\alpha\rfloor} \\
 & \downarrow & & \downarrow \\
\lfloor\beta\rfloor+2: & X_{\lfloor\beta\rfloor+2} & \xrightarrow{\cong} & Y_{\lfloor\beta\rfloor+2} \\
 & \downarrow & & \downarrow \\
\lfloor\beta\rfloor+1: & X_{\lfloor\beta\rfloor+1} & \xrightarrow{b} & Y_{\lfloor\beta\rfloor+1} \\
 & \downarrow & & \downarrow \\
\lfloor\beta\rfloor: & X_{\lfloor\beta\rfloor} & \xrightarrow{a} & Y_{\lfloor\beta\rfloor} \\
 & \downarrow & & \downarrow \\
\lfloor\beta\rfloor-1: & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \\
 & \vdots & & \vdots \\
 & \downarrow & & \downarrow \\
 & 0 & & 0
\end{array}
$$

1. *For the object given by the morphism "a" we obtain the short exact sequence*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X^1_{\lfloor\beta\rfloor} & \longrightarrow & X_{\lfloor\beta\rfloor} & \longrightarrow & X^2_{\lfloor\beta\rfloor} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_2} & & \\
0 & \longrightarrow & Y^1_{\lfloor\beta\rfloor} & \longrightarrow & Y_{\lfloor\beta\rfloor} & \longrightarrow & Y^2_{\lfloor\beta\rfloor} & \longrightarrow & 0
\end{array}
$$

where $X^2_{\lfloor\beta\rfloor} \xrightarrow{a_2} Y^2_{\lfloor\beta\rfloor} = 0 \in \mathcal{D}^{\uparrow}$ and moreover we have that the object $X^1_{\lfloor\beta\rfloor} \xrightarrow{a_1} Y^1_{\lfloor\beta\rfloor} \in \tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\lfloor\beta\rfloor+1,\lfloor\beta\rfloor+1}(\mathcal{D}^{\uparrow}))$ is an isomorphic arrow in the category $\tau^{\leq 0}_{1,\lfloor\beta\rfloor,\lfloor\beta\rfloor}(\tau^{\geq 1}_{1,\lfloor\beta\rfloor+1,\lfloor\beta\rfloor+1}(\mathcal{D}^{\uparrow}))$.

2. *For the object given by the morphism "b" we obtain the short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^1_{\lfloor\beta\rfloor+1} & \longrightarrow & X_{\lfloor\beta\rfloor+1} & \longrightarrow & X^2_{\lfloor\beta\rfloor+1} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle b_1} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle b_2} & & \\ 0 & \longrightarrow & Y^1_{\lfloor\beta\rfloor+1} & \longrightarrow & Y_{\lfloor\beta\rfloor+1} & \longrightarrow & Y^2_{\lfloor\beta\rfloor+1} & \longrightarrow & 0 \end{array}$$

where $X^1_{\lfloor\beta\rfloor+1} \xrightarrow{b_1} Y^1_{\lfloor\beta\rfloor+1} \in \tau^{\leq 0}_{1,\beta+1,\beta+1}(\tau^{\geq 1}_{1,\lfloor\beta\rfloor+2,\lfloor\beta\rfloor+2}(\mathcal{D}^\uparrow))$ *is an isomorphic arrow and* $X^2_{\lfloor\beta\rfloor+1} \xrightarrow{b_2} Y^2_{\lfloor\beta\rfloor+1} \in \tau^{\leq 0}_{1,\lfloor\beta\rfloor+1,\lfloor\beta\rfloor+1}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(\mathcal{D}^\uparrow))$ *is an epimorphic arrow in the category* $\tau^{\leq 0}_{1,\beta,\beta}(\tau^{\geq 1}_{1,\beta+1,\beta+1}(\mathcal{D}^\uparrow))$.

3. *For the object given by the morphism "c" we obtain the short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^1_{\lfloor\alpha\rfloor} & \longrightarrow & X_{\lfloor\alpha\rfloor} & \longrightarrow & X^2_{\lfloor\alpha\rfloor} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle c_1} & & \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle c_2} & & \\ 0 & \longrightarrow & Y^1_{\lfloor\alpha\rfloor} & \longrightarrow & Y_{\lfloor\alpha\rfloor} & \longrightarrow & Y^2_{\lfloor\alpha\rfloor} & \longrightarrow & 0 \end{array}$$

where $X^2_{\lfloor\alpha\rfloor} \xrightarrow{c_2} Y^2_{\lfloor\alpha\rfloor} \in \tau^{\leq 0}_{1,\lfloor\alpha\rfloor,\lfloor\alpha\rfloor}(\tau^{\geq 1}_{1,\alpha,\alpha}(\mathcal{D}^\uparrow))$ *is an epimorphic arrow and* $X^1_{\lfloor\alpha\rfloor} \xrightarrow{c_1} Y^1_{\lfloor\alpha\rfloor} \in \tau^{\leq 0}_{1,\alpha,\alpha}(\tau^{\geq 1}_{1,\lfloor\alpha\rfloor,\lfloor\alpha\rfloor}(\mathcal{D}^\uparrow))$.

Remark 5.1.33. This subsection was aimed at gaining a better understanding of t-structures obtained by recollement via the type 1,2 or 3 recollement-data (therefore linked to the semiorthogonal decompositions $\langle\mathcal{D}_1,\mathcal{D}_2\rangle, \langle\mathcal{D}_3,\mathcal{D}_1\rangle$ and $\langle\mathcal{D}_3,\mathcal{D}_1\rangle$ respectively). Our approach was to give a description of the involved objects in terms of standard-cohomology. While we almost completely succeeded, two interesting cases remain as demonstrated by the previous examples as the morphism a_1 in 5.1.29 and the morphism b_2 in 5.1.32 obtain their characterisation as being linked to a monomorphism and an epimorphism respectively from categories of the form $\mathcal{P}(\gamma,\gamma+1]$ where $\gamma \in \mathbb{Z}$ (with $\mathcal{P}(0,1] = \mathcal{A}^\uparrow$). Since these categories are not simple shifts of $\mathcal{A}^\uparrow$, we do not obtain a connection to mono-/epimorphisms in $\mathcal{A}^\uparrow$.

5.2 Connecting morphisms on $\mathcal{D}^\uparrow$

An important feature of the derived category is, that it has morphisms which are not a straightforward adaptation of those of the underlying abelian category $\mathcal{A}$ (and hence of the – also abelian – category $\mathcal{C}(\mathcal{A})$). On the other hand, it is via these additional morphisms that the triangulated structure is provided.

While the first and the second horizontal arrow in the exact triangle

$$i_2(\rho_2(X)) \to X \to i_1(\lambda_1(X)) \xrightarrow{+}$$

in $\mathcal{D}^{\uparrow}$ are obvious and are moreover taken from $\mathrm{mor}(\mathcal{C}(\mathcal{A})))$, the connecting morphism $\xrightarrow{+}$ is less obvious and therefore interesting. Since, as an implication of [59, Tag 05QT] this morphism has to be non-zero in general, it is also important to understand how it is constructed.

However, $\xrightarrow{+}$ has proved to be of particular importance in the context of $\mathcal{D}^{\uparrow}$. Lemma 2.2.11 has revealed another problem that appears in this context, provided by the question if the diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{der} & \mathcal{D} \\ {\scriptstyle\uparrow}\big\downarrow & & {\scriptstyle\uparrow}\big\downarrow \\ \mathcal{A}^{\uparrow} & \xrightarrow{der} & \mathcal{X} \end{array}$$

commutes. In other words, the question is if there is an equivalence of categories between $(\mathcal{D}^b(\mathcal{A}))^{\uparrow}$ and $\mathcal{D}^{\uparrow}$ provided by a trivial functor (or by any functor at all). Both the author and others believed this to be true for while (see [29]). However, the functor T of [29, Section 3.4] turns out not to be faithful, since it maps $\xrightarrow{+}$ to 0. We shall hence provide a description of $\xrightarrow{+}$ throughout this subsection, which allows us to see why $\xrightarrow{+}$ is non-zero and hence how it is constructed.

We will require two definitions. At first recall this – well known – definition.

Definition 5.2.1. For a morphism

$$E \xrightarrow{f} F$$

in $\mathcal{C}(\mathcal{A})$ we define the "mapping cone" by

$$\mathrm{Cone}(f)^n = E^{n+1} \oplus F^n$$

(meaning that without the differential $\mathrm{Cone}(f)$ would simply be $E[1] \oplus F$) with

$$\begin{pmatrix} -d_E^{n+1} & 0 \\ f^{n+1} & d_F^n \end{pmatrix}$$

as differential.

It is also well known, that with this object we obtain two canonical mappings $F \to \mathrm{Cone}(f)$ and $\mathrm{Cone}(f) \to E[1]$.

Definition 5.2.2. For a morphism

$$E \xrightarrow{f} F$$

define i_F and $p_{E[1]}$ to be the canonical mappings given by the mapping cone:

$$F \xrightarrow{i_F} \mathrm{Cone}(f) \xrightarrow{p_{E[1]}} E[1].$$

In other words, i_F embeds F into the F-component of $\mathrm{Cone}(f)$ and $p_{E[1]}$ projects the $E[1]$-component of $\mathrm{Cone}(f)$ onto $E[1]$.

The following lemma provides the fact that these mappings are – indeed – useful.

Lemma 5.2.3. *The maps defined in Definition 5.2.2 are maps of chain-complexes.*

Proof. To prove the lemma we must show the commutativity of the respective map with the boundary-operator.

For i_F we obtain:

$$\begin{pmatrix} -d_E^{n+1} & 0 \\ f^{n+1} & d_F^n \end{pmatrix} \begin{pmatrix} 0 \\ i_F^n \end{pmatrix} = \begin{pmatrix} 0 \\ d_F^n \circ i_F^n \end{pmatrix} = \begin{pmatrix} 0 \\ i_F^{n+1} \circ d_F^n \end{pmatrix}.$$

For $p_{E[1]}$ we easily check:

$$-d_E^{n+1} \circ p_{E[1]}^n = p_{E[1]}^{n+1} \begin{pmatrix} -d_E^{n+1} & 0 \\ f^{n+1} & d_F^n \end{pmatrix}.$$

□

Remark 5.2.4. Note that the object $\mathrm{Cone}(f)$ is – generally – not equal to the direct sum of E and F, which implies, that we cannot assume the existence of mappings p_F and $i_{E[1]}$ in analogy to i_F and $p_{E[1]}$. For example would we get

$$p_F^{n+1} \begin{pmatrix} -d_E^{n+1} & 0 \\ f^{n+1} & d_F^n \end{pmatrix} = (p_F^{n+1} \circ f^{n+1}, p_F^{n+1} \circ d_F^n) \neq (0, d_F^n \circ p_F^n)$$

whenever $f \neq 0$ and at the same time $F \neq 0$. We refer to [56, Section 10.5] for more details.

Lemma 5.2.5. *Let* $(A \xrightarrow{\varphi} B) \in \mathcal{D}^{\uparrow}$*. There is an exact triangle*

$$\begin{array}{ccccccc} A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & 0 & & \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle i_B \circ \varphi} & & \downarrow & \xrightarrow{+} & \\ B & \xrightarrow{i_B} & \mathrm{Cone}(\mathrm{id}_B) & \xrightarrow{p_{B[1]}} & B[1] & & \end{array}$$

in $\mathcal{D}^{\uparrow}$*.*

Proof. The sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle i_B \circ \varphi} & & \downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow{\mathrm{id}_B} & \mathrm{Cone}(\mathrm{id}_B) & \xrightarrow{p_{B[1]}} & B[1] & \longrightarrow & 0 \end{array}$$

is exact in $\mathcal{C}(\mathcal{A}^{\uparrow})$. □

We also have the following.

Lemma 5.2.6. *Let* $(A \xrightarrow{\varphi} B) \in \mathcal{D}^{\uparrow}$*, we have* $i_1(A) \cong_{\mathcal{D}^{\uparrow}} (A \xrightarrow{i_B \circ \varphi} \mathrm{Cone}(\mathrm{id}_B))$*.*

Proof. We have $i_1(A) = (A \to 0)$ which fits into the exact triangle

$$\begin{array}{cccccc} 0 & \xrightarrow{0} & A & \xrightarrow{\mathrm{id}_A} & A & \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 0} & \xrightarrow{+} \\ B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{0} & 0. & \end{array}$$

On the other hand, $(A \xrightarrow{i_B \circ \varphi} \mathrm{Cone}(\mathrm{id}_B))$ is the mapping cone of the morphism

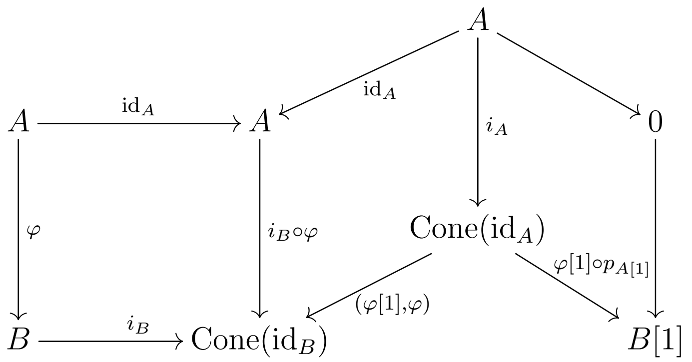
$$\begin{array}{ccc} 0 & \xrightarrow{0} & A \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\mathrm{id}_B} & B, \end{array}$$

in $\mathcal{C}(\mathcal{A}^{\uparrow})$ and combining [32, Section 1.1, (TR3)] with [32, Section 1.2, Proposition], finishes the proof.

□

Hence, we obtain the following.

Lemma 5.2.7. *Let* $(A \stackrel{\varphi}{\to} B) \in \mathcal{D}^{\uparrow}$*. There is an exact triangle*



(5.5)

in $\mathcal{D}^{\uparrow}$*.*

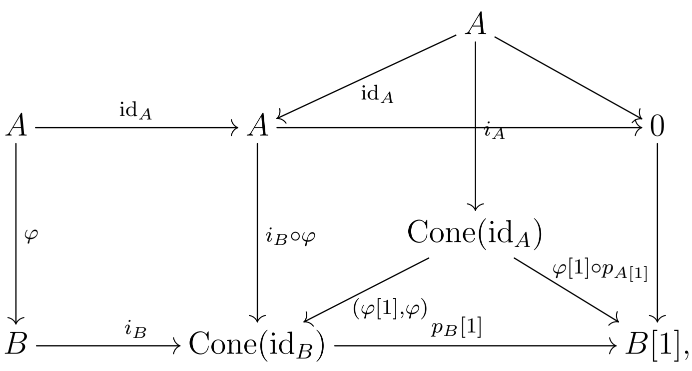
Proof. At first note that the morphism

$$\mathrm{Cone}(\mathrm{id}_A) \stackrel{(\varphi[1],\varphi)}{\to} \mathrm{Cone}(\mathrm{id}_B)$$

in the diagram above is obtained as the mapping cone of the morphism

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

and does – therefore – exist. Moreover, by lemma 5.2.6, both $A \stackrel{i_A}{\to} \mathrm{Cone}(\mathrm{id}_A)$ and $A \stackrel{i_B \circ \varphi}{\to} \mathrm{Cone}(\mathrm{id}_B)$ are isomorphic to $i_1(A)$, which implies that the morphism induced by $(\mathrm{id}_A, (\varphi[1], \varphi))$ on the cohomology objects equals to $(\mathrm{id}_A, 0)$ – but since both $\mathrm{Cone}(\mathrm{id}_A)$ and $\mathrm{Cone}(\mathrm{id}_B)$ are acyclic, we obtain that the map $(\mathrm{id}_A, (\varphi[1], \varphi))$ is a quasi-isomorphism and therefore an isomorphism in $\mathcal{D}^{\uparrow}$. Considering the exact triangle from lemma 5.2.5 we now obtain the commutative (in $\mathcal{C}(\mathcal{A}^{\uparrow})$) diagram



(5.6)

and the proof is finished. $\square$

We hence conclude this section with the following theorem.

Theorem 5.2.8. *Let* $X = (A \xrightarrow{\varphi} B)$*. The connecting morphism* $\xrightarrow{+}$ *of the exact triangle*

$$i_2(\rho_2(X)) \to X \to i_1(\lambda_1(X)) \xrightarrow{+}$$

is – up to isomorphisms – given by the chain-complex homomorphism φ *via the roof* (5.5) *described in lemma 5.2.7.*

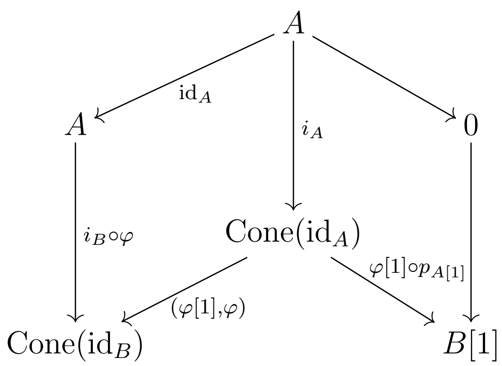
Proof. This is a consequence of lemma 5.2.7, obtained via the fact that we have $A \xrightarrow{i_B \circ \varphi} \mathrm{Cone}(\mathrm{id}_B)$ isomorphic to $A \to 0$ in $\mathcal{D}^{\uparrow}$ by lemma 5.2.6. We combine this with the commutativity of the diagram (5.6) which provides the equality of both morphisms from $(A \to \mathrm{Cone}(\mathrm{id}_B))$ to $(0 \to B[1])$. $\square$

Remark 5.2.9. It should be noted, that the mathematics in this chapter also provide a different approach to obtain the result of corollary 4.5.6. This can be seen in the following way, assume that $F = (A \xrightarrow{\varphi} B)$ such that $\varphi = 0$ in $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, B)$, then we have a decomposition

$$F = (A \xrightarrow{i_B \circ \varphi} \mathrm{Cone}(\mathrm{id}_B)) \oplus i_2(B) \tag{5.7}$$

and obtain the result of corollary 4.5.6 by lemma 5.2.7.

The reason for (5.7) to be true is that by [59, Tag 05QT] we need to prove that the morphism $\xi =$



in the exact triangle from lemma 5.2.7 equals $0 \in \mathrm{mor}(\mathcal{D}^{\uparrow})$ for $\varphi = 0$ in $\mathrm{mor}(\mathcal{D})$. Apply the functor $\mathrm{Hom}_{\mathcal{D}^{\uparrow}}(-, i_2(B[1]))$ to the (by lemma 5.2.5) exact triangle

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle i_A} & & \downarrow & \xrightarrow{+} \\ A & \xrightarrow{i_A} & \mathrm{Cone}(\mathrm{id}_A) & \xrightarrow{p_{A[1]}} & A[1] \end{array}$$

to get a well-defined mapping

$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(i_2(A[1]), i_2(B[1])) \stackrel{-\circ\nu}{\to} \mathrm{Hom}_{\mathcal{D}^\uparrow}(A \stackrel{i_A}{\to} \mathrm{Cone}(\mathrm{id}_A), i_2(B[1]))$$

where ν is the morphism

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow{\scriptstyle i_A} & & \downarrow & \xrightarrow{+} \\ \mathrm{Cone}(\mathrm{id}_A) & \xrightarrow{p_{A[1]}} & A[1] \end{array}$$

from the previous exact triangle. We obtain the well-defined mapping

$$\mathrm{Hom}_{\mathcal{D}}(A, B) \stackrel{[1]}{\to} \mathrm{Hom}_{\mathcal{D}}(A[1], B[1]) \stackrel{(0,-)}{\to}$$
$$\mathrm{Hom}_{\mathcal{D}^\uparrow}(i_2(A[1]), i_2(B[1])) \stackrel{-\circ\nu}{\to} \mathrm{Hom}_{\mathcal{D}^\uparrow}(A \stackrel{i_A}{\to} \mathrm{Cone}(\mathrm{id}_A), i_2(B[1]))$$

which maps φ to ξ and hence proves that if $\varphi = 0$ in $\mathrm{Hom}_{\mathcal{D}^\uparrow}(A, B)$, then so is ξ in $\mathrm{Hom}_{\mathcal{D}^\uparrow}(A \stackrel{i_A}{\to} \mathrm{Cone}(\mathrm{id}_A), i_2(B[1]))$. As mentioned earlier, this is what we need.

5.3 Exceptional collections

Another technique of finding hearts of bounded t-structures uses a particular kind of objects which may or may not exist in the category one is working with. These so called "Exceptional objects" were used by Macrí in [45] and [46] to obtain new pre-stability conditions.

We will provide a short introduction into the theory of exceptional collections on triangulated categories by providing the crucial definitions required. Following the notation of Macrí in [46] we introduce the notion of an exceptional object.

Definition 5.3.1. An object $E \in \mathcal{TR}$ is called "exceptional" if for all $i \neq 0$

$$\mathrm{Hom}^i(E, E) = 0,$$
$$\mathrm{Hom}^0(E, E) = \mathbb{C}.$$

We also need the following notation.

Notation 5.3.2. For $A, B \in \mathcal{TR}$ let $\mathrm{Hom}^\bullet(A,B) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(A,B)[-k]$.

We can now provide the definition of an exceptional collection.

Definition 5.3.3. A finite sequence of exceptional objects $(E_0, \dots, E_n)$ is called an "exceptional collection" in $\mathcal{TR}$ if for $i > j$ we have

$$\mathrm{Hom}^\bullet(E_i, E_j) = 0.$$

In general, the term exceptional collection is not strong enough to make use of it in order to find t-structures. One requires the exceptional collection to have additional features as provided by the following definitions.

Definition 5.3.4. An exceptional collection $\mathcal{E} = (E_0, \dots, E_n)$ is "complete" if it generates $\mathcal{TR}$ by shifts and extensions.

Definition 5.3.5. An exceptional collection $\mathcal{E} = (E_0, \dots, E_n)$ is "Ext", if for all $i \neq j$ we have

$$\mathrm{Hom}^{\leq 0}(E_i, E_j) = 0.$$

It is our objective to investigate the construction of exceptional collections on $\mathcal{D}^\uparrow$ in this subsection. We want to investigate what happens going from $\mathcal{D}$ to $\mathcal{D}^\uparrow$. That is, we want to extend concepts that are related to exceptional collections from $\mathcal{D}$ to $\mathcal{D}^\uparrow$. We start by illustrating this with the easy observation, that for a semiorthogonal decomposition $\mathcal{TR} = \langle \mathcal{TR}^1, \mathcal{TR}^2 \rangle$ and an exceptional collection $\mathcal{E} = (E_0, \dots, E_n)$ in $\mathcal{TR}^a$ we obtain an exceptional collection $i_a(\mathcal{E}) = (i_a(E_0), \dots, i_a(E_n))$ where $a \in \{1, 2\}$ and i_a as in lemma 3.1.4. This is due to the fact that i_a is fully faithful and commutes with the shift functor which implies that we obtain $\mathrm{Hom}_{\mathcal{TR}}(i_a(E), i_a(F)) \cong \mathrm{Hom}_{\mathcal{TR}^a}(E, F)$ for any $E, F \in \mathcal{TR}^a$.

It is essential, yet not surprising, that the exceptional collections $i_a(\mathcal{E})$ do in general not meet any criterion that allows one to use them in order to generate t-structures. It is evident, that an exceptional collection which can be used to construct t-structures on $\mathcal{TR}$ will have to be the result of a construction process that involves both $\mathcal{TR}^1$ and $\mathcal{TR}^2$. Based on this premise, we will hence try to construct more interesting exceptional collections on $\mathcal{TR}$.

Lemma 5.3.6. *Let $\mathcal{TR} = \langle \mathcal{TR}^1, \mathcal{TR}^2 \rangle$ be a semiorthogonal decomposition of $\mathcal{TR}$. Let $\mathcal{E} = (E_0, \dots, E_n)$ be an exceptional collection in $\mathcal{TR}^1$ and $\mathcal{F} = (F_0, \dots, F_n)$ be an exceptional collection in $\mathcal{TR}^2$. Then*

1. *$(i_1\mathcal{E}, i_2\mathcal{F})$ is an exceptional collection in $\mathcal{TR}$ and*

2. if $\mathcal{E}$ and $\mathcal{F}$ are complete, then $(i_1\mathcal{E}, i_2\mathcal{F})$ is complete.

Proof. We will prove the fact that $(i_1\mathcal{E}, i_2\mathcal{F})$ is an exceptional collection in $\mathcal{TR}$ before we prove the completeness.

1. Any object in $(i_1\mathcal{E}, i_2\mathcal{F})$ is exceptional because i_1 and i_2 are fully faithful and commute with the shift functor. Hence, it only remains to show that $\mathrm{Hom}^{\bullet}_{\mathcal{TR}}(i_2(F_a), i_1(E_b)) = 0$ for any $a, b \in \{0, \ldots, n\}$. Since $E_b \in \mathcal{TR}^1$ and $F_a \in \mathcal{TR}^2$ and $\mathcal{TR} = \langle \mathcal{TR}^1, \mathcal{TR}^2 \rangle$ is a semiorthogonal decomposition this follows by definition.

2. Since $\mathcal{E}$ is assumed to be complete, $\mathcal{E}$ generates $\mathcal{TR}^1$ and in the same way $\mathcal{F}$ generates $\mathcal{TR}^2$. Moreover, for any $G \in \mathcal{TR}$, there exists an exact triangle $i_2(\rho_2 G) \to G \to i_1(\lambda_1 G) \stackrel{+}{\to}$, which again derives from the fact that $\mathcal{TR} = \langle \mathcal{TR}^1, \mathcal{TR}^2 \rangle$ is a semiorthogonal decomposition. Hence, as G is an extension of two objects that can be generated by $i_1\mathcal{E}$ and by $i_2\mathcal{F}$ via shifts and extensions, $(i_1\mathcal{E}, i_2\mathcal{F})$ is indeed complete.

□

In order to establish corollary 5.3.8 we will need the following observation:

Lemma 5.3.7. *If for an $n \in \mathbb{N}$ we have that $\mathcal{E} = (E_1, \ldots, E_n)$ is an exceptional collection in a triangulated category $\mathcal{TR}$, then $\mathcal{E}[c] = (E_1[c], \ldots, E_n[c])$ is an exceptional collection in $\mathcal{TR}$.*

Proof. The fact that $\mathrm{Hom}(E, F) \cong \mathrm{Hom}(E[c], F[c])$ implies – on one hand – that the condition on each object of $\mathcal{E}[c]$ given in definition 5.3.1 is fulfilled, and – on the other – that the condition on the sequence given in definition 5.3.3 is fulfilled. □

Corollary 5.3.8. *Let $\mathcal{TR} = \langle \mathcal{TR}^1, \mathcal{TR}^2 \rangle$ be a semiorthogonal decomposition of $\mathcal{TR}$. Assume there is an equivalence of categories $\phi : \mathcal{TR}^1 \to \mathcal{TR}^2$ (as in lemma 3.1.8). Let $\mathcal{E} = (E_0, \ldots, E_n)$ be a semiorthogonal decomposition in $\mathcal{TR}^1$ and assume that there is an $m \in \mathbb{Z}$ such that for any $A_1 \in \mathcal{TR}^1$ and $A_2 \in \mathcal{TR}^2$ we have that $\mathrm{Hom}^i_{\mathcal{TR}^2}(\phi(A_1), A_2) \cong \mathrm{Hom}^i_{\mathcal{TR}}(i_1(A_1), i_2(A_2)[m])$. For $c \in \mathbb{Z}_{<m}$ we obtain*

1. *$(i_1\mathcal{E}, i_2\phi(\mathcal{E})[c]) = (i_1(E_0), \ldots, i_1(E_n), i_2\phi(E_0)[c], \ldots, i_2\phi(E_n)[c])$ is an exceptional collection in $\mathcal{TR}$,*

2. *if $\mathcal{E}$ is complete, then $(i_1\mathcal{E}, i_2\phi(\mathcal{E})[c])$ is complete and*

3. *if $\mathcal{E}$ is Ext, then $(i_1\mathcal{E}, i_2\phi(\mathcal{E})[c]))$ is Ext.*

Proof. We will proceed in the order of the statements of the corollary. We combine lemma 5.3.6 with lemma 5.3.7 to prove 1 and proceed similarly with 2.

The proof for 3 works as follows. In order to fulfil the definition for the exceptional collection to be Ext given in 5.3.5, we only have to prove that for an object F to the right of an object E in $(i_1\mathcal{E}, i_2\phi(\mathcal{E})[c])$ we get $\mathrm{Hom}^{\leq 0}(E, F) = 0$. The rest of the definition will be fulfilled by the fact that $(i_1\mathcal{E}, i_2\phi(\mathcal{E})[c])$ is an exceptional collection.

The semiorthogonal decomposition $\mathcal{E}$ being Ext means that for any objects $E_a, E_b \in \{E_0, \dots, E_n\}$ and $a < b$, we have $\mathrm{Hom}^{\leq 0}_{\mathcal{TR}^1}(E_a, E_b) = 0$. Hence

$$\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(i_1(E_a), i_1(E_b)) = \mathrm{Hom}^{\leq 0}_{\mathcal{TR}^1}(E_a, E_b) = 0$$

and a similar statement holds for the embedding i_2. Moreover, if we have $d \neq e$, then we obtain

$$\begin{aligned}\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(i_1(E_d), i_2(\phi(E_e)[c])) &= \mathrm{Hom}^{\leq 0}_{\mathcal{TR}^2}(\phi(E_d), \phi(E_e)[c-m]) \\ &\subset \mathrm{Hom}^{\leq 0}_{\mathcal{TR}^2}(\phi(E_d), \phi(E_e)) = 0.\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{Hom}^{\leq 0}_{\mathcal{TR}}(i_1(E_d), i_2(\phi(E_d)[c])) &= \mathrm{Hom}^{\leq 0}_{\mathcal{TR}^2}(\phi(E_d), \phi(E_d)[c-m]) \\ \cong \mathrm{Hom}^{\leq 0}_{\mathcal{TR}^2}(\phi(E_d), \phi(E_d)[c-m]) &\subset \mathrm{Hom}^{\leq 0}_{\mathcal{TR}^2}(\phi(E_d), \phi(E_d)[c-m]) = 0.\end{aligned}$$

□

Remark 5.3.9. Note that corollary 5.3.8 applies in the same manner for the collection $(i_1(\mathcal{E})[-c], i_2\phi(\mathcal{E}))$, as all definitions agree with the shift functor.

Corollary 5.3.10. *If $\mathcal{D}$ has a complete and Ext-exceptional collection, then so does $\mathcal{D}^{\uparrow}$.*

Proof. If $(E_0, \dots, E_n)$ is complete and Ext-exceptional collection in $\mathcal{D}$, then we can – for instance – choose

$$((E_0 \to 0), \dots, (E_n \to 0), (0 \to E_0), \dots, (0 \to E_n)) \subset \mathcal{D}^{\uparrow}$$

as an example of an exceptional collection that fulfils all the criteria of corollary 5.3.8. In other words, we apply corollary 5.3.8 in the situation of $\mathcal{D}$ and $\mathcal{D}^{\uparrow}$, where $\phi(E \to 0) = (0 \to E)$ and choose c to be equal to 0. By corollary 3.2.29 the condition $\mathrm{Hom}^i_{\mathcal{TR}^2}(\phi(A_1), A_2) \cong \mathrm{Hom}^i_{\mathcal{TR}}(i_1(A_1), i_2(A_2)[m])$ for $A_1 \in \mathcal{TR}^1$ and $A_2 \in \mathcal{TR}^2$ is fulfilled in the case of $\mathcal{D}$ and $\mathcal{D}^{\uparrow}$ with $m = 1$. □

A result by Macrí, given in [45] and in [46] now provides the use of exceptional collections for the stability space.

Proposition 5.3.11. *If $\mathcal{D}$ has a complete and Ext-exceptional collection $(E_0, \ldots, E_n)$ then there is an open, connected and simply connected $(n+1)$-dimensional submanifold $\Theta_{\mathcal{E}} \subset \operatorname{pre}\operatorname{Stab}(\mathcal{D}^{\uparrow})$.*

Proof. Since 5.3.10 provides that $\mathcal{D}^{\uparrow}$ has a complete and Ext-exceptional collection given that $\mathcal{D}$ has one, one can apply [45, Lemma 3.19]. □

Remark 5.3.12. Note that the support property was a feature not yet added to the set of conditions of a stability condition and is hence disregarded in [45] and in [46].

We can now conclude this subsection by providing a fact on the probably most common example regarding the theory of exceptional collections.

Corollary 5.3.13. *There is an open, connected and also simply connected $(2N+2)$-dimensional submanifold $\Theta_{\mathcal{E}} \subset \operatorname{pre}\operatorname{Stab}(\mathcal{D}^b((\mathbb{P}^N)^{\uparrow}))$.*

Proof. We can apply proposition 5.3.11 with regard to the exceptional collection given by $\mathcal{E} = \{\mathcal{O}, \ldots, \mathcal{O}(N)\}$ (see [9] or [31]). □

6 On the stability spaces of $\mathcal{D}^{\uparrow\uparrow}$ and $\mathcal{D}^{n\uparrow}$

The category $\mathcal{D}^{\uparrow}$ can be considered as the derived category of a simple graph obtained from $\mathcal{A}$ (**not** as graph obtained from $\mathcal{D}$), looking like this: $\cdot \to \cdot$). The question is natural, what one can say about the derived category of a more advanced graph such as $\cdot \to \cdot \to \cdot$, for start, and subsequently about one of the form

$$\underbrace{\cdot \to \cdots \to \cdot}_{n-arrows}.$$

In analogy to the naming of $\mathcal{A}^{\uparrow}$ the category of holomorphic triples if $\mathcal{A} = \operatorname{Coh}(C)$ one can now – more generally – talk about "holomorphic chains" of length n.

Evidently even this is only the first step in a prospective goal to understand the stability space of any given graph. This chapter investigates how far the previous findings on $\mathcal{D}^{\uparrow}$ can be generalised with regard to simple graphs of the kind mentioned above.

6.1 Gluing and recollement on $\mathcal{D}^{\uparrow\uparrow}$

Definition 6.1.1. In analogy to definition 2.1.12 we define $\mathcal{A}^{\uparrow\uparrow}$ to be the category for which $\mathrm{obj}(\mathcal{A}^{\uparrow})$ is the set of all arrows

$$A \to B \to C$$

between objects $A, B, C \in \mathcal{A}$. For $(A \xrightarrow{f} B \xrightarrow{g} C), (A' \xrightarrow{f'} B' \xrightarrow{g'} C') \in \mathrm{obj}(\mathcal{A}^{\uparrow})$ denote by

$$\mathrm{Hom}((A \xrightarrow{f} B \xrightarrow{g} C), (A' \xrightarrow{f'} B' \xrightarrow{g'} C'))$$

the set of all triples (ϕ, ϕ', ϕ'') of arrows such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{\phi'} & B' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{\phi''} & C'. \end{array}$$

Lemma 6.1.2. *The category $\mathcal{A}^{\uparrow\uparrow}$ is abelian.*

Proof. Similar to the proof of proposition 2.2.1, where one now uses the diagram

$$\begin{array}{ccccc} \mathrm{K}(\beta) & \xrightarrow{\widetilde{f}} & \mathrm{K}(\beta') & \xrightarrow{\widetilde{g}} & \mathrm{K}(\beta'') \\ \downarrow{\scriptstyle \ker(\beta)} & & \downarrow{\scriptstyle \ker(\beta')} & & \downarrow{\scriptstyle \ker(\beta'')} \\ B & \xrightarrow{f} & B' & \xrightarrow{f} & B'' \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta''} \\ C & \xrightarrow{g} & C' & \xrightarrow{g} & C''. \end{array}$$

□

Remark 6.1.3. Alternatively we can use definition 2.1.11 with $n = 2$.

Definition 6.1.4. Define $\mathcal{D}^{\uparrow\uparrow} = \mathcal{D}^b(\mathcal{A}^{\uparrow\uparrow})$.

We will now provide the generalisation of the theory developed in sections 3 and – with regard to recollement – in 4. Where our starting point then was the natural split-up of $\mathcal{D}^{\uparrow}$ into two copies of $\mathcal{D}$ we can now repeat this process in two different ways – either by considering three copies of $\mathcal{D}$ sitting in $\mathcal{D}^{\uparrow\uparrow}$ or by regarding $\mathcal{D}^{\uparrow\uparrow}$ as a composit of a copy of $\mathcal{D}$ and one of $\mathcal{D}^{\uparrow}$.

The first prospective is realised by the following definitions.

Definition 6.1.5. For an object $A \in \mathcal{D}$ and a morphism $f_1 \in \mathcal{D}$, define embeddings $i_1^{\uparrow\uparrow}, i_2^{\uparrow\uparrow}, i_3^{\uparrow\uparrow}$ and $\Delta^{\uparrow\uparrow}$ as the (trivial) extensions of the functors that restricted to $\mathcal{A}$ fulfil the equations

1.
$$\begin{aligned} i_1^{\uparrow\uparrow} &: \mathcal{D} \to \mathcal{D}^{\uparrow\uparrow} \\ i_1^{\uparrow\uparrow}(A) &= A \to 0 \to 0 \\ i_1^{\uparrow\uparrow}(f_1) &= (f_1, 0, 0), \end{aligned}$$

2.
$$\begin{aligned} i_2^{\uparrow\uparrow} &: \mathcal{D} \to \mathcal{D}^{\uparrow\uparrow} \\ i_2^{\uparrow\uparrow}(A) &= 0 \to A \to 0 \\ i_2^{\uparrow\uparrow}(f_1) &= (0, f_1, 0), \end{aligned}$$

3.
$$\begin{aligned} i_3^{\uparrow\uparrow} &: \mathcal{D} \to \mathcal{D}^{\uparrow\uparrow} \\ i_3^{\uparrow\uparrow}(A) &= 0 \to 0 \to A \\ i_3^{\uparrow\uparrow}(f_1) &= (0, 0, f_1), \end{aligned}$$

4.
$$\begin{aligned} \Delta^{\uparrow\uparrow} &: \mathcal{D} \to \mathcal{D}^{\uparrow\uparrow} \\ \Delta^{\uparrow\uparrow}(A) &= A \overset{\mathrm{id}}{\to} A \overset{\mathrm{id}}{\to} A \\ \Delta^{\uparrow\uparrow}(f_1) &= (f_1, f_1, f_1). \end{aligned}$$

Definition 6.1.6. Define projections $P_1^{\uparrow\uparrow}, P_2^{\uparrow\uparrow}$ and $P_3^{\uparrow\uparrow}$ as the (trivial) extensions of the functors that restricted to $\mathcal{A}^{\uparrow\uparrow}$ fulfil the equations

1.
$$\begin{aligned} P_1^{\uparrow\uparrow} &: \mathcal{D}^{\uparrow\uparrow} \to \mathcal{D} \\ P_1^{\uparrow\uparrow}(A \to B \to C) &= A \\ P_1^{\uparrow\uparrow}(f_1, f_2, f_3) &= f_1, \end{aligned}$$

2.

$$
\begin{aligned}
P_2^{\uparrow\uparrow} : \mathcal{D}^{\uparrow\uparrow} &\to \mathcal{D} \\
P_2^{\uparrow\uparrow}(A \to B \to C) &= B \\
P_2^{\uparrow\uparrow}(f_1, f_2, f_3) &= f_2,
\end{aligned}
$$

3.

$$
\begin{aligned}
P_3^{\uparrow\uparrow} : \mathcal{D}^{\uparrow\uparrow} &\to \mathcal{D} \\
P_3^{\uparrow\uparrow}(A \to B \to C) &= C \\
P_3^{\uparrow\uparrow}(f_1, f_2, f_3) &= f_3.
\end{aligned}
$$

The second perspective, however, is captured by the next definition.

Definition 6.1.7. Define embeddings $i_{1,2}^{\uparrow\uparrow}, i_{2,3}^{\uparrow\uparrow}, \Delta_1^{\uparrow\uparrow}$ and $\Delta_2^{\uparrow\uparrow}$ as the (trivial) extensions of the functors that restricted to $\mathcal{A}^{\uparrow}$ fulfil the equations

1.

$$
\begin{aligned}
i_{1,2}^{\uparrow\uparrow} : \mathcal{D}^{\uparrow} &\to \mathcal{D}^{\uparrow\uparrow} \\
i_{1,2}^{\uparrow\uparrow}(A \to B) &= A \to B \to 0 \\
i_{1,2}^{\uparrow\uparrow}(f_1, f_2) &= (f_1, f_2, 0),
\end{aligned}
$$

2.

$$
\begin{aligned}
i_{2,3}^{\uparrow\uparrow} : \mathcal{D}^{\uparrow} &\to \mathcal{D}^{\uparrow\uparrow} \\
i_{2,3}^{\uparrow\uparrow}(A \to B) &= 0 \to A \to B \\
i_{2,3}^{\uparrow\uparrow}(f_1, f_2) &= (0, f_1, f_2),
\end{aligned}
$$

3.

$$
\begin{aligned}
\Delta_1^{\uparrow\uparrow} : \mathcal{D}^{\uparrow} &\to \mathcal{D}^{\uparrow\uparrow} \\
\Delta_1^{\uparrow\uparrow}(A \to B) &= A \xrightarrow{\text{id}} A \to B \\
\Delta_1^{\uparrow\uparrow}(f_1, f_2) &= (f_1, f_1, f_2).
\end{aligned}
$$

4.

$$
\begin{aligned}
\Delta_2^{\uparrow\uparrow} : \mathcal{D}^{\uparrow} &\to \mathcal{D}^{\uparrow\uparrow} \\
\Delta_2^{\uparrow\uparrow}(A \to B) &= A \to B \xrightarrow{\text{id}} B \\
\Delta_2^{\uparrow\uparrow}(f_1) &= (f_1, f_2, f_2).
\end{aligned}
$$

At this point we can observe something awkward (which – in a more hidden form – already manifested itself in the definition of $i_2^{\uparrow\uparrow}$). This is that there is no functor $i_{1,3}^{\uparrow\uparrow}$ as the zero in the middle causes information-loss in the object. We will instead define a new functor.

Definition 6.1.8. Define the embedding $j_{1,3}^{\uparrow\uparrow}$ as the (trivial) extension of the functor that restricted to $\mathcal{A}^{\uparrow}$ fulfils the equations

$$\begin{aligned} j_{1,3}^{\uparrow\uparrow} &: \mathcal{D} \to \mathcal{D}^{\uparrow\uparrow} \\ j_{1,3}^{\uparrow\uparrow}(A \to B) &= A \to 0 \to B \\ i_{1,3}^{\uparrow\uparrow}(f_1, f_2) &= (f_1, 0, f_2). \end{aligned}$$

Remark 6.1.9. Note that $j_{1,3}^{\uparrow\uparrow}$ has potential significance for further studies however will not be used in the data provided in this subsection.

Definition 6.1.10. Define projections $P_{1,2}^{\uparrow\uparrow}, P_{1,3}^{\uparrow\uparrow}$ and $P_{2,3}^{\uparrow\uparrow}$ as the (trivial) extensions of the functors that restricted to $\mathcal{A}^{\uparrow\uparrow}$ fulfil the equations

1.
$$\begin{aligned} P_{1,2}^{\uparrow\uparrow} &: \mathcal{D}^{\uparrow\uparrow} \to \mathcal{D}^{\uparrow} \\ P_{1,2}^{\uparrow\uparrow}(A \to B \to C) &= A \to B \\ P_{1,2}^{\uparrow\uparrow}(f_1, f_2, f_3) &= (f_1, f_2), \end{aligned}$$

2.
$$\begin{aligned} P_{1,3}^{\uparrow\uparrow} &: \mathcal{D}^{\uparrow\uparrow} \to \mathcal{D}^{\uparrow} \\ P_{1,3}^{\uparrow\uparrow}(A \to B \to C) &= A \overset{A \to B \to C}{\longrightarrow} C \\ P_{1,3}^{\uparrow\uparrow}(f_1, f_2, f_3) &= (f_1, f_3), \end{aligned}$$

3.
$$\begin{aligned} P_{2,3}^{\uparrow\uparrow} &: \mathcal{D}^{\uparrow\uparrow} \to \mathcal{D}^{\uparrow} \\ P_{2,3}^{\uparrow\uparrow}(A \to B \to C) &= B \to C \\ P_{2,3}^{\uparrow\uparrow}(f_1, f_2, f_3) &= (f_2, f_3). \end{aligned}$$

With these functors established we must answer the question what the analogy to the adjunction-chain $\mathbb{K}[1] \dashv i_2 \dashv \rho_2 \dashv \Delta \dashv \lambda_1 \dashv i_1 \dashv \mathbb{K}$ of functors between $\mathcal{D}$ and $\mathcal{D}^{\uparrow}$ is.

Theorem 6.1.11. *There are chains of adjoint functors*

1. *between $\mathcal{D}$ and $\mathcal{D}^{\uparrow\uparrow}$:*

$$i_3^{\uparrow\uparrow} \dashv P_3^{\uparrow\uparrow} \dashv \Delta^{\uparrow\uparrow} \dashv P_1^{\uparrow\uparrow} \dashv i_1^{\uparrow\uparrow}$$

2. *and between $\mathcal{D}^{\uparrow}$ and $\mathcal{D}^{\uparrow\uparrow}$:*

$$i_{2,3}^{\uparrow\uparrow} \dashv P_{2,3}^{\uparrow\uparrow} \dashv \Delta_1^{\uparrow\uparrow} \dashv P_{1,3}^{\uparrow\uparrow} \dashv \Delta_2^{\uparrow\uparrow} \dashv P_{1,2}^{\uparrow\uparrow} \dashv i_{1,2}^{\uparrow\uparrow}.$$

Proof. In analogy to lemmas 3.2.3 and 3.2.4 we obtain the adjunctions in the abelian case. For example do we have

$$\mathrm{Hom}_{\mathcal{D}}(P_3^{\uparrow\uparrow\mathcal{A}}(A \xrightarrow{d} B \xrightarrow{e} C), D) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^{\uparrow\uparrow}}((A \xrightarrow{d} B \xrightarrow{e} C, \Delta^{\uparrow\uparrow\mathcal{A}}(D))$$

given by $f \mapsto (f \circ e \circ d, f \circ e, f)$. Additionally we use the exactness of these functors obtained analogous to lemma 3.2.6. □

We can achieve an extension of the chains provided in theorem 6.1.11 if we – once again – use the theory of Serre functors.

Lemma 6.1.12. *If $\mathcal{D}$ has a Serre functor then so has $\mathcal{D}^{\uparrow\uparrow}$.*

Proof. Similar to proposition 3.2.22 we can see that $\mathcal{D}^{\uparrow} = \langle i_{1,2}^{\uparrow\uparrow}(\mathcal{D}), i_3^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$. In analogy to lemma 4.2.12, we now observe that

$$i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}) = i_3^{\uparrow\uparrow}(\mathcal{D})^{\perp} = \mathrm{im}(i_3)^{\perp} = \ker(P_3^{\uparrow\uparrow}) = {}^{\perp}\,\mathrm{im}(\Delta^{\uparrow\uparrow}) = {}^{\perp}(\Delta^{\uparrow\uparrow}(\mathcal{D})).$$

Via the admissibility of $\Delta^{\uparrow\uparrow}(\mathcal{D})$ granted by $P_3^{\uparrow\uparrow} \dashv \Delta^{\uparrow\uparrow} \dashv P_1^{\uparrow\uparrow}$ in combination with lemma 4.2.1, we also obtain $i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})^{\perp} = \Delta^{\uparrow\uparrow}(\mathcal{D})$ and can therefore proceed analogously to lemma 4.2.16 to see that $i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})$ is admissible. We then repeat this argument with regard to $i_3^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})$ using that

$$i_3^{\uparrow\uparrow}(\mathcal{D}) = {}^{\perp}i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}) = {}^{\perp}\,\mathrm{im}(i_{1,2}^{\uparrow\uparrow}) = \ker(P_{1,2}^{\uparrow\uparrow}) = \mathrm{im}(\Delta_2^{\uparrow\uparrow})^{\perp} = \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})^{\perp}.$$

Hence, the result is obtained by theorem A.1.15. □

We can therefore extend theorem 6.1.11 and define new functors.

Lemma 6.1.13. *If $\mathcal{D}$ has a Serre-functor, we obtain adjunctions*

1. *between $\mathcal{D}$ and $\mathcal{D}^{\uparrow\uparrow}$:*

$$\mathbb{K}P_{2,3}^{\uparrow\uparrow}[1] \dashv i_3^{\uparrow\uparrow} \dashv P_3^{\uparrow\uparrow} \dashv \Delta^{\uparrow\uparrow} \dashv P_1^{\uparrow\uparrow} \dashv i_1^{\uparrow\uparrow} \dashv \mathbb{K}P_{1,2}^{\uparrow\uparrow} \dashv i_2^{\uparrow\uparrow}[1] \dashv \mathbb{K}P_{2,3}^{\uparrow\uparrow}[-1]$$

2. between $\mathcal{D}^{\uparrow}$ and $\mathcal{D}^{\uparrow\uparrow}$:

$$\mathbb{K}_2^{\uparrow\uparrow}[1] \dashv i_{2,3}^{\uparrow\uparrow} \dashv P_{2,3}^{\uparrow\uparrow} \dashv \Delta_1^{\uparrow\uparrow} \dashv P_{1,3}^{\uparrow\uparrow} \dashv \Delta_2^{\uparrow\uparrow} \dashv P_{1,2}^{\uparrow\uparrow} \dashv i_{1,2}^{\uparrow\uparrow} \dashv \mathbb{K}_1^{\uparrow\uparrow},$$

where $\mathbb{K}_1^{\uparrow\uparrow}$ and $\mathbb{K}_2^{\uparrow\uparrow}$ are given via the Serre-functor as $\mathbb{K}_1^{\uparrow\uparrow} = S_{\mathcal{D}^{\uparrow}} \circ P_{1,2}^{\uparrow\uparrow} \circ S_{\mathcal{D}^{\uparrow\uparrow}}^{-1}$ and $\mathbb{K}_2^{\uparrow\uparrow} = S_{\mathcal{D}^{\uparrow}}^{-1} \circ P_{2,3}^{\uparrow\uparrow}[-1] \circ S_{\mathcal{D}^{\uparrow\uparrow}}$.

Proof. To see how the adjunction $\mathbb{K}P_{2,3}^{\uparrow\uparrow}[1] \dashv i_3^{\uparrow\uparrow}$ and the chain of adjunctions of functors $i_1^{\uparrow\uparrow} \dashv \mathbb{K}P_{1,2}^{\uparrow\uparrow} \dashv i_2^{\uparrow\uparrow}[1] \dashv \mathbb{K}P_{2,3}^{\uparrow\uparrow}[-1]$ are obtained we will demonstrate in the case of $\mathbb{K}P_{1,2}^{\uparrow\uparrow} \dashv i_2^{\uparrow\uparrow}[1]$. We have

$$\begin{gathered}\mathrm{Hom}_{\mathcal{D}}(\mathbb{K}P_{1,2}^{\uparrow\uparrow}(X), Y) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(P_{1,2}^{\uparrow\uparrow}(X), i_2[1](Y)) = \\ \mathrm{Hom}_{\mathcal{D}^{\uparrow\uparrow}}(X, i_{1,2}^{\uparrow\uparrow}(i_2[1](Y))) = \mathrm{Hom}_{\mathcal{D}^{\uparrow\uparrow}}(X, i_2^{\uparrow\uparrow}[1](Y))\end{gathered}$$

where $X \in \mathcal{D}^{\uparrow\uparrow}$ and $Y \in \mathcal{D}$.

We obtain $\mathbb{K}_1^{\uparrow\uparrow}$ and $\mathbb{K}_2^{\uparrow\uparrow}$ by theorem A.1.16. □

Lemma 6.1.14. *There are exact triangles*

- $$i_{2,3}^{\uparrow\uparrow}P_{2,3}^{\uparrow\uparrow}(X) \to X \to i_1^{\uparrow\uparrow}P_1^{\uparrow\uparrow}(X) \xrightarrow{+}$$
- $$i_1^{\uparrow\uparrow}\mathbb{K}P_{1,2}^{\uparrow\uparrow}(X) \to X \to \Delta_1^{\uparrow\uparrow}P_{2,3}^{\uparrow\uparrow}(X) \xrightarrow{+}$$
- $$\Delta_1^{\uparrow\uparrow}P_{1,3}^{\uparrow\uparrow}(X) \to X \to i_2^{\uparrow\uparrow}\mathbb{K}P_{1,2}^{\uparrow\uparrow}(X)[1] \xrightarrow{+}$$
- $$i_2^{\uparrow\uparrow}\mathbb{K}P_{2,3}^{\uparrow\uparrow}(X) \to X \to \Delta_2^{\uparrow\uparrow}P_{1,3}^{\uparrow\uparrow}(X) \xrightarrow{+}$$
- $$\Delta_2^{\uparrow\uparrow}P_{1,2}^{\uparrow\uparrow}(X) \to X \to i_3^{\uparrow\uparrow}\mathbb{K}P_{2,3}^{\uparrow\uparrow}(X)[1] \xrightarrow{+}$$
- $$i_3^{\uparrow\uparrow}P_3^{\uparrow\uparrow}(X) \to X \to i_{1,2}^{\uparrow\uparrow}P_{1,2}^{\uparrow\uparrow}(X) \xrightarrow{+}$$
- $$i_{1,2}^{\uparrow\uparrow}\mathbb{K}_1^{\uparrow\uparrow}(X) \to X \to \Delta^{\uparrow\uparrow}P_3^{\uparrow\uparrow}(X) \xrightarrow{+}$$
- $$\Delta^{\uparrow\uparrow}P_1^{\uparrow\uparrow}(X) \to X \to i_{2,3}^{\uparrow\uparrow}\mathbb{K}_2^{\uparrow\uparrow}(X)[1] \xrightarrow{+}$$

in $\mathcal{D}^{\uparrow\uparrow}$.

Proof. For the exact triangles that include the functors $\mathbb{K}_1^{\uparrow\uparrow}, \mathbb{K}_2^{\uparrow\uparrow}$, we obtain the result similar (and generalising it) to lemma 4.2.26. In the cases of the exact sequences where $\mathbb{K}$ is involved we deduce the existence of the exact triangle from $\mathcal{D}^{\uparrow}$. For instance is the triangle

$$i_1^{\uparrow\uparrow}\mathbb{K}P_{1,2}^{\uparrow\uparrow}(X) \to X \to \Delta_1^{\uparrow\uparrow}P_{2,3}^{\uparrow\uparrow}(X) \xrightarrow{+}$$

with $X = (X_1 \to X_2 \to X_3)$ provided by

$$i_1(\mathbb{K}(X_1 \to X_2)) \to (X_1 \to X_2) \to \Delta(X_2) \xrightarrow{+},$$

since

$$\begin{aligned} &P_{1,2}^{\uparrow\uparrow}(i_1^{\uparrow\uparrow}\mathbb{K}P_{1,2}^{\uparrow\uparrow}(X) \to X \to \Delta_1^{\uparrow\uparrow}P_{2,3}^{\uparrow\uparrow}(X) \xrightarrow{+}) \\ = {}& i_1(\mathbb{K}(X_1 \to X_2)) \to (X_1 \to X_2) \to \Delta(X_2) \xrightarrow{+}, \end{aligned}$$

as illustrated by

$$\begin{array}{ccccccc} \mathbb{K}(X_1 \to X_2) & \longrightarrow & X_1 & \longrightarrow & X_2 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & X_2 & \xrightarrow{+} & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_3 & \longrightarrow & X_3. & & \end{array}$$

The exact triangles

$$i_{2,3}^{\uparrow\uparrow}P_{2,3}^{\uparrow\uparrow}(X) \to X \to i_1^{\uparrow\uparrow}P_1^{\uparrow\uparrow}(X) \xrightarrow{+} \text{ and } i_3^{\uparrow\uparrow}P_3^{\uparrow\uparrow}(X) \to X \to i_{1,2}^{\uparrow\uparrow}P_{1,2}^{\uparrow\uparrow}(X) \xrightarrow{+}$$

finally, are already short exact sequences in $\mathcal{C}(\mathcal{A}^{\uparrow\uparrow})$. □

Remark 6.1.15. Lemma 6.1.14 also allows us to provide a less abstract description of the functors $\mathbb{K}_1^{\uparrow\uparrow}, \mathbb{K}_2^{\uparrow\uparrow} : \mathcal{D}^{\uparrow\uparrow} \to \mathcal{D}^{\uparrow}$. To understand $\mathbb{K}_1^{\uparrow\uparrow}$ we consider the exact triangle

$$i_{1,2}^{\uparrow\uparrow}\mathbb{K}_1^{\uparrow\uparrow}(X) \to X \xrightarrow{\xi} \Delta^{\uparrow\uparrow}P_3^{\uparrow\uparrow}(X) \xrightarrow{+}$$

for $X = (X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3)$. The morphism ξ is then given by the chain map

$$\begin{array}{ccc}
X_1 & \xrightarrow{g\circ f} & X_3 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X_3}} \\
X_2 & \xrightarrow{g} & X_3 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X_3}} \\
X_3 & \xrightarrow[\mathrm{id}_{X_3}]{} & X_3.
\end{array}$$

This yields the description of the functor's action on objects as $\mathbb{K}_1^{\uparrow\uparrow}(X) \cong \mathrm{Cone}(g \circ f)[-1] \to \mathrm{Cone}(g)[-1]$.

To understand $\mathbb{K}_2^{\uparrow\uparrow}$ we consider the exact triangle

$$\Delta^{\uparrow\uparrow} P_1^{\uparrow\uparrow}(X) \to X \to i_{2,3}^{\uparrow\uparrow}\mathbb{K}_2^{\uparrow\uparrow}(X)[1] \xrightarrow{+}$$

for $X = (X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3)$. Similar to the case of $\mathbb{K}_1^{\uparrow\uparrow}$, that we have just investigated, this yields the description of the functors action on objects as $\mathbb{K}_2^{\uparrow\uparrow}(X) \cong \mathrm{Cone}(f)[-1] \to \mathrm{Cone}(g \circ f)[-1]$. Another way to see this is by considering the following identities on the functors that we introduced.

$$\begin{aligned}
\lambda_1 \circ \mathbb{K}_2^{\uparrow\uparrow} = \mathbb{K} \circ P_{1,2}^{\uparrow\uparrow}, \rho_2 \circ \mathbb{K}_2^{\uparrow\uparrow} = \mathbb{K} \circ P_{1,3}^{\uparrow\uparrow}, \mathbb{K} \circ \mathbb{K}_2^{\uparrow\uparrow} = \mathbb{K} \circ P_{2,3}^{\uparrow\uparrow}[-1], \\
\lambda_1 \circ \mathbb{K}_1^{\uparrow\uparrow} = \mathbb{K} \circ P_{1,3}^{\uparrow\uparrow}, \rho_2 \circ \mathbb{K}_1^{\uparrow\uparrow} = \mathbb{K} \circ P_{2,3}^{\uparrow}, \mathbb{K} \circ \mathbb{K}_1^{\uparrow\uparrow} = \mathbb{K} \circ P_{1,2}^{\uparrow\uparrow}
\end{aligned} \tag{6.1}$$

The following lemma shows that we obtain a "circle" of semiorthogonal decompositions like we did in the situation of $\mathcal{D}^{\uparrow}$.

Lemma 6.1.16. *We obtain semiorthogonal decompositions*

1. $\langle i_1^{\uparrow\uparrow}(\mathcal{D}), i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$
2. $\langle i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), \Delta^{\uparrow\uparrow}(\mathcal{D})\rangle$
3. $\langle \Delta^{\uparrow\uparrow}(\mathcal{D}), i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$
4. $\langle i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), i_3^{\uparrow\uparrow}(\mathcal{D})\rangle$
5. $\langle i_3^{\uparrow\uparrow}(\mathcal{D}), \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$
6. $\langle \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), i_2^{\uparrow\uparrow}(\mathcal{D})\rangle$
7. $\langle i_2^{\uparrow\uparrow}(\mathcal{D}), \Delta_1^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$

8. $\langle \Delta_1^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), i_1^{\uparrow\uparrow}(\mathcal{D})\rangle$

on $\mathcal{D}^{\uparrow\uparrow}$*.*

Proof. We obtain the necessary exact triangles from lemma 6.1.14. The vanishing of the homomorphisms can be deduced in a very similar manner to that in which it was done for $\mathcal{D}^{\uparrow}$. Consider, for example, $A \in \mathcal{D}^{\uparrow}$ and $B \in \mathcal{D}$. By lemma 6.1.13 we obtain

$$\mathrm{Hom}_{\mathcal{D}^{\uparrow\uparrow}}(i_{2,3}^{\uparrow\uparrow}(A), i_1^{\uparrow\uparrow}(B)) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, P_{2,3}^{\uparrow\uparrow}(i_1^{\uparrow\uparrow}(B))) = \mathrm{Hom}_{\mathcal{D}^{\uparrow}}(A, 0) = 0.$$

□

We sum up this subsection by stating the relation between gluing- and recollement-data on $\mathcal{D}^{\uparrow\uparrow}$ with regard to the semiorthogonal decompositions we have found.

Remark 6.1.17. Note that via the new style functor $j_{1,3}^{\uparrow\uparrow}$ it is possible to compute two more semiorthogonal decompositions, these are

- $$\langle \Delta_{1,2}^{\uparrow\uparrow}(\mathcal{D}), j_{1,3}^{\uparrow\uparrow}(\mathcal{D} \times \mathcal{D})\rangle$$

  and

- $$\langle j_{1,3}^{\uparrow\uparrow}(\mathcal{D} \times \mathcal{D}), \Delta_{2,3}^{\uparrow\uparrow}(\mathcal{D})\rangle$$

with $\Delta_{1,2}^{\uparrow\uparrow}$ and $\Delta_{2,3}^{\uparrow\uparrow}$ to be defined in remark 6.1.20.

Corollary 6.1.18. *If* $\mathcal{D}$ *has a Serre-functor, CP-gluing-data on* $\mathcal{D}^{\uparrow\uparrow}$ *given by a semiorthogonal decomposition of the kind provided in lemma 6.1.16 extends* *to recollement-data. The corresponding recollement-datas are:*

$$\begin{aligned}
&\underline{I}: \mathcal{Y} = i_1^{\uparrow\uparrow}(\mathcal{D}); \mathcal{X} = i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})); i^* = P_1^{\uparrow\uparrow}, i_* = i_! = i_1^{\uparrow\uparrow},\\
&i^! = \mathbb{K}\circ P_{1,2}^{\uparrow\uparrow}, j_! = i_{2,3}^{\uparrow\uparrow}, j^* = j^! = P_{2,3}^{\uparrow\uparrow}, j_* = \Delta_1^{\uparrow\uparrow}\\
&\underline{II}: \mathcal{Y} = i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}); \mathcal{X} = \Delta^{\uparrow\uparrow}(\mathcal{D})); i^* = \mathbb{K}_2^{\uparrow\uparrow}[1], i_* = i_! = i_{2,3}^{\uparrow\uparrow},\\
&i^! = P_{2,3}^{\uparrow\uparrow}, j_! = \Delta^{\uparrow\uparrow}, j^* = j^! = P_1^{\uparrow\uparrow}, j_* = i_1^{\uparrow\uparrow}\\
&\underline{III}: \mathcal{Y} = \Delta^{\uparrow\uparrow}(\mathcal{D}); \mathcal{X} = i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})); i^* = P_3^{\uparrow\uparrow}, i_* = i_! = \Delta^{\uparrow\uparrow},\\
&i^! = P_1^{\uparrow\uparrow}, j_! = i_{1,2}^{\uparrow\uparrow}, j^* = j^! = \mathbb{K}_1^{\uparrow\uparrow}, j_* = S_{\mathcal{D}^{\uparrow\uparrow}}\circ i_{1,2}^{\uparrow\uparrow}\circ S_{\mathcal{D}^{\uparrow}}\\
&\underline{IV}: \mathcal{Y} = i_{1,2}^{\uparrow\uparrow}(\mathcal{D}); \mathcal{X} = i_3^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})); i^* = P_{1,2}^{\uparrow\uparrow}, i_* = i_! = i_{1,2}^{\uparrow\uparrow},\\
&i^! = \mathbb{K}_1^{\uparrow\uparrow}, j_! = i_3^{\uparrow\uparrow}, j^* = j^! = P_3^{\uparrow\uparrow}, j_* = \Delta^{\uparrow\uparrow}\\
&\underline{V}: \mathcal{Y} = i_3^{\uparrow\uparrow}(\mathcal{D}); \mathcal{X} = \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})); i^* = \mathbb{K}\circ P_{2,3}^{\uparrow\uparrow}[1], i_* = i_! = i_3^{\uparrow\uparrow},\\
&i^! = P_3^{\uparrow\uparrow}, j_! = \Delta_2^{\uparrow\uparrow}, j^* = j^! = P_{1,2}^{\uparrow\uparrow}, j_* = i_{1,2}^{\uparrow\uparrow}\\
&\underline{VI}: \mathcal{Y} = \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}); \mathcal{X} = i_2^{\uparrow\uparrow}(\mathcal{D})); i^* = P_{1,3}^{\uparrow\uparrow}, i_* = i_! = \Delta_2^{\uparrow\uparrow},\\
&i^! = P_{1,2}^{\uparrow\uparrow}, j_! = i_2,^{\uparrow\uparrow}, j^* = j^! = \mathbb{K}\circ P_{2,3}^{\uparrow\uparrow}[-1], j_* = i_3^{\uparrow\uparrow}[2]\\
&\underline{VII}: \mathcal{Y} = i_2^{\uparrow\uparrow}(\mathcal{D}); \mathcal{X} = \Delta_1^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})); i^* = \mathbb{K}\circ P_{1,2}^{\uparrow\uparrow}[-1], i_* = i_! = i_2^{\uparrow\uparrow},\\
&i^! = \mathbb{K}\circ P_{2,3}^{\uparrow\uparrow}, j_! = \Delta_1^{\uparrow\uparrow}, j^* = j^! = P_{1,3}^{\uparrow\uparrow}, j_* = \Delta_2^{\uparrow\uparrow}\\
&\underline{VIII}: \mathcal{Y} = \Delta_1^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}); \mathcal{X} = i_1^{\uparrow\uparrow}(\mathcal{D})); i^* = P_{2,3}^{\uparrow\uparrow}, i_* = i_! = \Delta_1^{\uparrow\uparrow},\\
&i^! = P_{1,3}^{\uparrow\uparrow}, j_! = i_1^{\uparrow\uparrow}, j^* = j^! = \mathbb{K}\circ P_{1,2}^{\uparrow\uparrow}, j_* = i_2^{\uparrow\uparrow}[1]
\end{aligned} \tag{6.2}$$

resulting in t-structures:

$$
\begin{aligned}
\underline{I}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_{2,3}(Z) \in \mathcal{D}^{\leq 0}_1, P^{\uparrow\uparrow}_1(Z) \in \mathcal{D}^{\leq 0}_2\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_{2,3}(Z) \in \mathcal{D}^{\geq 1}_1, \mathbb{K} \circ P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\geq 1}_2\}\\
\\
\underline{II}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_1(Z) \in \mathcal{D}^{\leq 0}_2, \mathbb{K}^{\uparrow\uparrow}_2[1](Z) \in \mathcal{D}^{\leq 0}_2\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_1(Z) \in \mathcal{D}^{\geq 1}_2, \mathbb{K} \circ P^{\uparrow\uparrow}_{2,3}(Z) \in \mathcal{D}^{\geq 1}_1\}\\
\\
\underline{III}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}^{\uparrow\uparrow}_1 \in \mathcal{D}^{\leq 0}_1, P^{\uparrow\uparrow}_3(Z) \in \mathcal{D}^{\leq 0}_2\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}^{\uparrow\uparrow}_1 \in \mathcal{D}^{\geq 1}_1, P^{\uparrow\uparrow}_1(Z) \in \mathcal{D}^{\geq 1}_2\}\\
\\
\underline{IV}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_3(Z) \in \mathcal{D}^{\leq 0}_2, P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\leq 0}_1\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_3(Z) \in \mathcal{D}^{\geq 1}_2, \mathbb{K}^{\uparrow\uparrow}_1 \in \mathcal{D}^{\geq 1}_1\}\\
\\
\underline{V}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\leq 0}_1, \mathbb{K} \circ P^{\uparrow\uparrow}_{2,3}[1](Z) \in \mathcal{D}^{\leq 0}_2\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\geq 1}_1, P^{\uparrow\uparrow}_3(Z) \in \mathcal{D}^{\geq 1}_2\}\\
\\
\underline{VI}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K} \circ P^{\uparrow\uparrow}_{2,3}[1](Z) \in \mathcal{D}^{\leq 0}_2, P^{\uparrow\uparrow}_{1,3}(Z) \in \mathcal{D}^{\leq 0}_1\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K} \circ P^{\uparrow\uparrow}_{2,3}[1](Z) \in \mathcal{D}^{\geq 1}_2, P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\geq 1}_1\}\\
\\
\underline{VII}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_{1,3}(Z) \in \mathcal{D}^{\leq 0}_1, \mathbb{K} \circ P^{\uparrow\uparrow}_{1,2}[-1](Z) \in \mathcal{D}^{\leq 0}_2\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid P^{\uparrow\uparrow}_{1,3}(Z) \in \mathcal{D}^{\geq 1}_1, \mathbb{K} \circ P^{\uparrow\uparrow}_{2,3}(Z) \in \mathcal{D}^{\geq 1}_2\}\\
\\
\underline{VIII}: \mathcal{D}^{\leq 0} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K} \circ P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\leq 0}_2, P^{\uparrow\uparrow}_{2,3}(Z) \in \mathcal{D}^{\leq 0}_1\},\\
\mathcal{D}^{\geq 1} &= \{Z \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K} \circ P^{\uparrow\uparrow}_{1,2}(Z) \in \mathcal{D}^{\geq 1}_2, \mathbb{K} \circ P^{\uparrow\uparrow}_{1,3}(Z) \in \mathcal{D}^{\geq 1}_1\}
\end{aligned}
\tag{6.3}
$$

where $(\mathcal{D}^{\leq 0}_1, \mathcal{D}^{\geq 1}_1)$ *is a t-structure on* $\mathcal{D}^{\uparrow}$ *and* $(\mathcal{D}^{\leq 0}_2, \mathcal{D}^{\geq 1}_2)$ *is a t-structure on* $\mathcal{D}$*.*

Proof. On the bases of theorem 6.1.12 this follows mostly from lemma 6.1.13 and lemma 6.1.14. □

In order to find stability conditions we will now investigate the criterion for hearts of the previously constructed t-structures on $\mathcal{D}^{\uparrow\uparrow}$ to be CP-glued.

Lemma 6.1.19. *For hearts of bounded t-structures* $H_1 = \mathcal{P}_\mu(\alpha, \alpha+1], H_2 = \mathcal{P}_\mu(\beta, \beta+1], H_3 = \mathcal{P}_\mu(\gamma, \gamma+1]$ *on* $\mathcal{D}$*, we obtain hearts of t-structures*

1. *If* $\alpha \geq \beta \geq \gamma$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

2. *If* $\alpha \geq \gamma+1, \beta \geq \gamma+1$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

3. *If* $\alpha \geq \beta \geq \gamma+1$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

4. *If* $\alpha \geq \beta \geq \gamma+1$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

5. *If* $\alpha \geq \beta+1 \geq \gamma+2$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{2,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\}$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

6. *If* $\alpha \geq \beta+1 \geq \gamma+1$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

7. *If* $\alpha \geq \beta+1 \geq \gamma+1$ *then*
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$
is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ *obtained by CP-gluing.*

8. *If $\alpha \geq \beta \geq \gamma + 1$ then*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ obtained by CP-gluing.

9. *If $\alpha \geq \beta + 1 \geq \gamma + 2$ then*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\},$$

is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ obtained by CP-gluing.

10. *If $\alpha \geq \beta + 1 \geq \gamma + 1$ then*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \lambda_1(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ obtained by CP-gluing.

11. *If $\alpha \geq \beta + 1, \alpha \geq \gamma$ then*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \rho_2(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ obtained by CP-gluing.

12. *If $\alpha \geq \beta + 1 \geq \gamma + 2$ then*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}P_{2,3}^{\uparrow\uparrow}[1](X) \in H_1, \rho_2(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\},$$

is a heart of a bounded t-structure on $\mathcal{D}^{\uparrow\uparrow}$ obtained by CP-gluing.

Proof. This uses arguments analogous to those needed to prove corollary 3.2.31, one uses lemma 3.1.5 and the key is the vanishing of the homomorphisms. The condition on α, β and γ guarantees that the heart $H_{(i,j)}$ is obtained by CP-gluing on $\mathcal{D}^{\uparrow}$ in the first place. Let $H_{(i,j)}$ be the heart of the t-structure obtained by recollement from t-structures with hearts H_i, H_j, where $(i, j) \in \{(1, 2), (2, 3)\}$ on $\mathcal{D}^{\uparrow}$. In other words, H_{ij} is defined as either

- $H_{ij} = \{E \in \mathcal{D}^{\uparrow} \mid \lambda_1(E) \in H_i, \rho_2(E) \in \mathcal{H}_j\}$, or
- $H_{ij} = \{E \in \mathcal{D}^{\uparrow} \mid \mathbb{K}[1](E) \in H_i, \lambda_1(E) \in H_j\}$, or
- $H_{ij} = \{E \in \mathcal{D}^{\uparrow} \mid \rho_2(E) \in \mathcal{H}_i, \mathbb{K}(E) \in H_j\}$.

For the semiorthogonal decomposition

1. $\langle i_1^{\uparrow\uparrow}(\mathcal{D}), i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, P_{2,3}^{\uparrow\uparrow}(X) \in H_{(2,3)}\},$$

2. $\langle i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), \Delta^{\uparrow\uparrow}(\mathcal{D})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}_2^{\uparrow\uparrow}[1](X) \in H_{(1,2)}, P_1^{\uparrow\uparrow}(X) \in H_3\},$$

3. $\langle \Delta^{\uparrow\uparrow}(\mathcal{D}), i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}_1^{\uparrow\uparrow}(X) \in H_{(2,3)}\},$$

4. $\langle i_{1,2}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), i_3^{\uparrow\uparrow}(\mathcal{D})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_{1,2}^{\uparrow\uparrow}(X) \in H_{(1,2)}, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

5. $\langle i_3^{\uparrow\uparrow}(\mathcal{D}), \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}P_{2,3}^{\uparrow\uparrow}[1](X) \in H_1, P_{1,2}^{\uparrow\uparrow}(X) \in H_{(2,3)}\},$$

6. $\langle \Delta_2^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), i_2^{\uparrow\uparrow}(\mathcal{D})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_{1,3}^{\uparrow\uparrow}(X) \in H_{(1,2)}, \mathbb{K}P_{2,3}^{\uparrow\uparrow}(X) \in H_3\},$$

7. $\langle i_2^{\uparrow\uparrow}(\mathcal{D}), \Delta_1^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}P_{1,2}^{\uparrow\uparrow}[-1](X) \in H_1, P_{1,3}^{\uparrow\uparrow}(X) \in H_{(2,3)}\},$$

8. $\langle \Delta_1^{\uparrow\uparrow}(\mathcal{D}^{\uparrow}), i_1^{\uparrow\uparrow}(\mathcal{D})\rangle$ we obtain that the heart of the corresponding recollement t-structure if obtained by gluing has the form
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_{(2,3)}^{\uparrow\uparrow}(X) \in H_{(1,2)}, \mathbb{K}P_{1,2}^{\uparrow\uparrow}(X) \in H_3\}.$$

To see that the vanishing-condition regarding the homomorphisms holds take – for example – the heart

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

that, if $\alpha \geq \beta$, is obtained by CP-gluing via the semiorthogonal decomposition $\langle i_1^{\uparrow\uparrow}(\mathcal{D}), i_{2,3}^{\uparrow\uparrow}(\mathcal{D}^{\uparrow})\rangle$. By lemma 3.1.5, we have to prove

$$\mathrm{Hom}^{\leq 0}(i_1^{\uparrow\uparrow}(X), i_{2,3}^{\uparrow\uparrow}(Y)) = 0 \tag{6.4}$$

for any $X \in H_1$ and $Y \in H_{(2,3)}$. Consider the exact triangle

$$i_2(\rho_2(Y)) \to Y \to i_1(\lambda_1(Y)) \xrightarrow{+} .$$

Now $\alpha \geq \beta$ ensures

$$\mathrm{Hom}^{\leq 0}(i_1^{\uparrow\uparrow}(X), i_{2,3}^{\uparrow\uparrow}(i_2(\rho_2(Y)))) = 0$$

as well as

$$\mathrm{Hom}^{\leq 0}(i_1^{\uparrow\uparrow}(X), i_{2,3}^{\uparrow\uparrow}(i_1(\lambda_1(Y)))) = 0$$

and hence (6.4) holds.

The 8 types of hearts above for $H_{(i,j)}$ result in 24 types of hearts of t-structures in total as there are three ways of gluing $H_{(i,j)}$. However, 12 of the types of hearts that we obtain turn out to be simple re-writes of others – denoting by $x.y$ the type of heart glued via the semiorthogonal decomposition number x on $\mathcal{D}^{\uparrow\uparrow}$ (in the order in which they appear in the lemma) and the semiorthogonal decomposition number y on $\mathcal{D}^{\uparrow}$ (for $(\mathcal{D}_y, \mathcal{D}_y^{\perp})$) via which $H_{(i,j)}$ is glued, we obtain

$$\begin{aligned}
1.1 &= 4.1\\
1.2 &= 5.1\\
1.3 &= 6.1\\
2.1 &= 7.2\\
2.2 &= 5.2\\
2.3 &= 6.2\\
3.1 &= 6.3\\
3.2 &= 7.3\\
3.3 &= 8.3\\
4.2 &= 7.1\\
4.3 &= 8.1\\
5.3 &= 8.2
\end{aligned}$$

such that we are indeed left with the 12 types of hearts of t-structures described – however in the cases of 5.1, 6.2, 7.3 and 8.1 the hearts H_1, H_2 and H_3 are in a different order and hence suitable adjustments to the inequalities on α, β and γ have to be made.

Additionally, note that using the identities (6.1) of remark 6.1.15 on some of the (less convenient) functors we have

$$
\begin{aligned}
&\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \rho_2(\mathbb{K}_2^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\} \\
&= \{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},
\end{aligned}
$$

$$
\begin{aligned}
&\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{2,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\} \\
&= \{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}[1](\mathbb{K}_2^{\uparrow\uparrow}[1](X)) \in H_1, \lambda_1(\mathbb{K}_2^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\}, \\
&\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\}, \\
&= \{X \in \mathcal{D}^{\uparrow\uparrow} \mid \rho_2(\mathbb{K}_2^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(\mathbb{K}_2^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\}, \\
&\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\}, \\
&= \{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \lambda_1(\mathbb{K}_1^{\uparrow\uparrow}(X)) \in H_2, \rho_2(\mathbb{K}_1^{\uparrow\uparrow}(X)) \in H_3\}, \\
&\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_3\}, \\
&= \{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](\mathbb{K}_1^{\uparrow\uparrow}(X)) \in H_2, \lambda_1(\mathbb{K}_1^{\uparrow\uparrow}(X)) \in H_3\}, \\
&\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\}, \\
&= \{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \rho_2(\mathbb{K}_1^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(\mathbb{K}_1^{\uparrow\uparrow}(X)) \in H_3\}.
\end{aligned}
$$

□

Remark 6.1.20. The semiorthogonal decompositions with three embedded subcategories used to directly obtain the hearts of lemma 6.1.19 (triggering remark 2.1.7) are

1. $\langle i_1^{\uparrow\uparrow}(\mathcal{D}), \langle i_2^{\uparrow\uparrow}(\mathcal{D}), i_3^{\uparrow\uparrow}(\mathcal{D})\rangle$ for

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

2. $\langle i_1^{\uparrow\uparrow}(\mathcal{D}), i_3^{\uparrow\uparrow}(\mathcal{D}), \Delta_{2,3}^{\uparrow\uparrow}(\mathcal{D})\rangle$ for

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

3. $\langle i_1^{\uparrow\uparrow}(\mathcal{D}), \Delta_{2,3}^{\uparrow\uparrow}(\mathcal{D}), i_2^{\uparrow\uparrow}(\mathcal{D})\rangle$ for

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

4. $\langle i_2^{\uparrow\uparrow}(\mathcal{D}), i_3^{\uparrow\uparrow}(\mathcal{D}), \Delta^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},$$

5. $\langle i_3^{\uparrow\uparrow}(\mathcal{D}), \Delta_{2,3}^{\uparrow\uparrow}(\mathcal{D}), \Delta^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{2,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\}$$

6. $\langle \Delta_{2,3}^{\uparrow\uparrow}(\mathcal{D}), i_2^{\uparrow\uparrow}(\mathcal{D}), \Delta^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},$$

7. $\langle \Delta^{\uparrow\uparrow}(\mathcal{D}), i_1^{\uparrow\uparrow}(\mathcal{D}), i_2^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

8. $\langle \Delta^{\uparrow\uparrow}(\mathcal{D}), i_2^{\uparrow\uparrow}(\mathcal{D}), \Delta_{1,2}^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

9. $\langle \Delta^{\uparrow\uparrow}(\mathcal{D}), \Delta_{1,2}^{\uparrow\uparrow}(\mathcal{D}), i_1^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\},$$

10. $\langle i_2^{\uparrow\uparrow}(\mathcal{D}), \Delta_{1,2}^{\uparrow\uparrow}(\mathcal{D}), i_1^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \lambda_1(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

11. $\langle \Delta_{1,2}^{\uparrow\uparrow}(\mathcal{D}), i_1^{\uparrow\uparrow}(\mathcal{D}), i_3^{\uparrow\uparrow}(\mathcal{D})\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \rho_2(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

12. $\langle i_3^{\uparrow\uparrow}(\mathcal{D}), \Delta^{\uparrow\uparrow}(\mathcal{D}), i_1^{\uparrow\uparrow}(\mathcal{D}\rangle$ for
$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}P_{2,3}^{\uparrow\uparrow}[1](X) \in H_1, \rho_2(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\},$$

where for $A \in \mathcal{D}$ we define the functors $\Delta_{1,2}^{\uparrow\uparrow}$ and $\Delta_{2,3}^{\uparrow\uparrow}$ as

$$\Delta_{1,2}(A) := A \xrightarrow{\mathrm{id}_A} A \to 0$$
$$\Delta_{2,3}(A) := 0 \to A \xrightarrow{\mathrm{id}_A} A.$$

Lemma 6.1.21. *For stability conditions $\sigma_1 = (Z_1, H_1), \sigma_2 = (Z_2, H_2)$, and $\sigma_3 = (Z_3, H_3)$ on $\mathcal{D}$ with hearts of t-structures $H_1 = \mathcal{P}_\mu(\alpha, \alpha+1], H_2 = \mathcal{P}_\mu(\beta, \beta+1], H_3 = \mathcal{P}_\mu(\gamma, \gamma+1]$ let $Z_{(i,j)}$ be the group homomorphism induced by $Z_i, Z_j, (i,j) \in \{(1,2),(2,3)\}$ in the sense of [21, (2.5)].*

1. *The heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 1 \geq \gamma + 2$,

2. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \rho_2(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}[1](P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if if $\alpha \geq \gamma + 2, \beta \geq \gamma + 2$,

3. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \lambda_1(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 1 \geq \gamma + 3$,

4. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 1 \geq \gamma + 3$,

5. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{2,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}[1](X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\}$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2 \geq \gamma + 4$,

6. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,3}^{\uparrow\uparrow}[1](X)) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, P_1^{\uparrow\uparrow}(X) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2 \geq \gamma + 4$,

7. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2 \geq \gamma + 3$,

8. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,3}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 1 \geq \gamma + 3$,

9. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_3^{\uparrow\uparrow}(X) \in H_1, \mathbb{K}(P_{2,3}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2 \geq \gamma + 4$,

10. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \rho_2(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \mathbb{K}[1](P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2 \geq \gamma + 3$,

11. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_1, \lambda_1(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, P_3^{\uparrow\uparrow}(X) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2, \alpha \geq \gamma + 1$,

12. *the heart*

$$\{X \in \mathcal{D}^{\uparrow\uparrow} \mid \mathbb{K}P_{2,3}^{\uparrow\uparrow}[1](X) \in H_1, \rho_2(P_{1,2}^{\uparrow\uparrow}(X)) \in H_2, \mathbb{K}(P_{1,2}^{\uparrow\uparrow}(X)) \in H_3\},$$

together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a pre-stability condition if $\alpha \geq \beta + 2 \geq \gamma + 4$.

Proof. Similar to proposition 3.2.37 we see this from [21, Proposition 3.5] after applying lemma 6.1.19. □

In order to construct stability conditions and – hence – to prove proposition 6.1.24 we generalise lemma 4.9.1 for the two natural choices.

Lemma 6.1.22. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve. For stability conditions $\sigma_1 = (Z_1, H_1), \sigma_2 = (Z_2, H_2)$, and $\sigma_3 = (Z_3, H_3)$ on $\mathcal{D}$ with hearts of t-structures $H_1 = \mathcal{P}_\mu(\alpha, \alpha+1], H_2 = \mathcal{P}_\mu(\beta, \beta+1], H_3 = \mathcal{P}_\mu(\gamma, \gamma+1]$ let $H_{(i,j)}$ be the heart of the t-structure obtained by recollement from t-structures with hearts H_i, H_j and $Z_{(i,j)}$ be the group homomorphism induced by $Z_i, Z, j, (i,j) \in \{(1,2),(2,3)\}$ in the sense of [21, (2.5)].*

- *The pair*

$$H = \{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_1^{\uparrow\uparrow}(X) \in H_1, P_{2,3}^{\uparrow\uparrow}(X) \in H_{(2,3)}\}$$

  together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a stability condition if $\alpha \geq \beta + 1 \geq \gamma + 2$.

- *The pair*

$$H = \{X \in \mathcal{D}^{\uparrow\uparrow} \mid P_{1,2}^{\uparrow\uparrow}(X) \in H_{(1,2)}, P_3^{\uparrow\uparrow}(X) \in H_3\}$$

  together with the stability function induced by Z_1, Z_2 and Z_3 in the sense of [21, (2.5)] is a stability condition if $\alpha \geq \beta + 1 \geq \gamma + 2$.

Proof. We can generalise lemma 4.9.1 where now

$$Z_{(1,2)}(X) = Z(P_{1,2}^{\uparrow\uparrow}(X)) \text{ and } Z_{(2,3)}(X) = Z(P_{2,3}^{\uparrow\uparrow}(X)) \tag{6.5}$$

and we use the analogous quadratic forms

$$\begin{aligned} Q : \mathcal{N}(\mathcal{D}^{\uparrow\uparrow}) \otimes \mathbb{R} \to \mathbb{R} \text{ as} \\ Q(v) = \Im(Z_1(v))\Im(Z_{(2,3)}(v)) + \Re(Z_1(v))\Re(Z_{(2,3)}(v)) \\ \text{and} \\ Q : \mathcal{N}(\mathcal{D}^{\uparrow\uparrow}) \otimes \mathbb{R} \to \mathbb{R} \text{ as} \\ Q(v) = \Im(Z_{(1,2)}(v))\Im(Z_3(v)) + \Re(Z_{(1,2)}(v))\Re(Z_3(v)) \end{aligned} \tag{6.6}$$

together with the fact that our condition on the hearts grants that for any object $E = E_1 \xrightarrow{\varphi_1} E_2 \xrightarrow{\varphi_2} E_3$ in H and hence – in particular – for a σ-semistable one, the morphisms φ_1 and φ_2 are zero. □

Remark 6.1.23. The construction of lemma 6.1.22 can be adapted for the other six semiorthogonal decompositions with their respective hearts of lemma 6.1.22, for suitable changes of the functors in (6.5). It is however our intent to prove proposition 6.1.24, for which lemma 6.1.22 is perfectly accurate in the presented form.

Proposition 6.1.24. *Let $\mathcal{A} = \mathrm{Coh}(C)$ where C is a smooth projective curve. The space $\mathrm{Stab}(\mathcal{D}^{\uparrow\uparrow})$ is non-empty.*

Proof. This follows from lemma 6.1.22. □

6.2 Generalisation of the Jealousy Lemma to $\mathcal{D}^{\uparrow\uparrow}$

Since, as section 6.1 reveals, one has both gluing- and recollement-data available on the category $\mathcal{D}^{\uparrow\uparrow}$, it is natural to ask the question if there is a version of the Jealousy Lemma (theorem 4.4.6). Unfortunately, the situation is more complex since one has to embed $\mathcal{D}^{\uparrow}$ into $\mathcal{D}^{\uparrow\uparrow}$ as part of the semiorthogonal decomposition that provides the basis for the gluing- and recollement data. Since the stability-manifold of $\mathcal{D}^{\uparrow}$ is a lot more difficult then that of $\mathcal{D}$, one cannot hope to obtain such clear results. As part of the proof of the Jealousy Lemma we will also provide a description of the recollement data that, under the condition of $\mathcal{D}$ having a Serre-functor, one obtains. This is done in (6.2). Subsequently we will then also describe the t-structures that one obtains from this recollement data – this is done in (6.3).

Following the notation introduced as part of corollary 4.3.17, we introduce the next definition.

Definition 6.2.1. Let t-structures on $\mathcal{D}$ and $\mathcal{D}^{\uparrow\uparrow}$ be given by

$$\begin{aligned}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = (\mathcal{P}_\mu(\alpha, \infty), \mathcal{P}_\mu(-\infty, \alpha]),\\ (\mathcal{D}^{\leq 0}_{y,\beta,\gamma}, \mathcal{D}^{\geq 1}_{y,\beta,\gamma}), y \in \{1, 2, 3\},\end{aligned} \tag{6.7}$$

with $(\mathcal{D}^{\leq 0}_{y,\beta,\gamma}, \mathcal{D}^{\geq 1}_{y,\beta,\gamma})$ as in definition 4.3.18. Define $(\mathcal{D}^{\leq 0}_{x,\alpha,(\beta,\gamma)}, \mathcal{D}^{\geq 1}_{x,\alpha,(\beta,\gamma)}), x \in \{1, \dots, 8\}$ to be the t-structure that is obtained by recollement based on the semiorthogonal decomposition "number x" of lemma 6.1.16 from the t-structures in (6.7).

Remark 6.2.2. Note that the previous definition abuses terminology since semiorthogonal decompositions appear in a CP-gluing context and not as part of that of a recollement. However – as we have seen before – under favourable conditions, CP-gluing data can be extended to recollement data.

Notation 6.2.3. Denote by $H_{x,\alpha,(\beta,\gamma)}$ the heart of $(\mathcal{D}^{\leq 0}_{x,\alpha,(\beta,\gamma)}, \mathcal{D}^{\geq 1}_{x,\alpha,(\beta,\gamma)}), x \in \{1, \dots, 8\}$.

Lemma 6.2.4. *If there is a stability condition σ with heart $H_{x,\alpha,(\beta,\gamma)}$, then $H_{x,\beta,\gamma}$ is obtained by gluing.*

Proof. By theorem 4.4.6 $H_{x,\beta,\gamma}$ carries no stability condition if $\beta < \gamma$. □

We have the jealous-lemma for $\mathcal{D}^{\uparrow\uparrow}$.

Proposition 6.2.5. *If there is a stability condition σ with heart $H_{1,\alpha,(\beta,\gamma)}$ obtained via recollement from hearts $H \in \mathcal{D}$ and $H_{1,\beta,\gamma} \subset \mathcal{D}^{\uparrow}$, then $H_{1,\alpha,(\beta,\gamma)}$ is obtained by CP-gluing.*

Proof. We will outline the main building-blocks for the generalisation of theorem 4.4.6. First, note that by lemma 6.2.4 we may assume $\beta \geq \gamma$. We need to see that $\alpha << \beta$ triggers another gluing-situation, generalising proposition 4.3.21. To demonstrate how this generalisation works, we use the list of recollements provided in (6.2) and hence that of recollement t-structures provided in (6.3). This shows how proposition 4.3.21 is generalised – we group I with II (note that in the same manner III with IV and so on can be grouped together) and use that the $\mathcal{D}^{\leq 0}$ of the respective first t-structure together with the $\mathcal{D}^{\geq 1}$ of the second provide CP-gluing data.

Hence, we can now generalise the proof for theorem 4.4.6. At first, we need a version of lemma 4.4.2. In this case, we need to prove $i_1^{\uparrow\uparrow}(\mathcal{P}(\alpha, \alpha+1]) \subset H_{1,\alpha,(\beta,\gamma)}$, $i_2^{\uparrow\uparrow}(\mathcal{P}(\beta, \beta+1]) \subset H_{1,\alpha,(\beta,\gamma)}$ and $\Delta \circ i_1(\mathcal{P}(\beta, \beta+1]) \subset H_{1,\alpha,(\beta,\gamma)}$.

To prove $i_1^{\uparrow\uparrow}(\mathcal{P}(\alpha, \alpha+1]) \subset H_{1,\alpha,(\beta,\gamma)}$, note that $P_{2,3}^{\uparrow\uparrow} \circ i_1^{\uparrow\uparrow}(\mathcal{P}(\alpha, \alpha+1]) = 0 \in H_{1,\alpha,(\beta,\gamma)}$. Moreover $P_1^{\uparrow\uparrow} \circ i_1^{\uparrow\uparrow}(\mathcal{P}(\alpha, \alpha+1]) = \mathcal{P}(\alpha, \alpha+1] \subset \mathcal{P}(\alpha, \infty)$ and $\mathbb{K} \circ P_{1,2}^{\uparrow\uparrow} \circ i_1^{\uparrow\uparrow}(\mathcal{P}(\alpha, \alpha+1]) = \mathbb{K} \circ i_1(\mathcal{P}(\alpha, \alpha+1])) = \mathcal{P}(\alpha, \alpha+1] \subset \mathcal{P}(\alpha, \infty)$.

To prove $i_2^{\uparrow\uparrow}(\mathcal{P}(\beta, \beta+1]) \subset H_{1,\alpha,(\beta,\gamma)}$, note that $P_1^{\uparrow\uparrow} \circ i_2^{\uparrow\uparrow}(\mathcal{P}(\beta, \beta+1]) = 0 \in H_{1,\alpha,(\beta,\gamma)}$. Moreover $P_{2,3}^{\uparrow\uparrow} \circ i_2^{\uparrow\uparrow}(\mathcal{P}(\beta, \beta+1]) = i_2(\mathcal{P}(\beta, \beta+1]) \subset H_{1,\beta,\gamma}$ given by the fact that $H_{1,\beta,\gamma}$ is obtained by CP-gluing. Finally, we also have $\mathbb{K} \circ P_{1,2}^{\uparrow\uparrow} \circ i_2^{\uparrow\uparrow}(\mathcal{P}(\beta, \beta+1]) = \mathbb{K} \circ i_2(\mathcal{P}(\beta, \beta+1])) = \mathcal{P}(\beta, \beta+1][-1] = \mathcal{P}(\beta-1, \beta] \subset \mathcal{P}(-\infty, \alpha+1]$ provided by the fact that $\beta \leq \alpha+1$.

To prove $\Delta_1 \circ i_1(\mathcal{P}(\beta, \beta+1]) \subset H_{1,\alpha,(\beta,\gamma)}$, note that $\mathbb{K} \circ P_{1,2}^{\uparrow\uparrow} \circ \Delta_1 \circ i_1(\mathcal{P}(\beta, \beta+1]) = \mathbb{K} \circ \Delta(\mathcal{P}(\beta, \beta+1]) = 0 \subset H_{1,\alpha,(\beta,\gamma)}$. Moreover $P_1^{\uparrow\uparrow} \circ \Delta_1 \circ i_1(\mathcal{P}(\beta, \beta+1]) = \mathcal{P}(\beta, \beta+1] \subset \mathcal{P}(\beta, \infty)$. Finally $P_{2,3}^{\uparrow\uparrow} \circ \Delta_1 \circ i_1(\mathcal{P}(\beta, \beta+1]) = i_1\mathcal{P}(\beta, \beta+1] \subset H_{1,\beta,\gamma}$ given by the fact that $H_{1,\beta,\gamma}$ is obtained by CP-gluing.

We can now generalise lemma 4.4.4. Let Z be the stability function given by σ. We consider the imaginary part of Z which is given by

$$\begin{aligned}\Im(Z(X)) = {} & D_1(\deg(P_1(X))) + D_2(\deg(P_2(X))) + D_3(\deg(P_3(X))) \\ & + C_1(\operatorname{rank}(P_1(X))) + C_2(\operatorname{rank}(P_2(X))) + C_3(\operatorname{rank}(P_3(X)))\end{aligned}$$

for $X \in H_{x,\alpha,(\beta,\gamma)}$. We obtain

$$\begin{aligned}\Im(Z(X))|_{i_1^{\uparrow\uparrow}(\mathcal{P}(\alpha,\alpha+1])} &= \Im(Z(i_1^{\uparrow\uparrow}(X_1))) \\ &= D_1(\deg(X_1)) + C_1(\operatorname{rank}(X_1)), \\ \Im(Z(X))|_{i_2^{\uparrow\uparrow}(\mathcal{P}(\beta,\beta+1])} &= \Im(Z(i_2(X_1))) \\ &= D_2(\deg(X_1)) + C_2(\operatorname{rank}(X_1)), \\ \Im(Z(X))|_{\Delta \circ i_1(\mathcal{P}(\beta,\beta+1])} &= \Im(Z(\Delta(X_1))) \\ = (D_1 + D_2)(\deg(X_1)) &+ (C_1 + C_2)(\operatorname{rank}(X_1))\end{aligned}$$

for any X_1 in either $\mathcal{P}(\alpha, \alpha + 1])$ or $\mathcal{P}(\beta, \beta + 1])$. We now use the same argument as in lemma 4.4.4 to finish the proof. □

6.3 Further generalisations: The category $\mathcal{D}^{n\uparrow}$

We will now provide a generalisation of some of the findings of the previous chapters to the situation of n arrows. The category $\mathcal{D}^{n\uparrow}$ that we will now introduce can be thought of as the derived category of a certain quiver (see [23] for details) with vertices in $\mathrm{obj}(\mathcal{A})$ and edges in $\mathrm{mor}(\mathcal{A})$.

Definition 6.3.1. Define $\mathcal{D}^{n\uparrow}, n \in \mathbb{N}$ to be the derived category of $\mathcal{A}^{n\uparrow}$ (see definition 2.1.11).

Definition 6.3.2. Let $t, n \in \mathbb{N}, t < n$. Define $i^{n\uparrow}_{j_1,\dots,j_t} : \mathcal{D}^{t\uparrow} \to \mathcal{D}^{n\uparrow}$ in analogy to definition 6.1.7 part 1, 2 and 3 to be the embedding into components $j_1, \dots, j_t$.

Furthermore define $\Delta^{n\uparrow}_{a,b} : \mathcal{D}^{t\uparrow} \to \mathcal{D}^{n\uparrow}, a, b \in \mathbb{N}, a < b$ by the equation

$$\begin{aligned} &\Delta^{n\uparrow}_{a,b}(A_1 \to \cdots \to A_a \xrightarrow{f} A_{a+1} \to \cdots \to A_t) \\ = (A_1 \to \cdots \to A_{a-1} \to A_1 \to B_1 \to \cdots \to B_{b-a} &\xrightarrow{f} A_{a+1} \to \cdots \to A_t), \\ \text{where } B_1 \to \cdots \to B_{b-a} &= A_a \xrightarrow{\mathrm{id}_{A_a}} \cdots \xrightarrow{\mathrm{id}_{A_a}} A_a, \end{aligned}$$

with the morphisms defined accordingly via the morphisms obtained from restricting the functors to the respective abelian categories.

Furthermore define $P^{n\uparrow}_{j_1,\dots,j_t} : \mathcal{D}^{n\uparrow} \to \mathcal{D}^{u\uparrow}, t \le u < n$, in analogy to definition 6.1.10 to be the projection onto the components $j_1, \dots, j_t$. Both for $i^{n\uparrow}_{j_1,\dots,j_t}$ and $P^{n\uparrow}_{j_1,\dots,j_t}$ morphisms between neighbouring objects that after the embedding or the projection are still neighbouring one another are being kept whilst all others are being deleted (similar to the case of $j^{\uparrow\uparrow}_{1,3}$ versus $i^{\uparrow\uparrow}_{1,2}$ and $i^{\uparrow\uparrow}_{2,3}$ in definition 6.1.7).

Remark 6.3.3. Note that for convenience we will always refer to $\Delta^{n\uparrow}_{1,v} : \mathcal{D}^{t\uparrow} \to \mathcal{D}^{n\uparrow}(v+u = n)$ as $\Delta^{n\uparrow}$. Note furthermore, that with this notation established, we identify λ_1 with $P^{\uparrow}_1$ and ρ_2 with $P^{\uparrow}_2$.

Generalising our observations made for the cases of $\mathcal{D}, \mathcal{D}^{\uparrow}$ and $\mathcal{D}^{\uparrow\uparrow}$, the following lemma shows how to find the component that all adjunction chains between given categories $\mathcal{D}^{t\uparrow}$ and $\mathcal{D}^{n\uparrow}$ have in common.

Lemma 6.3.4. *Let $t, n \in \mathbb{N}, t < n, m < n$. We have the following chain of adjoint functors:*

$$\begin{aligned} i^{n\uparrow}_{n-(t-1),\dots,n} \dashv P^{n\uparrow}_{n-(t-1),\dots,n} \dashv \Delta^{n\uparrow}_{1,n-(t-1)} \dashv \\ \dots \dashv \Delta^{n\uparrow}_{t-m,n-m} \dashv P^{n\uparrow}_{1,\dots,t-m,n+1-m,\dots,n} \dashv \dots \\ \dashv \Delta^{n\uparrow}_{t,n} \dashv P_{1,\dots,t} \dashv i^{n\uparrow}_{1,\dots,t} \end{aligned}$$

between $\mathcal{D}^{t\uparrow}$ and $\mathcal{D}^{n\uparrow}$.

Proof. The idea is the same as in lemma 3.2.3. It – once again – suffices to execute our proof on the abelian level. To prove that the adjunction $i^{n\uparrow}_{n-(t-1),\dots,n} \dashv P^{n\uparrow}_{n-(t-1),\dots,n}$ holds we use the isomorphism

$$(0, \dots, 0, f_{n-(t-1)}, \dots, f_n) \mapsto (f_{n-(t-1)}, \dots, f_n)$$

where $f_i : A_i \to B_i$, to obtain

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{A}^{n\uparrow}}(0 \to \dots \to A_{n-(t-1)} \to \dots \to A_n, B_1 \to B_n) \\ &\cong \mathrm{Hom}_{\mathcal{A}^{t\uparrow}}(A_{n-(t-1)} \to \dots \to A_n, B_{n-(t-1)} \to B_n). \end{aligned}$$

The other adjunctions are obtained similarly. □

Corollary 6.3.5. *In particular this implies*

$$\Delta^{n\uparrow} \dashv P^{n\uparrow}_n \dashv i^{n\uparrow}_n \text{ and}$$
$$\Delta^{n\uparrow}_{n-1,n} \dashv P^{n\uparrow}_{1,\dots,n-1} \dashv i^{n\uparrow}_{1,\dots,n-1}$$

Proof. This is a straightforward implication of lemma 6.3.4. □

We are now able to prove an important consequence, namely, that a category, that extends the definition of $\mathcal{D}^{\uparrow}$ to a chain of any given length n will always inherit a Serre-functor from the category $\mathcal{D}$ that one starts with.

Proposition 6.3.6. *If $\mathcal{D}$ has a Serre functor then so has $\mathcal{D}^{n\uparrow}$.*

Proof. We generalise the proof of lemma 6.1.12 where we use the equations

$$\begin{aligned} i^{n\uparrow}_{1,\dots,n-1}(\mathcal{D}^{(n-1)\uparrow}) &= i^{n\uparrow}_n(\mathcal{D})^{\perp} = \mathrm{im}(i^{n\uparrow}_n)^{\perp} \\ &= \ker(P^{n\uparrow}_n) = {}^{\perp}\mathrm{im}(\Delta^{n\uparrow}) = {}^{\perp}\Delta^{n\uparrow}(\mathcal{D}) \end{aligned}$$

and

$$\begin{aligned} i^{n\uparrow}_n(\mathcal{D}) &= {}^{\perp}i^{n\uparrow}_{1,\dots,n-1}(\mathcal{D}^{(n-1)\uparrow}) = {}^{\perp}\mathrm{im}(i^{n\uparrow}_{1,\dots,n-1}) \\ &= \ker(P^{n\uparrow}_{1,\dots,n-1}) = \mathrm{im}(\Delta^{n\uparrow}_{n-1})^{\perp} = \Delta^{n\uparrow}_{n-1}(\mathcal{D}^{n\uparrow})^{\perp}, \end{aligned}$$

that we are able to establish with the aid of corollary 6.3.5. This allows us to proceed in the very same way as we did in 6.1.12 and therefore finishes the proof. □

In line with the main topic of interest of this thesis we would like to establish that stability conditions on $\mathcal{D}^{n\uparrow}$ can be constructed. As previously done in this subsection, the basic concepts can be generalised to the case of $\mathcal{D}^{n\uparrow}$.

Proposition 6.3.7. *Let* $\mathcal{A} = \mathrm{Coh}(C)$ *where* C *is a smooth projective curve. The space* $\mathrm{Stab}(\mathcal{D}^{n\uparrow})$ *is non-empty.*

Proof. We can generalise lemma 6.1.22 via induction in order to obtain a $\sigma \in \mathrm{Stab}(\mathcal{D}^{n\uparrow})$. Using the quadratic form analogous to (6.6) and investigating the σ-semistable object

$$E = E_1 \xrightarrow{\varphi_1} E_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} E_{n+1} \in H.$$

The heart H on $\mathcal{D}^{n\uparrow}$ can be glued from hearts

$$H_1 = \mathcal{P}_\mu(\alpha_1, \alpha_1 + 1] \dots H_{n+1} = \mathcal{P}_\mu(\alpha_{n+1}, \alpha_{n+1} + 1]$$

like in lemma 6.1.19 but with the stronger restrictions

$$\alpha_1 \geq \alpha_2 + 1 \geq \dots \geq \alpha_{n+1} + n$$

of the kind of lemma 6.1.21, using stability functions $Z_1, \dots Z_{n+1}$ on the respective hearts such that we obtain that σ is obtained by CP-gluing the stability conditions

$$\sigma_1 = (H_1, Z_1), \dots \sigma_{n+1} = (H_{n+1}, Z_{n+1})$$

inductively. □

A Appendix

A.1 Serre functors

Serre functors were first introduced by Bondal and Kapranov in [12]. The proper definition, however, as it is given in definition A.1.8 can be found in [13]. Serre functors serve as a generalisation of the well known concept of Serre duality – we refer to Hartshorne, [34, Section III, subsection 7] for details. Serre functors are an important tool in homological algebra. They can be used to compute new functors out of given ones and have – therefore – very useful applications in relation to recollements. Bondal and Kapranov provided a – crucial – theorem (A.1.15) regarding Serre functors in [12]. In the following we will prepare its introduction by providing the following vital terminology that will allow us to formulate the main theorem (A.1.15) of this section.

Definition A.1.1. The right orthogonal of a full subcategory $\mathcal{B}$ of an additive category $\mathcal{A}$, denoted $\mathcal{B}^{\perp}$ is the full subcategory given by

$$\mathcal{B}^{\perp} = \{C \in \mathcal{A} \mid \mathrm{Hom}(B, C) = 0 \text{ for all } B \in \mathcal{B}\}.$$

Definition A.1.2. The left orthogonal of a full subcategory $\mathcal{C}$ of an additive category $\mathcal{A}$, denoted ${}^{\perp}\mathcal{C}$ is the full subcategory given by

$${}^{\perp}\mathcal{C} = \{B \in \mathcal{A} \mid \mathrm{Hom}(B, C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

We can now provide the following – important – definitions.

Definition A.1.3. A strictly full triangulated subcategory $\widetilde{\mathcal{TR}}$ of a triangulated category $\mathcal{TR}$ is right-admissible if for any $X \in \mathcal{TR}$ there is an exact triangle

$$\widetilde{T} \to X \to C \xrightarrow{+}$$

where $\widetilde{T} \in \widetilde{\mathcal{TR}}$ and $C \in \widetilde{\mathcal{TR}}^{\perp}$.

In analogy to this we obtain the next definition.

Definition A.1.4. A strictly full triangulated subcategory $\widetilde{\mathcal{TR}}$ of a triangulated category $\mathcal{TR}$ is left-admissible if for any $X \in \mathcal{TR}$ there is an exact triangle

$$D \to X \to \widetilde{T} \xrightarrow{+}$$

where $\widetilde{T} \in \widetilde{\mathcal{TR}}$ and $D \in {}^{\perp}\widetilde{\mathcal{TR}}$.

And hence,

Definition A.1.5. A triangulated subcategory $\mathcal{TR}'$ of a triangulated category $\mathcal{TR}$ is admissible if it is right-admissible and left-admissible.

The next lemma brings different terminology together.

Lemma A.1.6. *Every right- (left-) admissible triangulated subcategory $\mathcal{A}$ of a category $\mathcal{B}$ provides a semiorthogonal decomposition given by $\mathcal{B} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$ ($\mathcal{B} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$).*

Proof. Apply definition 2.1.6. □

Definition A.1.7. A category $\mathcal{A}$ is called "k-linear" if for any $A, B \in \mathcal{A}$, the set $\mathrm{Hom}(A, B)$ has the structure of a k-vector space and if additionally composition of morphisms is k-bilinear.

Definition A.1.8. Let k be a field and $\mathcal{A}$ be a k-linear category with finite-dimensional Hom-sets. A "Serre functor" $S : \mathcal{A} \to \mathcal{A}$ is an additive equivalence of categories that has bi-functorial isomorphisms

$$\phi_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(B, S(A))^*$$

of k-vector spaces for any two objects A and B in $\mathcal{A}$.

Remark A.1.9. Note that by [12, Proposition 3.4 b], Serre functors are unique up to functor isomorphisms. To see this, consider Serre functors S, S' and bi-functorial isomorphisms

$$\phi_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(B, S(A))^*$$

and

$$\phi'_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(B, S'(A))^*$$

of k-vector spaces for any two objects A and B in $\mathcal{A}$. We obtain

$$\phi'_{A,B} \circ \phi^{-1}_{A,B} : \mathrm{Hom}_{\mathcal{A}}(B, S(A))^* \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(B, S'(A))^*, \tag{A.1}$$

which means that $S(A) \cong S'(A)$ for any $A \in \mathcal{A}$. Moreover, letting $B = S(A)$ in (A.1), we obtain

$$\phi'_{A,S(A)} \circ \phi^{-1}_{A,S(A)} : \mathrm{Hom}_{\mathcal{A}}(S(A), S(A))^* \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(S(A), S'(A))^*,$$

when – on the other hand – letting $B = S(A')$ and $A = A'$ in (A.1) provides us with

$$\phi'_{A',S(A')} \circ \phi^{-1}_{A',S(A')} : \mathrm{Hom}_{\mathcal{A}}(S(A'), S(A'))^* \overset{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(S(A'), S'(A'))^*.$$

Hence, the diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{\cong} & S'(A) \\ {\scriptstyle S(f)}\downarrow & & {\scriptstyle S'(f)}\downarrow \\ S(A') & \xrightarrow{\cong} & S'(A') \end{array}$$

commutes for any $f \in \mathrm{Hom}(A, A')$ proving that we also obtain $S(f) \cong S'(f)$.

We will now provide a couple of important tools that will serve to prove the main theorem of this section (A.1.15) which is due to Bondal and Kapranov. In order to do so, we require the following lemma also due to Bondal and Kapranov.

Lemma A.1.10. *A category $\mathcal{A}$ has a Serre functor if and only if* $\mathrm{Hom}_{\mathcal{A}}(X,-)^*$ *and* $\mathrm{Hom}_{\mathcal{A}}(-,X)^*$ *are representable for any* $X \in \mathcal{A}$.

Proof. See [12, Proposition 3.4 a]. □

Additionally we require the next lemma, which is as well provided by Bondal and Kapranov ([12, Lemma 2.4 a]). In order to prepare it, we provide the following definition.

Definition A.1.11. For a category $\mathcal{A}$ and an object $X \in \mathcal{A}$, we define

$$h_X(A) = \mathrm{Hom}_{\mathcal{A}}(A, X).$$

Lemma A.1.12. *For a category $\mathcal{A}$, $h : \mathcal{A} \to \mathcal{S}ets$ a contravariant functor and $X \in \mathcal{A}$, there is a natural identification of the set of natural transformations $h_X \to h$ with $h(X)$.*

Proof. To prove this, fix an element $\delta \in h(X)$. For an object $A \in \mathcal{A}$ and a morphism $f \in h_X(A) = \mathrm{Hom}_{\mathcal{A}}(A, X)$, a natural transformation $\Phi^\delta : h_X \to h$ is now given by $\Phi^\delta_A(f) = h(f)(\delta)$. If we – on the other hand – now consider a given natural transform $\Phi : h_X \to h$, we obtain the required $\delta \in h(X)$ as $\Phi_X(\mathrm{id}_X)$. □

Hence, we also have the following corollary.

Corollary A.1.13. *Let $\mathcal{A}$ be a triangulated category, $\widetilde{X} \in \mathcal{A}$, $h : \mathcal{A} \to \mathcal{S}ets$ a contravariant cohomological functor and assume that*

$$B \to A \to C \xrightarrow{+}$$

is an exact triangle. The exact sequences

$$\ldots h(C) \to h(A) \to h(B) \to \ldots$$

and

$$\ldots h_{\widetilde{X}}(C) \to h_{\widetilde{X}}(A) \to h_{\widetilde{X}}(B) \to \ldots$$

are embedded in a commutative diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & h(C) & \longrightarrow & h(A) & \longrightarrow & h(B) & \longrightarrow & \ldots \\ & & \uparrow{\scriptstyle \Phi^\delta_C} & & \uparrow{\scriptstyle \Phi^\delta_A} & & \uparrow{\scriptstyle \Phi^\delta_B} & & \\ \ldots & \longrightarrow & h_{\widetilde{X}}(C) & \longrightarrow & h_{\widetilde{X}}(A) & \longrightarrow & h_{\widetilde{X}}(B) & \longrightarrow & \ldots \end{array} \tag{A.2}$$

for any $\delta \in h(\widetilde{X})$ and $\Phi^\delta : h_{\widetilde{X}} \to h$ a natural transformation constructed in the way it was outlined in the proof of lemma A.1.12.

Proof. By lemma A.1.12, Φ^δ is a natural transformation. Additionally the functors h and $h_{\widetilde{X}}$ are cohomological functors. □

Finally, we require the following.

Lemma A.1.14. *For vector spaces $V_1, V_2, V_3, V_4, V_1', V_2', V_3', V_4'$, let*

$$\begin{array}{ccccccc} V_1 & \xleftarrow{c} & V_2 & \xleftarrow{b} & V_3 & \xleftarrow{a} & V_4 \\ \alpha_1\uparrow & & \alpha_2\uparrow & & \alpha_3\uparrow & & \alpha_4\uparrow \\ V_1' & \xleftarrow{c'} & V_2' & \xleftarrow{b'} & V_3' & \xleftarrow{a'} & V_4' \end{array}$$

be a commutative diagram with exact rows. Assume that α_1 is a monomorphism and α_4 an epimorphism. Assume additionally that there is a $J \in V_3$ and an $I \in V_2'$, such that $\alpha_2(I) = b(J)$. Then there is a $\delta \in V_3'$ such that $b'(\delta) = I$ and $\alpha_3(\delta) = J$.

Proof. Let $K \in V_2$ be the image of I under α_2 and of J under b. The latter implies, that due to the exactness of the rows, $K \in \operatorname{im}(b)$, which gives $c(K) = 0$ and therefore, using the commutativity of the diagram,

$$\alpha_1(c'(I)) = c(\alpha_2(I)) = c(K) = 0.$$

Since α_1 is a monomorphism this means that $c'(I) = 0$ and hence – due to exactness – that there exists a $\delta' \in V_3'$ such that $b'(\delta') = I$. Therefore

$$b(\alpha_3(\delta')) = \alpha_2(b'(\delta')) = K$$

holds. Hence, if we let J' be the image of δ' under α_3, we obtain $b(J') = K$. This implies that

$$b(J' - J) = b(J') - b(J) = K - K = 0.$$

Therefore $J' - J \in \ker(b)$ and hence $J' - J \in \operatorname{im}(a)$. This means that there is a $d \in V_4$ such that $a(d) = J' - J$. Since α_4 is an epimorphism, we obtain a preimage d' of d under α_4 such that $a(\alpha_4(d')) = J' - J$ and hence

$$\alpha_3(a'(d')) = a(\alpha_4(d')) = J' - J.$$

Now, define $\beta = a'(d)$ then

$$b'(\beta) = b'(a'(d')) = 0.$$

Now let $\delta = \delta' - \beta$. On one hand we now obtain

$$b'(\delta) = b'(\delta' - \beta) = b'(\delta') - b'(\beta) = b'(\delta') - 0 = b'(\delta') = I$$

and on the other

$$\alpha_3(\delta) = \alpha_3(\delta' - \beta) = \alpha_3(\delta') - \alpha_3(\beta) = J' - (J' - J) = J.$$

This – hence – is the δ we were looking to find and the proof is finished.

□

As it was our aim, we can now provide the following theorem from [12]. Since the proof given by Bondal and Kapranov in [12] omits certain details we will, for the convenience of the reader, now provide a fully worked out version of the proof.

Theorem A.1.15. *Let $\mathcal{A}$ be a triangulated category. Assume $\mathcal{B} \subset \mathcal{A}$ to be an admissible subcategory and define $\mathcal{C} = \mathcal{B}^{\perp}$. Assume furthermore that $\mathcal{C}$ is admissible and that $\mathcal{B}$ and $\mathcal{C}$ have Serre functors. Then the category $\mathcal{A}$ also has a Serre functor.*

Proof. Let $h = \mathrm{Hom}_{\mathcal{A}}(X, -)^*$ for $X \in \mathcal{A}$. We will construct a representing object for h in order to apply lemma A.1.10. First, consider the exact triangle

$$R \to X \stackrel{\beta}{\to} B \stackrel{+}{\to}$$

with $R \in {}^{\perp}\mathcal{B}$ and $B \in \mathcal{B}$, that we obtain from the fact that $\mathcal{B}$ is admissible and hence in particular left-admissible. For any $B_1 \in \mathcal{B}$, the application of $\mathrm{Hom}(-, B_1)$, combined with the fact that $\mathrm{Hom}(R, B_1) = 0$ by definition A.1.4 we obtain

$$\mathrm{Hom}_{\mathcal{A}}(X, B_1) \stackrel{\cong}{\leftarrow} \mathrm{Hom}_{\mathcal{A}}(B, B_1)$$

which, since $\mathcal{B} \subset \mathcal{A}$ is a full subcategory provides us with

$$\begin{aligned} h(B_1) = \mathrm{Hom}_{\mathcal{A}}(X, B_1)^* &\stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(B, B_1)^* \\ = \mathrm{Hom}_{\mathcal{B}}(B, B_1)^* &= \mathrm{Hom}_{\mathcal{B}}(B_1, S_{\mathcal{B}}(B)) \end{aligned} \tag{A.3}$$

where $S_{\mathcal{B}}$ is the Serre functor on $\mathcal{B}$ (note that by remark A.1.9 Serre functors are essentially unique).

Similarly, consider the exact triangle

$$B' \to X \to C' \stackrel{+}{\to}$$

where $B' \in \mathcal{B}$ and $C' \in \mathcal{C}$. Since $\mathcal{C} = \mathcal{B}^{\perp}$ we can use definition A.1.3 to obtain

$$\begin{aligned} h(C_1) = \mathrm{Hom}_{\mathcal{A}}(X, C_1)^* &\stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(C', C_1)^* \\ = \mathrm{Hom}_{\mathcal{C}}(C', C_1)^* &= \mathrm{Hom}_{\mathcal{C}}(C_1, S_{\mathcal{C}}(C')) \end{aligned} \tag{A.4}$$

for $C_1 \in \mathcal{C}$ in a similar manner as before.

We now define $E = S_{\mathcal{B}}(B) \in \mathcal{B} \subset \mathcal{A}$ and – as before – use that $\mathcal{C}$ was assumed to be admissible to embed E into the exact triangle

$$C'' \xrightarrow{\gamma} E \to L \xrightarrow{+} \tag{A.5}$$

where $C'' \in \mathcal{C}$ and $L \in \mathcal{C}^{\perp}$. Now – again as before – we obtain

$$\mathrm{Hom}_{\mathcal{C}}(C, C'') \overset{\gamma\circ -}{\underset{\cong}{\to}} \mathrm{Hom}_{\mathcal{A}}(C, E)$$

for all $C \in \mathcal{C}$. Applying h to γ now supplies us with

$$h(C'') \xleftarrow[h(\gamma)]{} h(E)$$

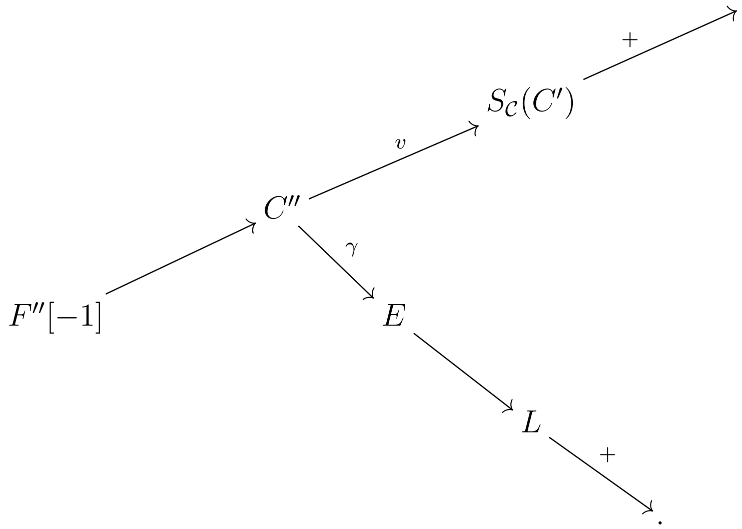
which – by substituting C'' for C_1 in equation (A.4) extends to

$$\begin{array}{ccc} h(C'') & \xleftarrow[h(\gamma)]{} & h(E) \\ \downarrow{\scriptstyle\cong} & & \\ \mathrm{Hom}_{\mathcal{C}}(C'', S_{\mathcal{C}}(C')) & & \end{array}$$

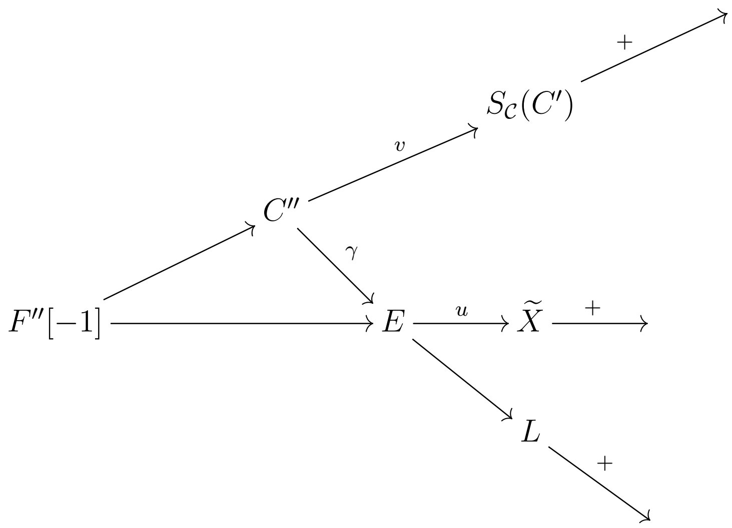
such that – using (A.3) this time, letting $B_1 = E$ – we obtain

$$\begin{array}{ccc} h(C'') & \xleftarrow[h(\gamma)]{} & h(E) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathrm{Hom}_{\mathcal{C}}(C'', S_{\mathcal{C}}(C')) & & \mathrm{Hom}_{\mathcal{B}}(E, E). \end{array}$$

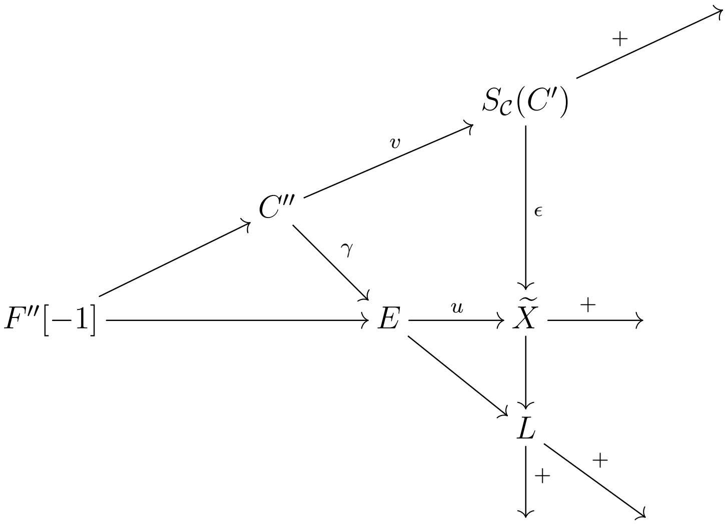
Hence the identity $\mathrm{id}_E \in \mathrm{Hom}_{\mathcal{B}}(E, E)$ has an image $v \in \mathrm{Hom}_{\mathcal{C}}(C'', S_{\mathcal{C}}(C'))$ and therefore, using the exact triangle (A.5) and defining $F'' = \mathrm{Cone}(v)$, we obtain the diagram



Next, we compose γ with the arrow $F''[-1] \to C''$ to get an arrow $F''[-1] \to E$. We let $\widetilde{X}$ be the object and u the arrow that – together – define the cone of $F''[-1] \to E$, obtain



and – using the octahedral axiom – finally acquire arrows $\widetilde{X} \to L$ and ϵ such that the following is a commutative diagram of exact triangles.



We have at this point constructed the representing object $\widetilde{X}$ for the functor h. It will subsequently be our objective to prove that $\widetilde{X}$ does indeed represent the functor h. First, we will provide an important consequence of the octahedral diagram. We use – similar as above – the exact triangle

$$E \stackrel{u}{\to} \widetilde{X} \to F'' \stackrel{+}{\to}$$

and combine it with the fact that $F'' = \mathrm{Cone}(v)$, which, since $\mathcal{C}$ was assumed to be triangulated and since $C'', S_{\mathcal{C}}(C') \in \mathcal{C}$ implies $F'' \in \mathcal{C}$. We acquire

$$\mathrm{Hom}_{\mathcal{B}}(\widehat{B}, E) = \mathrm{Hom}_{\mathcal{A}}(\widehat{B}, E) \cong \mathrm{Hom}_{\mathcal{A}}(\widehat{B}, \widetilde{X}) \tag{A.6}$$

for an arbitrary $\widehat{B} \in \mathcal{B}$. Since $\mathcal{B}$ be has a Serre functor, we additionally obtain

$$h(\widehat{B}) \stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{B}}(\widehat{B}, E) \tag{A.7}$$

by (A.3) and hence

$$h(\widehat{B}) \stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{B}}(\widehat{B}, E) = \mathrm{Hom}_{\mathcal{A}}(\widehat{B}, E) \cong \mathrm{Hom}_{\mathcal{A}}(\widehat{B}, \widetilde{X}). \tag{A.8}$$

Taking the inverses of the isomorphisms in (A.8), we define a natural transformation $\varphi : \mathrm{Hom}_{\mathcal{A}}(-, \widetilde{-}) \to h(E)$ between functors defined in $\mathcal{B}$. Since the isomorphism in (A.6) was obtained by applying $\mathrm{Hom}_{\mathcal{A}}(\widehat{B}, -)$, it is natural. So is the isomorphism obtained by (A.7), to see this we examine (A.3) and obtain that the isomorphism $\mathrm{Hom}_{\mathcal{A}}(X, B_1)^* \stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{A}}(B, B_1)^*$ being obtained from the application of the functor $\mathrm{Hom}_{\mathcal{A}}(-, B_1)$ and the dualising-functor is natural. So is $\mathrm{Hom}_{\mathcal{B}}(B, B_1)^* = \mathrm{Hom}_{\mathcal{B}}(B_1, S_{\mathcal{B}}(B))$, by definition A.1.8. Hence φ is a natural transformation. In the same manner, use

$$S_{\mathcal{C}}(C') \xrightarrow{\epsilon} \widetilde{X} \to L \xrightarrow{+}$$

to obtain a natural transformation $\psi : \mathrm{Hom}_{\mathcal{A}}(-, \widetilde{X}) \xrightarrow{\cong} h(-)$. Now, we define $h_{\widetilde{X}} = \mathrm{Hom}_{\mathcal{A}}(-, \widetilde{X})$ and are able to complete our previous diagram via

$$\begin{array}{ccc}
h(C'') & \xleftarrow{\;h(\gamma)\;} & h(E) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{Hom}_{\mathcal{C}}(C'', S_{\mathcal{C}}(C')) & & \mathrm{Hom}_{\mathcal{B}}(E, E) \\
{\scriptstyle \epsilon\circ-}\downarrow{\scriptstyle\cong} & & {\scriptstyle u\circ-}\downarrow{\scriptstyle\cong} \\
h_{\widetilde{X}}(C'') & & h_{\widetilde{X}}(E)
\end{array}$$

to

$$\begin{array}{ccccc}
& h(C'') & \xleftarrow{\;h(\gamma)\;} & h(E) & \\
& \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & \\
\psi_{C''}\uparrow & \mathrm{Hom}_{\mathcal{C}}(C'', S_{\mathcal{C}}(C')) & & \mathrm{Hom}_{\mathcal{B}}(E, E) & \uparrow\varphi_E \\
& {\scriptstyle \epsilon\circ-}\downarrow{\scriptstyle\cong} & & {\scriptstyle u\circ-}\downarrow{\scriptstyle\cong} & \\
& h_{\widetilde{X}}(C'') & & h_{\widetilde{X}}(E). &
\end{array}$$

We have – however – previously introduced u as the preimage of id_E via the upper half of the diagram. Considering the images $u \circ \mathrm{id}_E = u$ and $\epsilon \circ v$ under the respective maps $u \circ -$ and $\epsilon \circ -$, the diagram provides us with the equation

$$h(\gamma)(\varphi_E(u)) = \psi_{C''}(\epsilon \circ v). \tag{A.9}$$

We can now proceed with our actual objective – which was, of course, to prove that $\widetilde{X}$ is indeed the representing object for h. To this end, consider an arbitrary object $A \in \mathcal{A}$. Again using the admissibility property we can embed it into an exact triangle

$$B \to A \to C \stackrel{+}{\to}$$

where $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Using the functors h and $h_{\widetilde{X}}$ we now obtain exact sequences

$$\cdots \to h(C) \to h(A) \to h(B) \to \ldots$$

and

$$\cdots \to h_{\widetilde{X}}(C) \to h_{\widetilde{X}}(A) \to h_{\widetilde{X}}(B) \to \ldots$$

respectively that by corollary A.1.13 are embedded in the commutative diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & h(C) & \longrightarrow & h(A) & \longrightarrow & h(B) & \longrightarrow & \ldots \\ & & {\scriptstyle\Phi^\delta_C}\uparrow & & {\scriptstyle\Phi^\delta_A}\uparrow & & {\scriptstyle\Phi^\delta_B}\uparrow & & \\ \ldots & \longrightarrow & h_{\widetilde{X}}(C) & \longrightarrow & h_{\widetilde{X}}(A) & \longrightarrow & h_{\widetilde{X}}(B) & \longrightarrow & \ldots \end{array}$$

for any $\delta \in h(\widetilde{X})$. It is – hence – sufficient to prove that there is a $\delta \in h(X)$ such that $\Phi^\delta_B = \varphi_B$ and at the same time $\Phi^\delta_C = \psi_C$. We claim that the first is implied if $\varphi_E(u) = h(u)(\delta)$ and the latter if $\psi_{S_{\mathcal{C}}(C')}(\epsilon) = h(\epsilon)(\delta)$. To see this consider the construction of Φ^δ outlined in lemma A.1.12, we deduce immediately that what we need to prove is $\varphi_B(f) = h(f)(\delta)$ for $f \in \mathrm{Hom}_{\mathcal{A}}(B, \widetilde{X}) = h_{\widetilde{X}}(B)$. Consider the isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(B, E) \overset{u\circ -}{\underset{}{\stackrel{\cong}{\to}}} \mathrm{Hom}_{\mathcal{A}}(B, \widetilde{X})$$

obtained by the usual argument. Hence, there is a preimage $g \in \mathrm{Hom}_{\mathcal{B}}(B, E)$ of f such that $f = u \circ g$. We now consider the commutative diagram

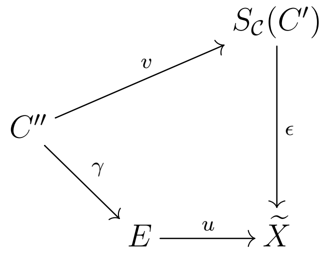
$$\begin{array}{ccc} h_{\widetilde{X}}(B) & \xrightarrow{\varphi_B} & h(B) \\ {\scriptstyle h_{\widetilde{X}}(g)}\uparrow & & {\scriptstyle h(g)}\uparrow \\ h_{\widetilde{X}}(E) & \xrightarrow{\varphi_E} & h(E) \end{array}$$

given by the natural transformation φ. Since $u \in h_{\widetilde{X}}(E)$ we now obtain $\varphi_B((h_{\widetilde{X}}(g))(u)) = h(g)(\varphi_E(u))$. Using our assumption that $\varphi_E(u) = h(u)(\delta)$ this provides us with

$$\Phi^\delta_B(f) = h(f)(\delta) = h(u \circ g)(\delta)$$
$$= h(g)h(u)(\delta) = h(g)(\varphi_E(u)) = \varphi_B((h_{\widetilde{X}}(g))(u)) = \varphi_B(u \circ g) = \varphi_B(f).$$

Similarly $\psi_{S_{\mathcal{C}}(C')}(\epsilon) = h(\epsilon)(\delta)$ implies $\Phi_C^\delta = \psi_C$.

Now, we use the following commutative "square" of the octahedral diagram



,

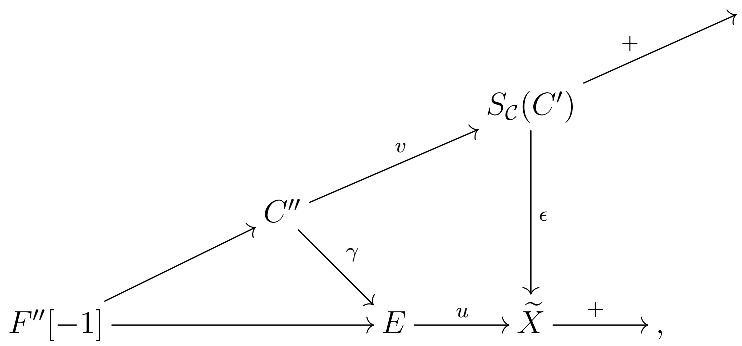
to obtain the – commutative – diagram

$$\begin{array}{ccc} h(C'') & \xleftarrow{h(v)} & h(S_{\mathcal{C}}(C')) \\ {\scriptstyle h(\gamma)}\uparrow & & {\scriptstyle h(\epsilon)}\uparrow \\ h(E) & \xleftarrow{h(u)} & h(\widetilde{X}). \end{array} \tag{A.10}$$

We have $\varphi_E(u) \in h(E)$ and $\psi_{S_{\mathcal{C}}(C')}(\epsilon) \in h(S_{\mathcal{C}}(C'))$ and know – from the fact that ψ is a natural transformation – that $\psi_{C''}(\epsilon \circ v) = h(v)(\psi_{S_{\mathcal{C}}(C')}(\epsilon))$. Combining this with equation (A.9), we obtain

$$h(\gamma)(\varphi_E(u)) = \psi_{C''}(\epsilon \circ v) = h(v)(\psi_{S_{\mathcal{C}}(C')}(\epsilon)). \tag{A.11}$$

Given the commutativity of the previous diagram, this makes it possible to look for a $\delta \in h(\widetilde{X})$ such that $h(u)(\delta) = \varphi_E(u)$ and $h(\epsilon)(\delta) = \psi_{S_{\mathcal{C}}(C')}(\epsilon)$. The octahedral diagram in particular supplies us with



which we will now rewrite as a morphism of exact triangles in the following manner

$$\begin{array}{ccccccc}
F''[-1] & \longrightarrow & C'' & \xrightarrow{v} & S_{\mathcal{C}}(C') & \longrightarrow & F'' \\
\| & & \gamma\downarrow & & \epsilon\downarrow & & \| \\
F''[-1] & \longrightarrow & E & \xrightarrow{u} & \widetilde{X} & \longrightarrow & F''.
\end{array}$$

Hence, by applying h, we can embed diagram (A.10) into the diagram

$$\begin{array}{ccccccc}
h(F''[-1]) & \longleftarrow & h(C'') & \xleftarrow{h(v)} & h(S_{\mathcal{C}}(C')) & \longleftarrow & h(F'') \\
\| & & h(\gamma)\uparrow & & h(\epsilon)\uparrow & & \| \\
h(F''[-1]) & \longleftarrow & h(E) & \xleftarrow{h(u)} & h(\widetilde{X}) & \longleftarrow & h(F'').
\end{array}$$

with exact rows. Lemma A.1.14, for the fact that the left equality in the diagram is – in particular – a mono- and the right an epimorphism, now provides us, together with equation (A.11), with the required δ. Hence, diagram (A.2) that we previously obtained via corollary A.1.13 now becomes

$$\begin{array}{ccccccccc}
h(B[1]) & \longrightarrow & h(C) & \longrightarrow & h(A) & \longrightarrow & h(B) & \longrightarrow & h(C[-1]) \\
\varphi_{B[1]}\uparrow & & \psi_C\uparrow & & \Phi_A^{\delta}\uparrow & & \varphi_B\uparrow & & \psi_{C[-1]}\uparrow \\
h_{\widetilde{X}}(B[1]) & \longrightarrow & h_{\widetilde{X}}(C) & \longrightarrow & h_{\widetilde{X}}(A) & \longrightarrow & h_{\widetilde{X}}(B) & \longrightarrow & h(C[-1])
\end{array}$$

in which $\psi_C, \varphi_B, \psi_{C[-1]}$ and $\varphi_{B[1]}$ are isomorphisms, due to the fact that $B \in \mathcal{B}, C \in \mathcal{C}$ and – since $\mathcal{B}$ and $\mathcal{C}$ are triangulated subcategories – also $B[1] \in \mathcal{B}$ and $C[-1] \in \mathcal{C}$. By the 5-lemma, this implies that Φ_A^{δ} is an isomorphism – and hence that h is representable.

One now simply dualises the very same techniques that were used to prove the representability of h to show that the covariant functor $\mathrm{Hom}(-,X)^*$ is representable. From the exact triangle

$$C'' \to X \to L \xrightarrow{+}$$

with $C'' \in \mathcal{C}$ and $L \in \mathcal{C}^{\perp}$ we obtain

$$\mathrm{Hom}_{\mathcal{C}}(S_{\mathcal{C}}^{-1}(C''), C_1) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(C_1, X)$$

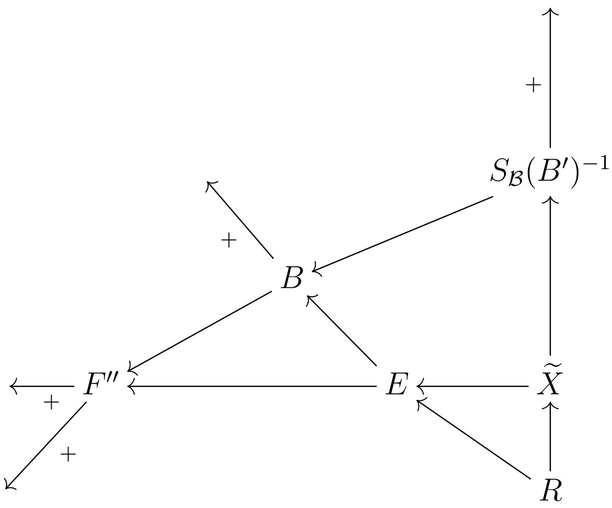
for any $C_1 \in \mathcal{C}$. Similarly we obtain $S_{\mathcal{B}}^{-1}(B')$ from

$$B' \to X \to C' \xrightarrow{+}$$

where $B' \in \mathcal{B}$ and $C' \in \mathcal{C}$. Letting $E = S_{\mathcal{C}}^{-1}(C'')$ we obtain

$$R \to E \to B \xrightarrow{+}$$

with $R \in \mathcal{C}$ and $B \in \mathcal{B}$. We get a morphism in $\mathrm{Hom}(S_{\mathcal{B}}^{-1}(B'), B)$ as the image of $\mathrm{id}_E \in \mathrm{Hom}_{\mathcal{C}}(E, E)$ and finally construct $\widetilde{X}$ by the octahedral diagram



where F'' is the cone of the mapping $S_{\mathcal{B}}(B')^{-1} \to B$.

We – hence – have constructed representing objects for both $h = \mathrm{Hom}(X, -)^*$ and for $\mathrm{Hom}(-, X)^*$. By lemma A.1.10, the proof is now finished.

□

Theorem A.1.15, however, provides a statement on the existence, yet not on the use of Serre functors. Their usefulness and hence with it the usefulness of theorem A.1.15 in general and in particular in our situation will be provided by the following well known theorem which is a straightforward implication of A.1.15.

Theorem A.1.16. *Let $\mathcal{A}$ be a triangulated category that has a Serre functor S. Assume $F \dashv G$ is an adjoint pair of functors $F : \mathcal{A} \to \mathcal{A}$ and $G : \mathcal{A} \to \mathcal{A}$. Then $G \dashv SFS^{-1}$ and $S^{-1}GS \dashv F$.*

Proof. The first claimed adjunction of the functors is proved by the equation

$$\begin{aligned}\mathrm{Hom}(X, SFS^{-1}(Y)) &= \mathrm{Hom}(S^{-1}(X), FS^{-1}(Y) = \mathrm{Hom}(FS^{-1}(Y), X)^* \\ &= \mathrm{Hom}(S^{-1}(Y), G(X))^* = (\mathrm{Hom}(G(X), Y)^*)^* = \mathrm{Hom}(G(X), Y).\end{aligned}$$

The other adjunction is proven similarly. □

A.2 Iwasawa decomposition

It is a useful fact that matrices in $\mathrm{GL}_2^+(\mathbb{R})$ can be split in a convenient way that – in particular – allows one to see a possible involved rotation.

Definition A.2.1. We define the following matrices

$$K_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix},$$

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \text{ and}$$

$$N_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for $\phi \in [0, 2\pi), x \in \mathbb{R}$ and $a \in \mathbb{R}_{>0}$.

From [36] we obtain the following lemma.

Lemma A.2.2. *For every $T \in \mathrm{GL}_2^+(\mathbb{R})$ there are $\phi \in \mathbb{R}, k, a \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$ such that*

$$T = kK_\phi A_a N_x.$$

Proof. See [36, Section 16.3]. □

References

[1] L. Angeleri Hügel, D. Happel and H.Krause, *Handbook of tilting theory*, London mathematical society lecture note series, 332, Cambridge University Press (2007).

[2] M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc., 3, VII (1957), 414–452.

[3] S. Awodey, *Category Theory*, Oxford logic guides 52 (2010).

[4] A. Bayer, *A short proof of the deformation property of Bridgeland stability conditions*, Math. Ann. 375, No. 3-4, 1597–1613 (2019).

[5] A. Bayer, *A tour to stability conditions on derived categories*, notes, (2011). (https://www.math.utah.edu/dc/dc-lecture-notes.pdf)

[6] A. Bayer and E. Macrí, *Projectivity and birational geometry of Bridgeland moduli spaces*, AMS (2014), 707–752.

[7] A. Bayer and E. Macrí, *The space of stability conditions on the local projective plane*, Duke Math. J., 160(2) (2011), 263–322.

[8] A. Bayer, E. Macrí, P Stellari, *The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds*, Invent. Math., Volume 206, Issue 3 (2016), 869–933.

[9] A. Beilinson, *Coherent sheaves on* $\mathbb{P}^n$ *and problems in linear algebra.* Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68–69; English transl. in Functional Anal. Appl. 12 (1978), 214–216.

[10] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux Pervers*, Asterisque 100, Société Mathématique de France (1983).

[11] A. I. Bondal, *Representation of associative algebras and coherent sheaves*, Math. USSR Izvestiya, Vol. 53, No. 1 (1989), 25–44.

[12] A. I. Bondal and M. Kapranov, *Representable functors, Serre functors, and mutations*, Math. USSR Izvestiya, Vol. 35, No. 3 (1990), 519–541.

[13] A. I. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalence*, Compos. Math. 125 (03) (2001), 327–344.

[14] S. Bradlow and O. García-Prada, *Stable triples, equivariant bundles and dimensional reduction*, Math. Ann., 304(1) (1996), 225–252.

[15] S. Bradlow, O. García-Prada and P. Gothen, *Moduli spaces of holomorphic triples over compact Riemann surfaces*, Math. Ann., 328(12) (2004), 299–351.

[16] T .Bridgeland, *Spaces of stability conditions*, Algebraic Geometry – Seattle 2005. Part I, 1–21.Proc. Sympos. Pure Math. 80, Part 1, AMS (2009), 1–21.

[17] T. Bridgeland, *Stability conditions on K3 surfaces*, Duke Math. J., 141(2) (2008) 241–291.

[18] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. Math., Vol. 166, No. 2 (2007), 317–345.

[19] T. Bridgeland and A. Maciocia, *Fourier-Mukai transforms for K3 and elliptic fibrations*, J. Algebr. Geom. 11, No. 4 (2002), 629–657.

[20] I. Burban and B. Kreussler, *Derived Categories of irreducible projective curves of arithmetic genus one*, Compos. Math. 142, No. 5 (2006), 1231–1262.

[21] J. Collins and A. Polishchuk, *Gluing stability conditions*, Adv. Theor. Math. Phys., Vol. 14, No. 2 (2010), 563–608.

[22] C. Daly, *Stability concepts in algebraic geometry*, Master thesis, Mary Immaculate College, University of Limerick (2007). (http://www.maths.mic.ul.ie/kreussler/CiaraDalyMA.pdf)

[23] H. Derksen and J. Weyman, *Quiver representations*, Not. Am. Math. Soc. 52(2) (2005), 200–206.

[24] S. E. Dickson, *A torsion theory for Abelian categories*, Trans. Amer. Math. Soc. 121, No. 1 (1966), 223–235.

[25] M. R. Douglas, *D-branes, categories and n =1 supersymmetry*, J. Math. Phys., 42(7) (2001), 2818–2843.

[26] M.R.Douglas, *Dirichletbranes, homological mirror symmetry and stability*, In Proceedings of the International Congress of Mathematicians, Vol.III, Higher Ed. Press (2002), 395–408.

[27] O. García-Prada, *Dimensional reduction of stable bundles, vortices and stable pairs* Int. J. Math. 5 (1994), 1–52.

[28] S. Gelfand and Yu. Manin, *Methods of homological algebra*, Springer monographs in mathematics (2003).

[29] P. O. Gneri, and M. Jardim, *Derived categories of functors and Fourier-Mukai transform for quiver sheaves*, preprint arXiv:1209.4307 (2012).

[30] A. L. Gorodentsev, S. A. Kuleshov, and A. N. Rudakov, *Stability data and t-structures on a triangulated category*, Izvestiya: Mathematics, 68(4) (2004), 749–781.

[31] A.L. Gorodentsev and A.N. Rudakov, *Exceptional vector bundles on projective spaces*. Duke Math. J. 54 (1987), no. 1, 115–130.

[32] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, Lond.Math.Soc., Vol. 119 (1988).

[33] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Volume 575. AMS (1996).

[34] R. Hartshorne, *Algebraic geometry*, Graduate texts in Mathematics (1977).

[35] J. Huizenga, *Birational geometry of moduli spaces of sheaves and Bridgeland stability*, Surveys on recent developments in algebraic geometry 95, AMS (2017), 101–148.

[36] D. Husemöller, M. Joachim, B. Jurco, M. Schottenloher, S. Echterho, and B. Krötz. *Basic bundle theory and K-cohomology invariants* [electronic resource], Springer (2007).

[37] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, USA: Oxford University Press (2006).

[38] D. Huybrechts *Introduction to stability conditions* in: Brambila-Paz, Leticia (ed.) et al., Moduli spaces. Based on lectures of a programme on moduli spaces at the Isaac Newton Institute for Mathematical Sciences, Cambridge, UK, January 4 July 1, 2011. Cambridge: Cambridge University Press, Lond. Math. Soc. 411 (2014), 179-229.

[39] A. Kuznetsov, *Calabi-Yau and fractional Calabi-Yau categories*, J. Reine Angew. Math. (2017), 239–267.

[40] A. Kuznetsov, *Derived categories of families of sextic del pezzo surfaces*, preprint arXiv:1708.00522 (2017).

[41] S. Lang, *Algebra*, 3rd edition, Springer (2002).

[42] J. Le Potier, *Lectures on Vector Bundles*, Cambridge studies in advanced mathematics, Cambridge University Press (1997).

[43] Q. Liu and J. Vitoria, *T-structures via recollements for piecewise hereditary algebras*, J. Pure Appl. Algebra 216 (2012), 837–849.

[44] S. Mac Lane, *Categories for the working mathematician*, Graduate texts in mathematics (1978).

[45] E. Macrí, *Stability conditions on curves*, Math. Res. Lett. 14, No. 4 (2007), 657–672.

[46] E. Macrí, *Some examples of spaces of stability conditions on derived categories*, arXiv:math/0411613 (2007).

[47] G. Maltsiniotis, *Quillen's adjunction theorem for derived functors, revised*, Quillen's adjunction theorem for derived functors revisited, C. R., Math., Acad. Sci. Paris 344, No. 9 (2007), 549-552.

[48] E. Martínez-Romero, *Stability of Arakelov bundles over arithmetic curves and Bridgeland stability conditions on holomorphic triples*, Ph.D. thesis, Freie Universität Berlin (2018).

[49] E. Martínez-Romero, A. Rincón-Hidalgo, and A. Rüffer, *Bridgeland stability condition on holomorphic triples over curves*, preprint arXiv:1905.04240 (2020).

[50] A. Rincón-Hidalgo, *Bridgeland stability conditions on the category of holomorphic triples*, Ph.D. thesis, Freie Universität Berlin (2019).

[51] S. Meinhardt, *Stability Conditions on generic complex tori*, Int. J. Math., Vol. 23, No. 05, 1250035 (2012), 1250035–1250052.

[52] S. Mukai, *An Introduction to Invariants and Moduli*, Cambridge studies in advanced mathematics, Cambridge University Press (2003).

[53] S. Okada, *Stability manifold of* $\mathbb{P}^1$, J. Algebr. Geom, 15(3) (2006), 487–505.

[54] A. Polishchuk, *Noncommutative two-tori with real multiplication as noncommutative projective varieties*, J. Geom. Phys. 50 (2004), 162–187.

[55] M. J. Redondo and A. Solotar, *Derived categories and their applications*, Revista de la union mathematica argentina, Volumen 48, Numero 3 (2007), 1–26.

[56] J. J. Rotman, *An introduction to homological algebra*, Universitex (2008).

[57] J.-P. Schneiders, *Quasi-abelian categories and sheaves*, Mémoires de la Société Mathématique de France 76 (1999), III1–VI140.

[58] R.P. Thomas, *Derived categories for the working mathematician* in: Vafa, Cumrun (ed.) et al., Winter school on mirror symmetry, vector bundles and Lagrangian submanifolds. Proceedings of the winter school on mirror symmetry, Cambridge, MA, USA, January 1999. Providence, RI: AMS. AMS/IP Stud. Adv. Math. 23 (2001), 349-361.

[59] The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu, (accessed on 16.07.2020).

[60] C. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics, Vol. 38 (1997).

[61] J. Woolf, *Stability conditions, torsion theories and tilting*, Lond. Math. Soc. (2)82, no. 3 (2010), 663–682.

[62] A. Yekutieli, *Introduction to derived categories* in: Eisenbud, David (ed.) et al., Commutative algebra and noncommutative algebraic geometry. Volume I: Expository articles. Cambridge: Cambridge University Press., Publ. Res. Inst. Math. Sci. 67, 431-451 (2015).